MAT200: Logic Language and Proof

Lecture 17 - April 5 2021

- Continue talking about theory of counting for infinite sets

Highlights:

- Theorem: $\mathbb{R} - \mathbb{Q}$ is nonempty.

- Theorem: $\mathbb{R} - \mathbb{A}$ is nonempty.

  Here

  $\mathbb{A} = \{x :$ there exists polynomial p with integer coefficients such that $p(x) = 0.\}$

- There are different infinities.

Let A, B be sets.

1) We define $|A| \leq |B|$ to mean that there is an injective function $f : A \to B$.

2) We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

3) We define $|A| < |B|$ to to mean that $|A| \leq |B|$ and not $(|A| = |B|)$

Last time: Let $A = \mathbb{R}$ and $B = \{x : 0 < x < 1\}$. Then $|A| = |B|$.

Today we'll do more examples

$f : \mathbb{R} \to (0,1)$

$f(x) = \dfrac{\arctan(x) + \frac{\pi}{2}}{\pi}$.

is a bijection.

$|\mathbb{N}| = |X|$

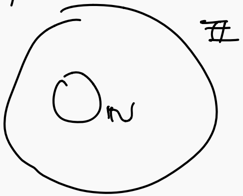
$|\mathbb{Z}| = |\mathbb{N}|$

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Theorem: $|\mathbb{N}| = |\mathbb{Z}|$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

$\mathbb{Z} = \{0, -1, 1, 2, -2, \ldots\}.$



Proof:

Define $f : \mathbb{N} \to \mathbb{Z}$

by:

$$f(x) = \begin{cases} \frac{x-1}{2} & x \text{ odd} \\ -\frac{x}{2} & x \text{ even} \end{cases}$$

This is a bijection.

Inverse is

$$g(x) = \begin{cases} 2x+1 & x \geq 0 \\ -2x & x < 0 \end{cases}$$

Easy to check $f \circ g = \mathrm{Id}_{\mathbb{Z}}$

$g \circ f = \mathrm{Id}_{\mathbb{N}}.$

So f is a bijection ■

To save time, we will define functions from $\mathbb{N} \to X$ by simply listing elements of X.

E.g. $f : \mathbb{N}_4 \to \{1, 5, 9, 4\}$

$f(1) = 1$
$f(2) = 9$
$f(3) = 5$
$f(4) = 4$

Will be written as

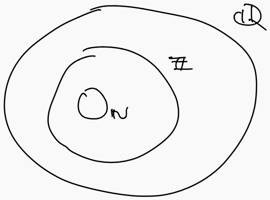
$f: \quad 1, 9, 5, 4$

Rough idea:

$f(1) = 0$
$f(2) = -1$
$f(3) = 1$
$f(4) = -2$
$f(5) = 2$
$\vdots$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.



Theorem: $|\mathbb{N}| = |\mathbb{Q}|$

Attempted

Proof: Define

$f : \mathbb{N} \to \mathbb{Q}$

$f(1) = 0$

$f(2) = \frac{1}{1}$

$f(3) = \frac{1}{2}$

$f(4) = \frac{1}{3}$

$f(5) = \frac{1}{4}$

$\vdots$

Is it actually a bijection?

No. Because

$f(x) \neq -1$

$f(x) \neq 2$

So not even surjective.

$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

Attempted

Proof: Define

$f : \mathbb{N} \to \mathbb{Q}$

$f(1) = 0$

$f(2) = \frac{1}{1}$

$f(3) = \frac{-1}{1}$

$f(4) = \frac{1}{2}$

$f(5) = \frac{-1}{2}$

$f(6) = \frac{2}{1}$

$f(7) = \frac{-2}{1}$

$f(8) = 1/3$

$f(9) = -1/3$

$f(10) = \frac{3}{1}$

$f(11) = \frac{-3}{1}$

$\vdots$

Is it actually a bijection?

No, because you only ever get numbers of the form $\pm\frac{1}{p}$ or $\pm p$.

Missing $\frac{2}{3}$.

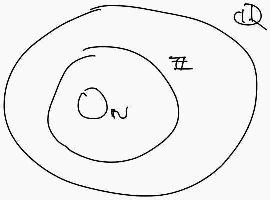
Alternatively:

$0, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{-1}{3}, \ldots$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

We define $|A| = |B|$ to mean that there is a bijection between $f: A \to B$.

Theorem: $|\mathbb{N}| = |\mathbb{Q}|$



Proof: Define

$f: \mathbb{N} \to \mathbb{Q}^+$

$0,$

$\frac{1}{1},$

$\frac{1}{2}, \frac{2}{1},$

$\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1},$

$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \frac{4}{2}, \frac{4}{1},$

$\frac{1}{5}, \frac{2}{5}, \ldots$

$\frac{1}{q}, \frac{2}{q}, \ldots$

Is this a bijection?

It is surjective:

To get p/q, just wait until the q^{th} "level".

Not injective:

$\frac{2}{1}$ and $\frac{4}{2}$ appear twice.

But this easy to fix, just don't list the repeats.

So from this I can get a bijection.

Streamlined Proof (Idea: describe f by describing an algorithm recipe procedure for computing it).

Define $f: \mathbb{N} \to \mathbb{Q}^+$ by the following algorithm:

* Listing all the rational numbers with denominator less than 1.
* Listing all the rational numbers with denominator 2, and their reciprocals.

⋮

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Theorem: $|\mathbb{N}| = |\mathbb{Q}|$

Proof: Define

$f : \mathbb{N} \to \mathbb{Q}^+$

$0,$

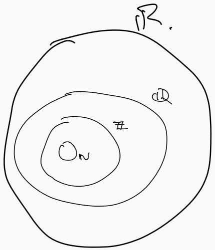
$\frac{1}{1},$

$\frac{1}{2}, \frac{2}{1},$

$\frac{1}{3}, \frac{2}{3}, \frac{3}{2}, \frac{3}{1},$

$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{3}, \frac{4}{2}, \frac{4}{1},$

$\frac{1}{5}, \frac{2}{5}, \ldots\ldots$



$\frac{1}{q}, \frac{2}{q}, \ldots$

Streamlined Proof (Idea: describe f by describing an algorithm recipe procedure for computing it).

Define $f : \mathbb{N} \to \mathbb{Q}^+$ by the following algorithm:

* Listing all the rational numbers with denominator less than 1.
* Listing all the rational numbers with denominator 2, and their reciprocals.

$\vdots$

* Skip numbers that have already been listed.

It's surjective because

To get p/q, just wait until we are listing the numbers with denominator q.

1 2 3 4 5 , — —

x_1 x_2 x_3 x_4 x_5 .

We define $|A| = |B|$ to mean that there is a bijection between $f: A \to B$.

informally - can make a list.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

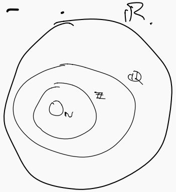
Theorem: If X and Y are countably infinite then $X \cup Y$ is countably infinite.

Proof: X countably infinite $\Rightarrow$ there is a bijection

f: $x_1, x_2, x_3, \ldots$

Y countably infinite $\Rightarrow$ there is a bijection

g: $y_1, y_2, y_3, \ldots$



Now form the bijection.

h: $x_1, y_1, x_2, y_2, x_3, y_3 \ldots$:

QED.

* See HW for more formal proof.
* There is a small mistake in this proof.

Example: $\mathbb{R}$ not countably infinite (not proved yet)

$1, 2, 3, \pi, e, \pi^2, \sqrt{2}, 4, \ldots$
$\sqrt{3}, \sqrt{73}, \ldots$

In this sense there are "different infinities".

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Definition: If $X_1, X_2, \ldots$ is an infinite sequence of sets then $\cup_i X_i$ is defined to be the set

$$\cup_i X_i = \{x : \exists i, \quad x \in X_i\}$$

Theorem: If $X_1, X_2, \ldots$ is an infinite sequence of countable sets then $\cup_i X_i$ is countable.

Take-away:
Very "difficult" to get uncountable sets.

Proof: X_i countable means $\exists$ bijection.

f_i: $x_{i1}, x_{i2}, x_{i3}, x_{i4} \ldots$

for each i.

E.g.

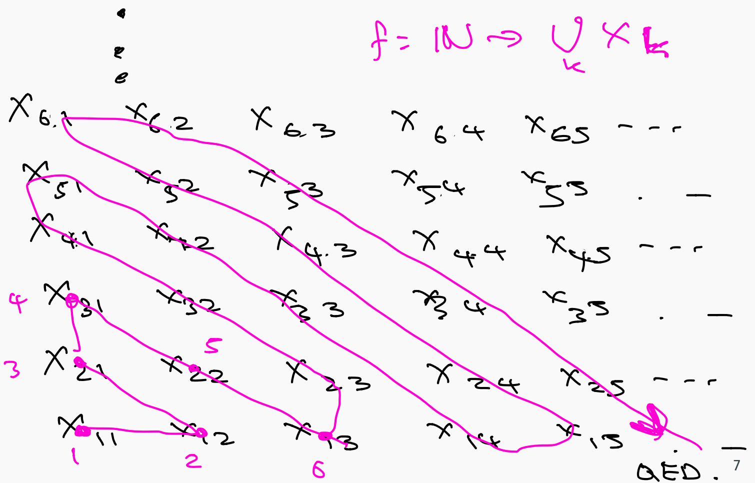
X_k = rational numbers with denom k

Then

$\bigcup_k X_k$ = rational numbers with any denom.

Consider this

$f : \mathbb{N} \to \bigcup_k X_k$



QED.

We define $|A| = |B|$ to mean that there is a bijection between $f: A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Theorem: If X and Y are countably infinite then $X \times Y$ is countably infinite.

E.g. $\mathbb{Z} \times \mathbb{Z}$ is countable.

Proof: very similar to previous slide.

E.g.

$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$

is countable

because

$\underbrace{(\mathbb{Z} \times \mathbb{Z})}_{\text{countable}} \times \underbrace{(\mathbb{Z} \times \mathbb{Z})}_{\text{countable.}}$

There are lots of countable sets.

Power set of $\{1, 2, 3\}$ is finite.

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Theorem: If there exists a surjection $f : \mathbb{N} \to A$, then A is countable, or finite.

We have the following theorems:

- $\mathbb{Z}$ is countable
- $\mathbb{Q}$ is countable
- Any product of countable sets is countable
- Any union of countable sets is countable
- The set of all possible English/Chinese/Human language texts is countable

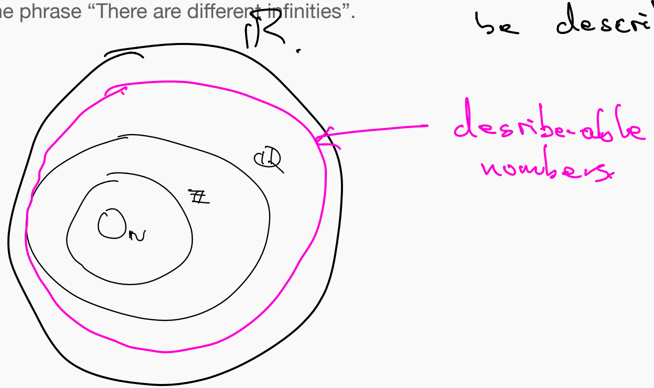
* The set of all numbers describe-able in english, is countable.

Can we even think of a set that's not countably infinite?

Consequence: some real numbers cannot be described.

Yes. $\mathbb{R}$,

This explains the phrase "There are different infinities".



Proof: (for english)

* List all texts of length 1 (26)

* List all texts of length 2 (26^2)

* List all texts of length 3 (26^3)

⋮

This infinite list contains all the possible english texts.

Is $P(\mathbb{N})$ countable?

recall $P(\mathbb{N}) =$ set of all subsets of $\mathbb{N}$.

Attempted Proof:

* ϕ, $\mathbb{N}$,
* $\{1\}$, $\mathbb{N} - \{1\}$
* $\{2\}$ $\{1,2\}$, $\mathbb{N} - \{1,2\}$,
* $\{3\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, $\mathbb{N} - \{1,2,3\}$, $\mathbb{N} - \{2,3\}$, ...
* $\{4\}$, ... all subsets of $\{1,2,3,4\}$, ...

* $\{9,2\}$ ⋮ all subsets of $\{1,2,3,6,\dots,9\}$

⋮

Is this a bijection?

No, The set of even numbers is not in the list.

No. Not surjective.

because all the sets in my list are finite. Missing $\mathbb{N}$.

$P(A) = \{\phi, \{1\}, \{0\}, A\}$.

$A = \{0,1\}$