MAT200: Logic Language and Proof

Lecture 10 - March 8 2021

- HW4 Review
- Functions ← Ch 8.

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1) $P(C) = \{x : x \subset C\}$
2) $C \cap D = \{x : x \in C \text{ and } x \in D\}$.
3) $E = F$ mean $x \in E \Leftrightarrow x \in F$

Problem 5 (10 points)

Consider any sets A and B.

(a) Show that $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

(b) Show that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Under what conditions does equality hold?

Common mistakes:

- Writing meaningless statements
- Writing false statements

sets
A or B
$P \cap Q$
propositions.

Your proofs should obey the following rules:

- All statements are meaningful
- All statements are true

  (unless you are doing proof by contradiction).

  - Don't abuse the = and $\Longleftrightarrow$ signs.

(that's why we prefer not use by contradiction)

$P(n) = n = 5q$, $\quad n = 5q$, where q is integer

$P(q) = \frac{5q}{q}$ number.

prop.

$P(1) = \frac{5(1)}{5} = 1$

$P(n) \implies P(n+5)$

$\frac{n+5}{5} = P(n+5)$

$\frac{5q+5}{5} = P(n+5)$ distributivity

$\frac{5(q+1)}{5} = P(n+5)$

$q + 1 = P(n+5)$ True

$P(n+5)$ is shown above and it's true

Almost complete proof:

(a) By definition of set equality, we have to show that $X \in \mathcal{P}(A) \cap \mathcal{P}(B) \iff X \in \mathcal{P}(A \cap B)$.

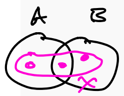
$$\begin{aligned} X \in \mathcal{P}(A) \cap \mathcal{P}(B) &\iff X \in \mathcal{P}(A) \text{ and } X \in \mathcal{P}(B) \\ &\iff X \subseteq A \text{ and } X \subseteq B \\ &\iff X \subseteq A \cap B \\ &\iff X \in \mathcal{P}(A \cap B) \end{aligned}$$

The first equivalence is from definition of $\cap$. The second eqeuivalence is from definition of $\mathcal{P}$. The third equivalence is from definition of $\cap$. The last equivalence is from definition of $\mathcal{P}$.

Hence, $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

(b) By definition of set equality, we have to show that $X \in \mathcal{P}(A) \cup \mathcal{P}(B) \iff X \in \mathcal{P}(A \cup B)$. We have the following chain of equivalences, obtained by expanding/contracting the relevant definitions:

$$\begin{aligned} X \in \mathcal{P}(A) \cup \mathcal{P}(B) &\iff X \in \mathcal{P}(A) \text{ or } X \in \mathcal{P}(B) \\ &\iff X \subseteq A \text{ or } X \subseteq B \\ &\implies X \subseteq A \overset{\text{or}}{\cancel{\cup}} B \\ &\iff X \in \mathcal{P}(A \cup B) \end{aligned}$$



So $X \in P(A) \cup P(B) \Rightarrow X \in P(A \cup B)$
So $P(A) \cup P(B) \subset P(A \cup B)$.

Part of proof writing skill is determining what to focus on.

When you have two proofs that are so similar, the interesting parts lie in the differences.

A function is an unambiguous rule for assigning inputs to outputs.

It consists of three parts:

- Domain (a set)
- Range (a set) also called codomain.
- The actual rule

The rule can be anything, as long as it is unambiguous.
It does not have to come from a 'formula'

Example:

$sq' : \mathbb{Z} \to \mathbb{R}, \quad sq'(x) = x \cdot x$

$\longrightarrow \boxed{sq'} \longrightarrow$

sq' is different from sq.

Example

$\underbrace{sq}_{\text{name.}} : \underbrace{\mathbb{R}}_{\text{domain}} \to \underbrace{\mathbb{R}}_{\text{range.}}, \quad \underbrace{sq(x) = x \cdot x}_{\text{rule.}}$

$x \longrightarrow \boxed{sq} \longrightarrow sq(x).$

To evaluate this:

$\underbrace{sq}_{\text{function.}}(\underset{\uparrow}{3}) = 3 \cdot 3 = \underset{\uparrow}{9}.$

input output

Example:

motherOf : humans $\to$ humans.

motherOf(x) = the biological mother of x.

$x \longrightarrow \boxed{\text{motherOf}} \longrightarrow \text{motherOf}(x).$

Warning: Don't mix up $\underbrace{\text{"}sq\text{"}}_{\text{function.}}$ and $\underbrace{\text{"}sq(x)\text{"}}_{\text{number.}}$

$\underbrace{\text{"motherOf"}}_{\text{function}}$ $\quad$ $\underbrace{\text{"motherOf}(x)\text{"}}_{\text{human.}}$

Example:

$sq'' : \mathbb{R} \to \mathbb{R},$

$sq''(x) = (x+1)^2 - 2x - 1.$

What does it mean for 2 functions to be equal?

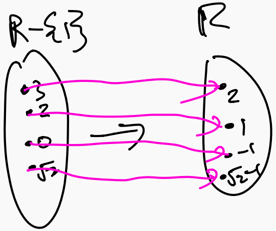
Defn:

$f : X \to Y,$
$g : X' \to Y'$
are equal if:

1) Same domain $X = X'$
2) Same range $Y = Y'$
3) Rule is same:
$\forall x \in X, \; f(x) = g(x).$

$f : \mathbb{R} - \{1\} \to \mathbb{R}$

$f(x) = \dfrac{(x-1)^2}{x-1}$



$x=1?$

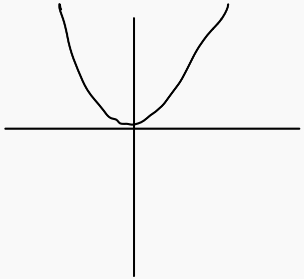
$f : \mathbb{R} \to \mathbb{R}$

$f(x) = \dfrac{(x-1)^2}{x-1}$

Not "well defined"

$f(x) = x^2$

Graph:



motherOf : humans → humans.

motherOf(x) = the biological mother of x.

Graph?

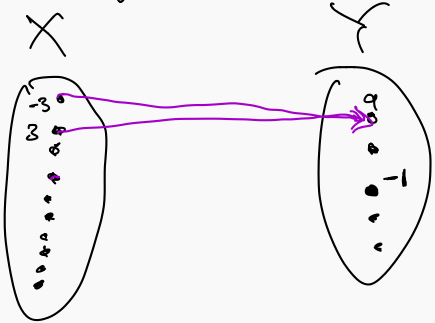
Fine:

$x=1$

$f : \mathbb{R} \to \mathbb{R}$

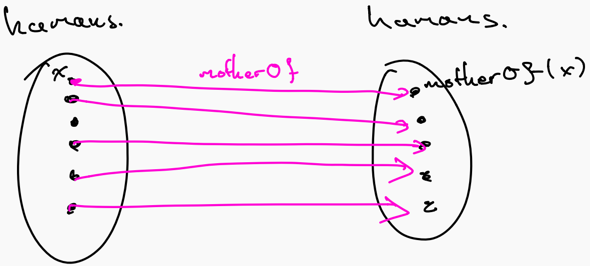
$f(x) = \begin{cases} \dfrac{(x-1)^2}{x-1} & x \neq 1 \\ 7 & x = 1 \end{cases}$

$sq : \mathbb{R} \to \mathbb{R}$



Nothing is pointing towards −1.

motherOf : humans → humans.



When you define functions, (domain, range, rule).
they need to "well-defined".

a) 0) Must specify domain, range, rule.

b) 1) The rule should describe the output unambiguously.

c) 2) Every input must result in an output.
The output must lie in the range.

Not well defined

childOf: Humans → Humans

Examples:

a) childof(x) = the child of x.

b) childof(Donald Trump). = the child of Donald Trump
WHICH ONE??

c) childOf(Peter Liu) = the child of Peter Liu = ?

Example:

$$\sqrt{\ }: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

$\sqrt{x}$ = The real a satisfying $a \cdot a = x$.

a) 0) Must specify domain, range, rule.
b) 1) The rule, with domain/range should describe the output unambiguously.

fine. $\sqrt{9} = \cancel{\pm} 3$

c) 2) Every input must result in an output. The output must lie in the range.

(how would you prove that?).

This is well defined.

Example:

$$\sqrt{\ }: \mathbb{R}_{\geq 0} \to \mathbb{R}$$

$\sqrt{x}$ = The real a satisfying $a \cdot a = x$.

a) 0) Must specify domain, range, rule.
b) 1) The rule, with domain/range should describe the output unambiguously.

$\sqrt{9} = \begin{matrix} 3 \\ -3 \end{matrix}$?

c) 2) Every input must result in an output. The output must lie in the range.

(how would you prove that?)

This is not well defined!

Example:

$$\sqrt{\ }: \mathbb{R} \to \mathbb{R}_{\geq 0}$$

Not well defined.

$\sqrt{x}$ = The real a satisfying $a \cdot a = x$.

a) 0) Must specify domain, range, rule.
b) 1) The rule, with domain/range should describe the output unambiguously.

c) 2) Every input must result in an output. The output must lie in the range.

$\sqrt{-1} =$?

This is well defined.

If $g: X \to Y$ and $f: Y \to Z$ are functions, then the composition of f and g is the function:

$$f \circ g : X \to Z \qquad f\circ g\,(x) = f(g(x))$$ ← defn of composition.

* f is a function.
* g is a function
* $f\circ g$ is a function.

E.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ $\qquad g: \mathbb{R} \to \mathbb{R}$, $g(x) = x+1$

Then

$f \circ g : \mathbb{R} \to \mathbb{R}$ defn of '$\circ$'

$$f\circ g\,(x) = f(g(x)) = (g(x))^2 = (x+1)^2.$$

E.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -x^2$ $\qquad g: \mathbb{R}_{\geq 0} \to \mathbb{R}$, $g(x) = x+1$

$g\circ f\,(x) = -x^2 + 1$

WRONG

$g \circ f$ is not well defined.

Summary

- The concept of a **function**
  - The parts of a function:
    - Domain, range, rule
  - What it means to be **well-defined**
    - The rule has to be applicable to everything in the domain
    - The rule should be unambiguous
- Operations on functions
  - Composition
  - Restriction
  - Multiplication, addition
- Special function: Identity function