MAT200: Logic Language and Proof

Lecture 5 - February 17 2021

Recently: Proving basic things using only the axioms

Today: Proof by induction

- I will not mention some things but you should think about the implicit assumptions that we make.

HW2 - Problem 5c)

Prove that

$x^2 = 0 \implies x = 0.$

Incorrect (not even wrong) Proof:

$x^2 = 0$ then $x = 0$ if $x \neq 0$
where $x^2 = 0 = p$ and x=0=q
then $x^2 \neq 0$ x is an integer if $x \neq 0$
an integer,q, times an integer q is an integer.
Therefor $x^2 = x(x) = aninteger$, therefor $x^2 \neq 0$
The contrapositive

$$\neg q = \neg p$$

is true making $p = q$ true.

Correct Proof:

We wish to prove that $x^2 = 0 \implies x = 0.$

Suppose for contradiction that $x^2 = 0$ and $x \neq 0.$

Since $x \neq 0,$ we have by division axiom there exists a real number x^{-1} such that $xx^{-1} = 1.$

Multiplying both sides of $x^2 = 0$ by x^{-1} yields $x \cdot x \cdot x^{-1} = 0 \cdot x^{-1}.$

The left side is equal to $x \cdot 1,$ which is equal to x by the unity law.

The right hand side is equal to 0 because we proved that $0 \cdot a = 0$ for any real a .

Therefore $x = 0.$

This contradicts the assumption that $x \neq 0.$ Thus we have proved the original claim.

HW2 - Problem 3c)

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Correct Proof:

We wish to prove that $x^2 = 0 \implies x = 0.$

We will proceed by contradiction: assume that $x^2 = 0$ and $x \neq 0.$

Since $x \neq 0$, then by the division axiom, there exists a real number x^{-1} such that $x \cdot x^{-1} = 1.$

Then

$$
\begin{aligned}
x^2 = 0 &\implies x \cdot x = 0 && \text{by definition of square} \\
&\implies (x \cdot x) \cdot x^{-1} = 0 \cdot x^{-1} && \text{doing the same operation to both sides} \\
&\implies x \cdot (x \cdot x^{-1}) = 0 \cdot x^{-1} && \text{associativity} \\
&\implies x \cdot 1 = 0 && \text{left: definition of } x^{-1}\text{, right: using given fact } a \cdot 0 = 0 \text{ for } a \text{ real} \\
&\implies x = 0 && \text{by unity}
\end{aligned}
$$

This contradicts our assumption that $x \neq 0$, so we have our desired contradiction.

Let $P(n)$ be the statement

"n is even"

Let $P(n)$ be the statement

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Example: Let P(n) be the statement:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

1) Prove the statement P(1).
2) Prove the statement $P(n) \implies P(n+1)$.

WARNING: $P(n)$ is not the same statement as $P(n) \implies P(n+1)$

Example: For all positive integers n,

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

Getting started with induction:

- Write down exactly what P(n) is
- Verify a few examples by hand, for specific n.
  - Write down the statement P(1) and verify that it is true.
- Write down the statement “P(n)=>P(n+1)”
- Prove the statement “P(n)=>P(n+1)”

Induction template: • First we verify the base case P(1):

- Now we prove the inductive step P(n)=>P(n+1):
  - Suppose P(n) is true. This means that…
  - We have thus shown that P(n+1) is true.
- The result follows by the principle of induction.

Theorem: For all positive integers $n,$ the number $n^3 + 2n$ is divisible by 3.

Theorem: For all positive integers n, for all real x, we have the inequality $(1+x)^n > 1 + nx$.

Theorem: For all positive integers n,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

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Induction template:

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- Now we prove the inductive step P(n)=>P(n+1):
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- The result follows by the principle of induction.

Example:

There do not exist integers m and n such that

$$14m + 20n = 101.$$

- Suppose for contradiction that m, n are integers such that $14m + 20n = 101.$
- Then $2(7m + 10n) = 101.$
- Thus $101 = 2q$ for some integer $q,$ so 101 is even.
- This contradicts the fact that 101 is odd.

Example:

101 is odd.

We need to prove that 101 is not even.

Suppose for contradiction that 101 is even.

Then $101 = 2q$

Then $1 = 101 - 100 = 2(q - 50)$.

If $q - 50 \leq 0$ then $1 \leq 0$, a contradiction.

If $q - 50 \geq 1$ then $1 \geq 2$, also a contradiction.

So 101 cannot be even.

Recall:

$P \implies Q$ is equivalent to $\neg Q \implies \neg P$

This can used to transform statements into something easier to prove.

Example:

$$ac \leq bc \implies c \leq 0$$

We prove the contrapositive: if $c > 0$ then $ac > bc$.

This is axiom 3.1.2 ii) in the textbook.

Recall:

$P \implies Q$ is equivalent to $\neg Q \implies \neg P$

This can used to transform statements into something easier to prove.

Example:

If n^2 is odd then n is odd.

We prove the contrapositive: if n is even then n^2 is even.
Suppose n is even, then n=2q where q is an integer.
Then $n^2 = 4q^2 = 2(2q^2)$, so we have shown that n^2 is even.

We prove the contrapositive: if n is even then n^2 is even.
This follows from the fact we proved last time:
If a and b are even, then ab is even.
Indeed we can take a=b=n in the above statement.

Example:

$$ab = 0 \implies a = 0 \text{ or } b = 0$$

It suffices to prove that

$ab = 0$ and $a \neq 0 \implies$ b=0.

Suppose a, b are real numbers such that $ab = 0$, and suppose $a \neq 0$.

Multiplying both sides by a^{-1} yields $a^{-1}ab = a^{-1} \cdot 0$.

The left hand side is equal to b.

The right hand side is equal to 0.

So b=0 as desired.

- Proof by contradiction
- Proof by cases
- Proof by contrapositive
- Proving an ‘or’ statement.

Warning: many proofs do not fit neatly into one of these categories.

Especially try to avoid proof by contradiction (usually proof by contrapositive is better).