MAT200: Logic Language and Proof

Lecture 17 - April 5 2021

- Continue talking about theory of counting for infinite sets

Highlights:

- Theorem: $\mathbb{R} - \mathbb{Q}$ is nonempty.

- Theorem: $\mathbb{R} - \mathbb{A}$ is nonempty.

  Here

  $\mathbb{A} = \{x :$ there exists polynomial p with integer coefficients such that $p(x) = 0.\}$

- There are different infinities.

Let A, B be sets.

We define $|A| \leq |B|$ to mean that there is an injective function $f : A \to B$.

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

We define $|A| < |B|$ to to mean that $|A| \leq |B|$ and not ($|A| = |B|$)

Last time: Let $A = \mathbb{R}$ and $B = \{x : 0 < x < 1\}$. Then $|A| = |B|$.

Today we'll do more examples

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Theorem: $|\mathbb{N}| = |\mathbb{Z}|$

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Theorem: $|\mathbb{N}| = |\mathbb{Q}|$

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Theorem: If X and Y are countably infinite then $X \cup Y$ is countably infinite.

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Definition: If $X_1, X_2, \ldots$ is an infinite sequence of sets then $\cup_i X_i$ is defined to be the set

$$\cup_i X_i = \{x : \exists i, \quad x \in X_i\}$$

Theorem: If $X_1, X_2, \ldots$ is an infinite sequence of sets then $\cup_i X_i$ is countable.

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Theorem: If X and Y are countably infinite then $X \times Y$ is countably infinite.

We define $|A| = |B|$ to mean that there is a bijection between $f : A \to B$.

Definition: If $|\mathbb{N}| = |X|$ then X is said to be **countably infinite**.

Theorem: If there exists a surjection $f : \mathbb{N} \to A$, then A is countable.

We have the following theorems:

- $\mathbb{Z}$ is countable
- $\mathbb{Q}$ is countable
- Any product of countable sets is countable
- Any union of countable sets is countable
- The set of all possible English/Chinese/Human language texts is countable

Can we even think of a set that's not countably infinite?

Yes.

This explains the phrase "There are different infinities".