Review #3 Monday May 11

* What problems do you have questions about?
* Let me know if already covered in previous review today.

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545. Find anti derivatives of $g(x) = \sqrt{x} - \frac{1}{x^2}$

Recall:

f	f'
x^n	nx^{n-1}

Solution

Rewrite

$$g(x) = x^{1/2} - x^{-2}$$

So one antiderivative is

$$\frac{2}{3}x^{3/2} + x^{-1}$$

All the antiderivatives are

$\int g\,dx = \frac{2}{3}x^{3/2} + x^{-1} + C$

(i.e. differentiative above yields $x^{1/2} - x^{-2}$).

Verify:

$$\frac{d}{dx}\frac{2}{3}x^{3/2} + x^{-1} + C$$

$$= \frac{2}{3}\cdot\frac{3}{2}\cdot x^{1/2} - x^{-2} + 0$$

$$= x^{1/2} - x^{-2}.$$

Notation:

antiderivative of $g \iff \int g\,dx$

2f Midterm 2

Find $\frac{dy}{dx}$ if $y = \ln((2x)^x)$

Key: use log laws to simplify first:

* $\ln(a^b) = b\ln(a)$
* $\frac{d}{dx}\ln x = \frac{1}{x}$

Solution:

$$y = x\ln(2x)$$ (using log law)

So

$$\frac{dy}{dx} = 1 \cdot \ln(2x) + x\frac{2}{2x}$$ (product rule)

$$\boxed{= \ln(2x) + 1.}$$

1e) Midterm

$$\lim_{x \to 0} \frac{\tan x}{\sin x} = ?$$

Can't plug in 0 $\left(\frac{0}{0}\right)$.

Recall:

$$\tan x = \frac{\sin x}{\cos x}$$

$$\lim_{x \to 0} \frac{\tan x}{\sin x} = \lim_{x \to 0} \frac{\frac{\cancel{\sin x}}{\cos x}}{\cancel{\sin x}}$$

$$= \lim_{x \to 0} \frac{1}{\cos x}$$

$$= \frac{1}{\cos(0)}$$

$$= 1$$

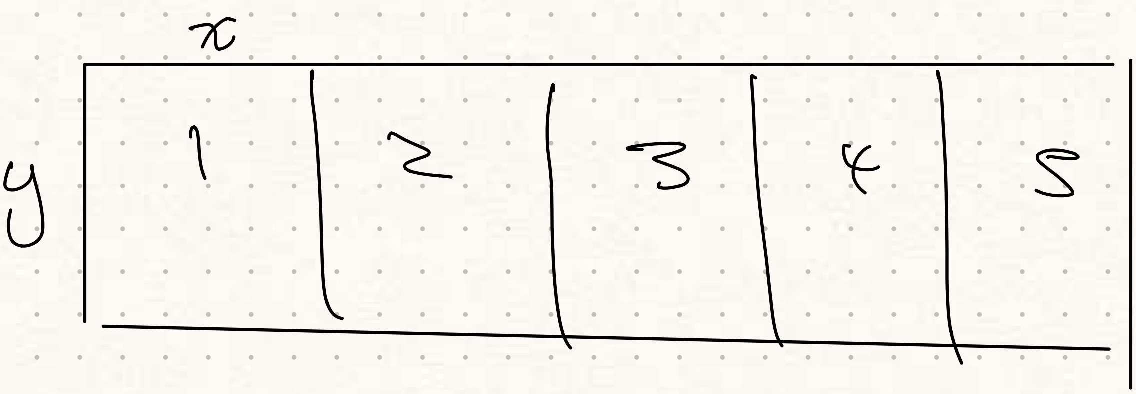
4.6: 295, 304

(Removed everything else).

4.7 354

Five adjacent, identical pens

total area 1000.



What dimensions to minimize

total amount of fencing?

① Draw picture & label quantities.

② Constraint:

Area = 1000.

width × height = $\boxed{5xy = 1000}$

Goal: Minimize Fencing.

Total Fencing = $\underbrace{6 \cdot y}_{\text{vertical fencing}} + \underbrace{5x + 5x}_{\text{Top and bottom fencing.}}$

i.e. $\boxed{\begin{array}{l}\text{Minimize} \\ \quad F = 6y + 10x\end{array}}$

(3) Sub. constraint into Goal.

Constraint: $5xy = 1000$

$xy = 200$

$\Rightarrow y = \frac{200}{x}$

Goal:

$F = 6y + 10x$

$= \frac{6 \cdot 200}{x} + 10x$

$= 6 \cdot 200 x^{-1} + 10x$

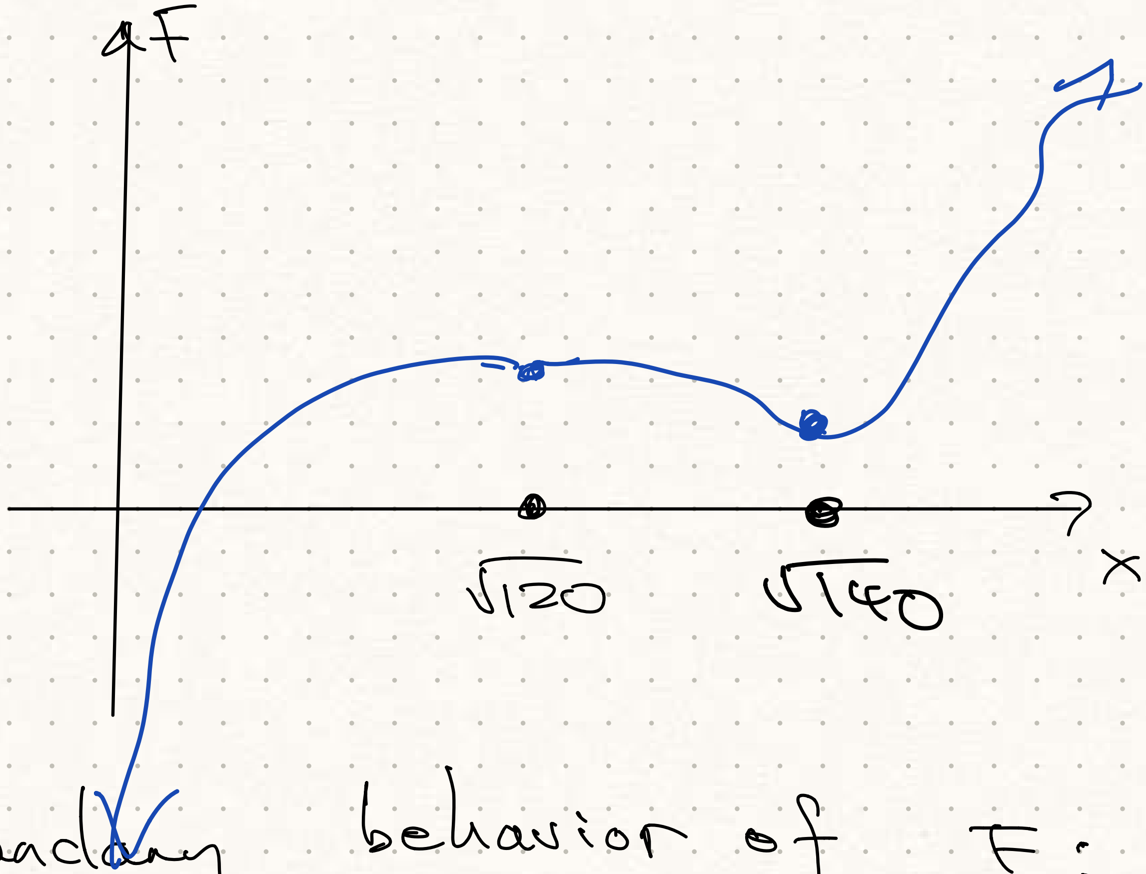
(4) Minimize F $(x > 0)$

Critical points:

$F' = 0 \Leftrightarrow -6 \cdot 200 x^{-2} + 10 = 0$

$\Leftrightarrow 10 = 1200 x^{-2}$

$\Leftrightarrow x^2 = 120 \Leftrightarrow x = \sqrt{120}$



Boundary behavior of F:

$$\lim_{x \to 0^+} F(x) = \lim_{x \to 0^+} \frac{6 \cdot 200}{x} + 10x$$

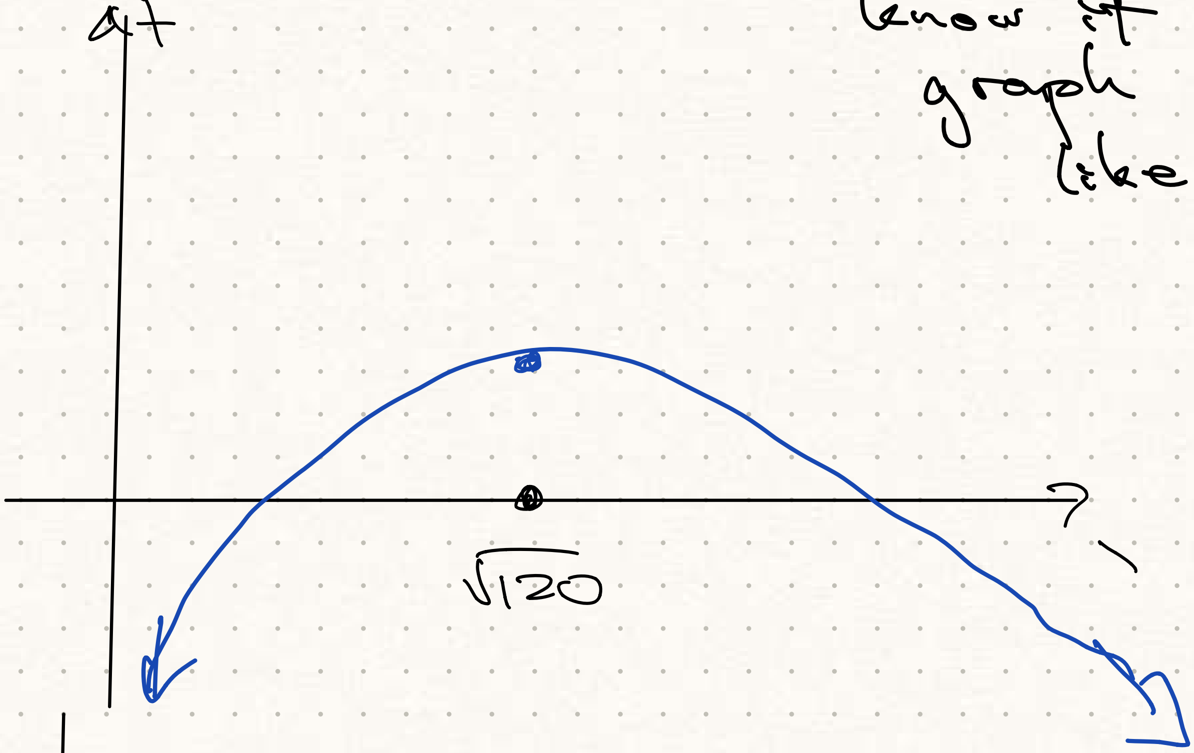
$$= \infty$$

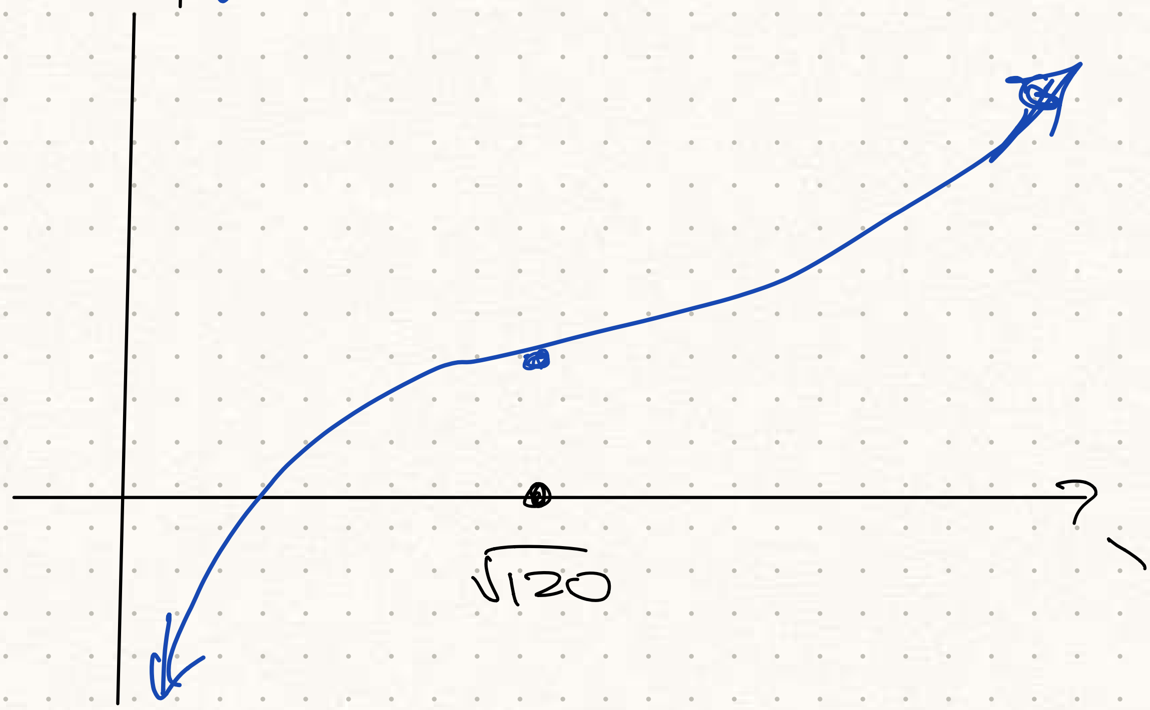
$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \frac{6 \cdot 200}{x} + 10x$$

$$= \infty$$

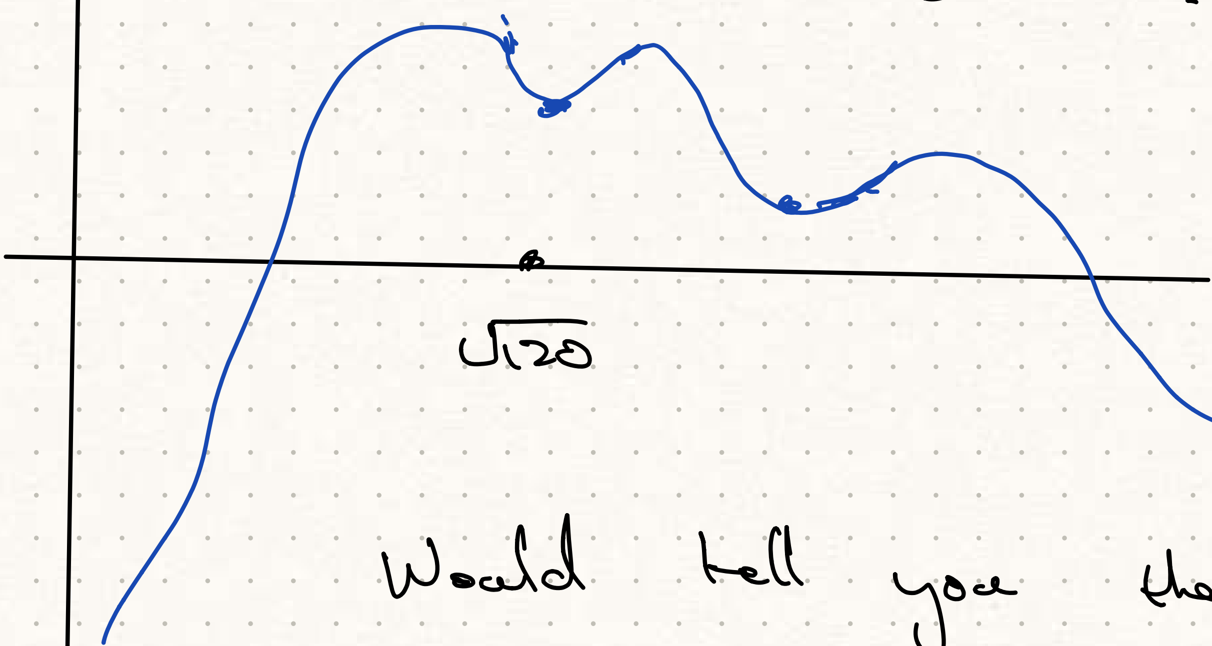
Looking at graph, we conclude that
$x = \sqrt{120}$ is minimum

If we hadn't looked at boundary, we don't know it graph looks like:





Second derivative test is not reliable:



Would tell you there is local min. But need more thought to ensure is abs. min.

$$x = \sqrt{120} = \sqrt{4 \cdot 30} = 2\sqrt{30}.$$

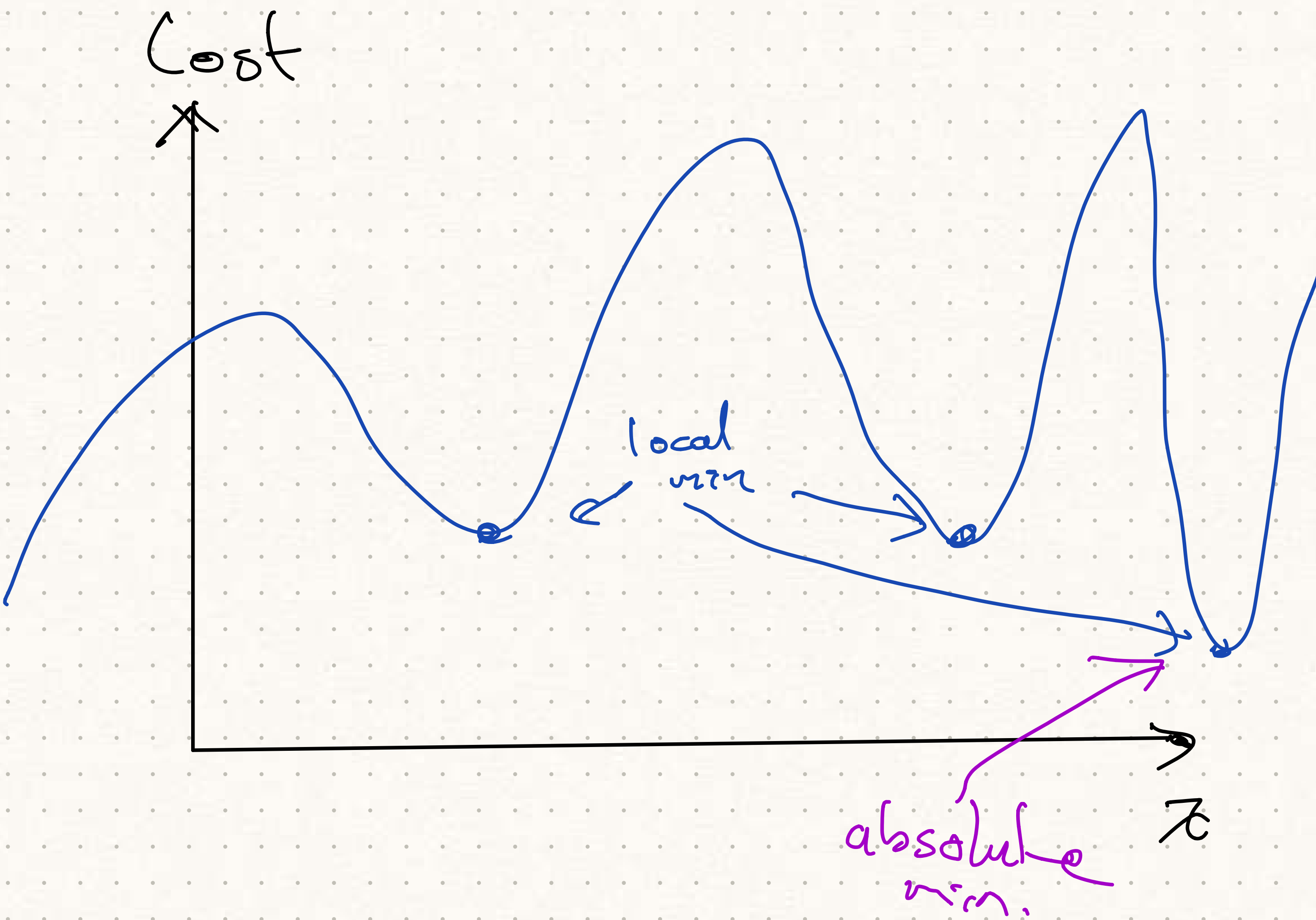
So best dimensions are

$$\boxed{\begin{aligned} x &= \sqrt{120} \\ y &= \frac{200}{\sqrt{120}} \end{aligned}}$$

(use constraint

$5xy = 1000$

$\Rightarrow y = \frac{200}{x}$)



To find absolute min/max
(Ch 4.5)

1) Find ~~all~~ critical points

2) Find boundary behavour

3) Compare results and
pick smallest/biggest

x	$F(x)$
$\sqrt{120}$	$\frac{6\cdot 200}{\sqrt{120}} + 10\sqrt{120}$
$\sqrt{140}$	$\frac{6\cdot 200}{\sqrt{140}} + 10\sqrt{140}$
0	∞
∞	∞

4.1 Problem 21

Radius of sphere increasing at 9 cm/sec.

Find radius of sphere when volume and radius are increasing at same numerical rate.

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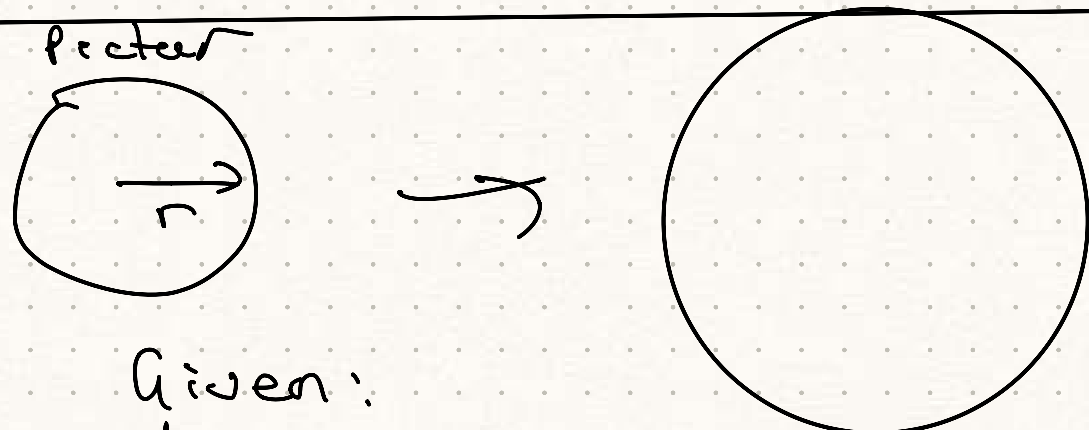
① Picture



Given:

② $\frac{dr}{dt} = 9$

Goal:

Find r when $\frac{dV}{dt} = \frac{dr}{dt}$

③ Relationship between V and r:

$$V = \frac{4\pi r^3}{3}$$

④ Differentiate relationship:

(implicit diff, diff both sides).

$$\frac{dV}{dt} = \frac{3 \cdot 4\pi r^2}{3} \frac{dr}{dt}$$

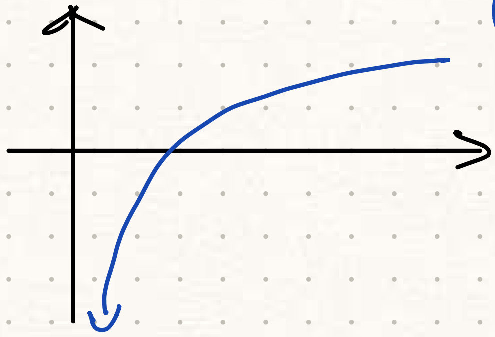
⑤ Achieve our goal: Find r.

$$9 = 4\pi r^2 \, 9$$

$$1 = 4\pi r^2$$

$$\Rightarrow \quad r^2 = \frac{1}{4\pi} \Rightarrow r = \sqrt{\frac{1}{4\pi}}.$$

(d) MT2



$$\lim_{x\to 0^+} x^2 \ln x$$

Plug in 0: $\quad 0 \cdot (-\infty)$

Use L'hopital's:

Key step

$$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{x^{-2}}$$

L'hopital's

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2x^{-3}} =$$

$$\lim_{x\to 0^+} \frac{x^2}{-2} = \boxed{0}.$$

$$\lim \frac{f}{g} = \lim \frac{f'}{g'}$$

$$\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}, \frac{\infty}{-\infty}$$

$$\lim \frac{f}{g}.$$

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1 c). $\lim_{x \to \infty} \frac{e^x}{x^3}$

$$= \lim_{x \to \infty} \frac{e^x}{3x^2}$$

$$\frac{d}{dx} x^3 = 3x^2$$

$$x^2 \ln x = \frac{\ln(x)}{x^{-2}}$$

Why? if

$$\frac{\ln(x)}{x^{-2}} = \frac{\ln(x)}{x^{-2}} \frac{x^2}{x^2}$$

$$= \frac{\ln(x)\, x^2}{1}$$

More generally,

$$\frac{a}{b^{-1}} = \frac{a}{b^{-1}} \frac{b}{b} \qquad = ab$$

380 Ch3

Derivative of y if

$$x^2 y = y + 2 + xy \sin x$$

Solution

Implicit differentiation, solve for dy/dx

$$2xy + x^2 y' = y' + (xy)' \sin x + (xy) \cos x$$

$$= y' + (y + xy') \sin x + (xy) \cos x$$

So

$$2xy + x^2 y' = y' + y \sin x + xy' \sin x + xy \cos x$$

$$x^2 y' - y' - xy' \sin x =$$
$$y \sin x + xy \cos x - 2xy$$

So

$$y' (x^2 - 1 - x \sin x) =$$
$$y \sin x + xy \cos x - 2xy$$

So

$$y' = \frac{y \sin x + xy \cos x - 2xy}{x^2 - 1 - x \sin x}$$

304 Ch 4.6.

$y = x \ln x$

(Find
* crit points,
* inc/dec.
* conc. up/down.
* asymptotic behavior.

Derivatives:

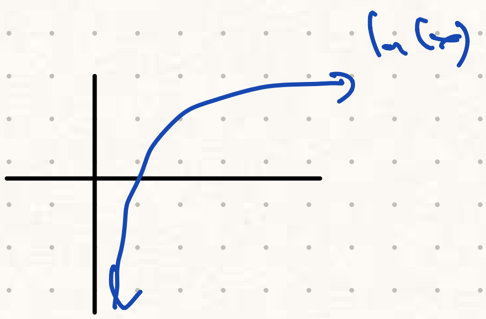
$y' = 1 \cdot \ln x + x \frac{1}{x} = \ln x + 1$

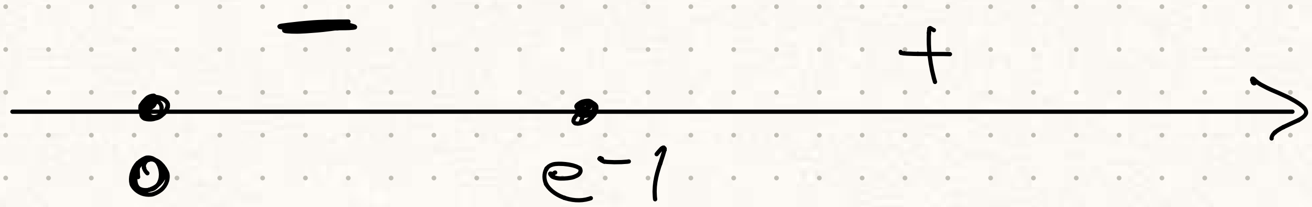
$y'' = \frac{1}{x}$

What does y' tell us?

* Critical points $y'=0$

$$y'=0 \iff \ln x + 1 = 0$$
$$\iff \ln x = -1$$
$$\iff x = e^{-1}$$





pos/neg y'?

$y'>0$ on (e^{-1},∞)

$y'<0$ on $(0,e^{-1})$.

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increasing on (e^{-1},∞)
critical point at $x=e^{-1}$
decreasing on $(0,e^{-1})$.

To evaluate sign of

$\ln(x)+1$ when $x < e^{-1}$,

notice that

$$\underbrace{\ln(0.0001)}_{\text{very negative}} + 1 = \text{negative.}$$

<u>y'' tells us:</u>

$y'' > 0$ for $x > 0$.

So

y concave up for $x > 0$

<u>Asymptotics</u> $x \in (0, \infty)$

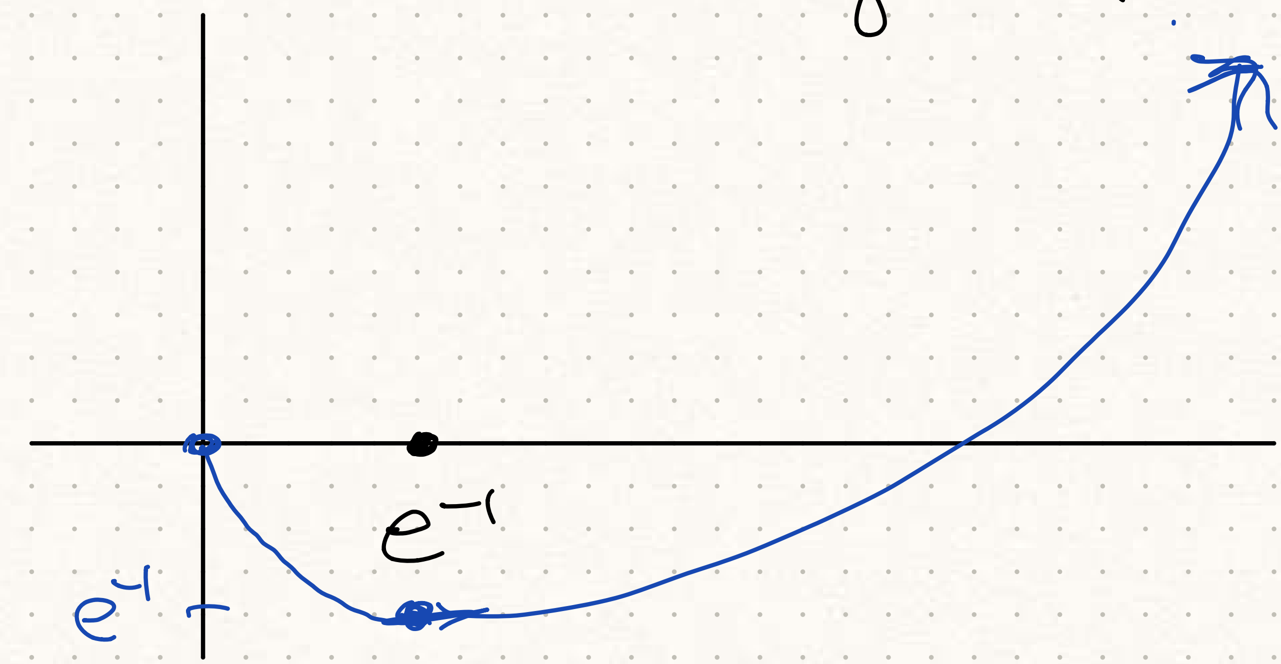
$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{x^{-1}}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x \to 0^+} -x = 0$$

$$\lim_{x \to \infty} x \ln x = \infty$$

Sketch graph:

Using all info so far:



at $x = e^{-1}$,

$$y = x \ln x$$
$$y = e^{-1}(-1)$$
$$= -e^{-1}$$