Lecture 36

For most of the course:

Given $f \xrightarrow{\text{Take derivative}}$ Get f'

function values $\longrightarrow$ Rate of change of f

<u>Today</u> (Antiderivatives) Ch 4.10

Going in the opposite direction:

function values $\longleftarrow$ Rate of change of f

Get $f \xleftarrow{\text{Take antiderivative}}$ Given f'

Example

if $f'(x) = 2x$, what is f?

One Answer:

$f(x) = x^2$ (because $f'(x) = 2x$ by power rules).

Another answer?

$f(x) = x^2 + 1$ (because $f'(x) = 2x + 0$.).

More answers:

- $f(x) = x^2 - 2$
- $f(x) = x^2 + 10000$
- $f(x) = x^2 + \pi$
- $f(x) = x^2 + 3e$
- $f(x) = x^2 + \sqrt{2}$

More generally:

$f(x) = x^2 + C$ where $\underbrace{C \in \mathbb{R}}$

C is any number.

Are all answers actually of the form $x^2 + C$?

Q: if $f'(x) = 2x$
does $f(x) = x^2 + C$?

(OR is there a completely different function which also works?).

<u>Definition</u> if $G'(x) = g(x)$

Then G is called the <u>antiderivative</u> of g

E.g. x^2 is an <u>antiderivative</u> of $2x$.

(because $2x$ is the deriv. of x^2).

Theorem 4.14

Let G be an antiderivative of g.

1) For $c \in \mathbb{R}$, $G+c$ is another antiderivative.

2) if F is another antiderivative, $F = G + c$ for some $c \in \mathbb{R}$.

Will come back to explain 2).

<u>Example</u> Find all the antiderivatives of $f(x) = \frac{1}{x}$.

Ans:

$F(x) = \ln(x)$ is an antideriv.

All antiderivatives are of the form
$$F(x) = \ln(x) + C \quad , \quad C \in \mathbb{R}$$

Verify:
$$F'(x) = \frac{1}{x} + 0$$
deriv of const. is 0.
$$= \frac{1}{x}.$$

You can always add $D \in \mathbb{R}$ to a function without changing the derivative.

To take antiderivatives, you should be good at taking derivatives.

Example

if $\frac{dy}{dx} = \sin x$,

what are the possible choices for y?

Soln: (because $\frac{d}{dx} \cos x = -\sin x$).

$y = -\cos x$ is one choice,

because $\frac{dy}{dx} = -(-\sin x) = \sin x$.

All other choices are of the form $y = -\cos x + C$.

Verify: $\frac{dy}{dx} = \sin x + 0$

You can always verify that you the correct antiderivative, by taking the derivative.

Initial Value Problems

Continuing on from example, Suppose you're also told

$\boxed{y = 5 \text{ when } x = 0.}$

What are the possible choices for y?

Answer: Already know:

$y = -\cos x + C$

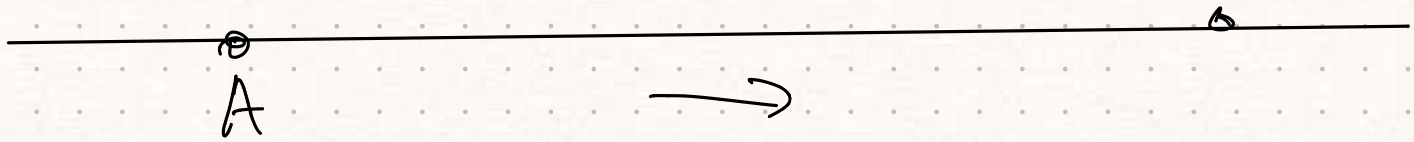
But

$5 = -\cos 0 + C$

$5 = -1 + C$

$\Rightarrow C = 6.$

$\boxed{y = -\cos x + 6}$



Just knowing $\frac{dx}{dt} = 20\,\text{mi/h}$

doesn't allow us to figure out x at $t=10$.

But if in addition, you know $x = A$ when $t=0$, then you can figure out where x is at $t=10$.

Example

if $f'(x) = 3x^{-2}$,

and $f(1) = 2$,

What is $f(x)$?

Soln Because $f'(x) = 3x^{-2}$,

$$f(x) = -3x^{-1} + C$$

deriv. is $-x^{-2}$

Verify:

Take derivative: $f'(x) = -3(-x^{-2}) + 0$

$$= 3x^{-2}.$$

To find C: use $f(1) = 2$

Plug into $f(x) = -3x^{-1} + C$,

$$2 = -3(1)^{-1} + C$$

So $C = 2 + 3 = 5$.

So

$$\boxed{f(x) = -3x^{-1} + 5}$$

$$\binom{2n}{n} = \left| P_n(\mathbb{N}_{2n}) \right| \quad \mathbb{N}_{2n} \to \{1,2,\dots,2n\}.$$

$$Z = \binom{n}{0}\binom{n}{n} \uplus$$

$$P_0(\mathbb{N}_n) \times P_n(\mathbb{N}_n) \cup \dots$$

$\{1,2,3,4\}$.

$n = 4$.

$\emptyset$

$$P_0(\mathbb{N}_0) \times P_4(\mathbb{N}_4)$$

$$= \{ (\emptyset, \{1,2,3,4\}) \}$$

$$\underline{P_4(N_3) \times P_3(N_4)}$$

$$\{ (\{1\}, \{1,2,4\}),$$

$$(\{1\}, \{1,2,3\}) \quad \}$$

$$\vdots$$

$$f(\{2,3,7,9,12,3,4,14\})$$

$$= (\{2,5,7,9,14\}, \{3,4,6\})$$

$(\{2,5,7,9,4\}, \{2,1,7\})$

$\Big\downarrow f^{-1}$

$\cancel{n=15}$

$n=8$

$\{2,5,7,\cancel{1},4, \; 10, 9, 15 \}$

$\Big\downarrow f$

$f(A) = (\{x \in A : x \leq n\}, \{ \; , \; \})$

$\{x \in A : x > n\}$

$\{x \in A : x \leq n\}$

$\{x+1 : x \in A\}.$

$\underline{\underline{A \times B}}$ $\qquad (3,5)$

$\neq (5,3)$

$(\_ , \_) \quad \in \mathbb{R}^2$

$\left\{ x - n \; : \; x \in \underline{A \cap \{n+1, \ldots, 2n\}} \right\}$

$f(\{1,2,3,\ldots,8\})$

$= (\{1,\ldots,8\}, \phi)$

$f(\{9, 1\ldots\ldots$

$(\phi, \{1 \ldots\ldots, 8\}).$

$(1,3)$, $\cancel{(0,4)}$, $(2,2)$,

$(3,1)$

$\{1\}$, $\{2\}$, $\{3\}$.

triple/ 3-tuple

$\cancel{(2,3)}$

$\{(1,2,2)$, $(2,1,2)$, $(2,2,1)$,

$(3,1,1)$ $(1,3,1)$, $(1,1,3)\}$,

d).

$P_k(X)$ = subsets of X with k elements

P

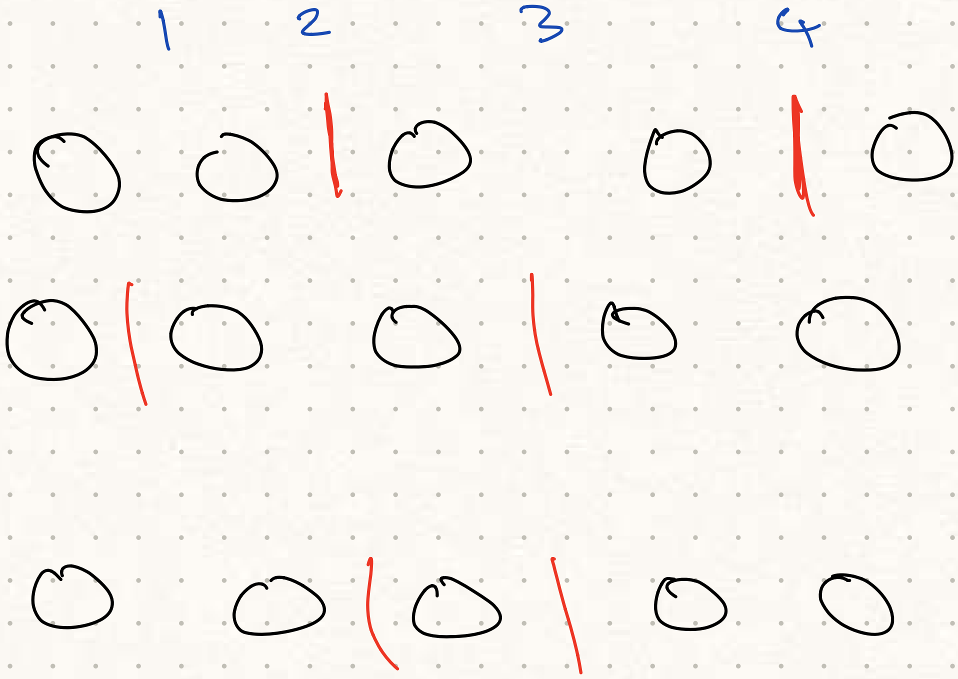
$P_2(\mathbb{N}_4)$ = subsets of $\mathbb{N}_4$ with 2 elements

= subsets of $\{1,2,3,4\}$ with 2 elements

$$= \{ \{1,2\}, \{1,3\}, \{1,4\}, \{3,4\}, \_\_, \_\_ \}.$$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\times 3\times \cancel{2\times 1}}{\cancel{2}\times 2}$$

$$= 6$$



$$A(\underset{k}{3},\underset{n}{5})$$

$$\{(1,2,2),\ (2,1,2),\ (2,2,1),\ (3,1,1),\ (1,3,1),\ (1,1,3)\} \quad A(3,5).$$

$$\{\{1,2\},\ \{1,3\},\ \{1,4\},\ \{3,4\},\ \{2,4\},\ \{2,3\}\}. \quad P_2(\mathbb{N}_4)$$

$$\binom{4}{2} = 6.$$

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2$$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!}$$

$$\binom{n}{1} = |P_1(N_n)|$$

= subsets of $\{1, \ldots, n\}$ with k elements

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n \times \cancel{(n-1)} \times \cancel{(n-2)} \ldots \cancel{1}}{1\,\cancel{(n-1)}\,\cancel{(n-2)}\,\cancel{(n-3)} \ldots 1}$$

$$= n$$

$$P_1(\{1, \ldots, n\}) = \{\{1\}, \{2\}, \{3\}, \ldots, \{n\}\}$$

$$\binom{n}{2} = |P_2(\mathbb{N}_n)|$$

$$\mathbb{N}_4 = \{1,2,3,4\}.$$

$$\underline{\underline{n = 4.}}$$

$$P(\mathbb{N}_4)$$

$$P_2(\mathbb{N}_4) = \{ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \}.$$

$$\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\times3\times2\times1}{2\times1\times2\times1}$$

$$= \frac{12}{2}$$

$$= 6$$

$$\binom{n}{k} = |P_k(\mathbb{N}_n)|$$

$$\binom{n}{k} = |P_k(\mathbb{N}_n)|$$

$$\binom{2n}{n} = |P_n(\mathbb{N}_{2n})|$$

$P_1(\mathbb{N}_4)$
$= \{\{1\}$
$\}$

$$\binom{n}{1}^2 = |P_1(\mathbb{N}_n)^2|$$

$$P_1(\mathbb{N}_4)^2 = \{ \quad \}$$

$$\underline{P_1(\mathbb{N}_4) \times P_1(\mathbb{N}_4)} = \{(\{1\}, \{1\}),$$
$$- - - - \}$$

$\{1,2\} \times \{3\}$

$A \times B = \{(1,3), (2,3)\}.$

$$\binom{n}{k}^2 = \left| P_k(\mathbb{N}_n)^2 \right|$$

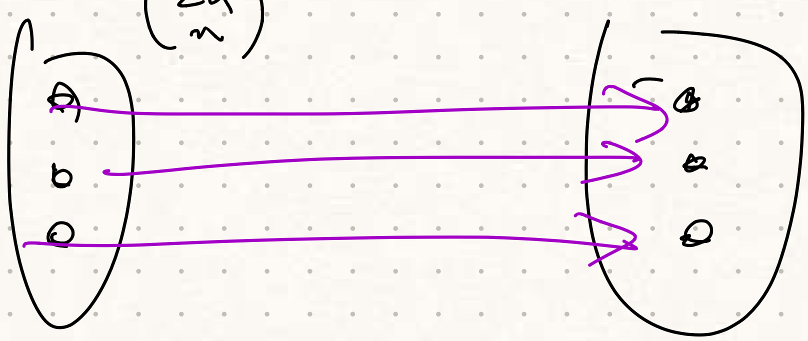
$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

$$= \Big| \underbrace{P_0(\mathbb{N}_n)^2 \cup P_1(\mathbb{N}_n)^2 \cup \cdots \cup P_n(\mathbb{N}_n)^2}_{Z =} \Big|$$

$$|A \cup B| = |A| + |B|$$

$$f : P_n(\mathbb{N}_{2n}) \longrightarrow Z.$$

$\binom{2n}{n}$



$n = 2.$

* List all the elements of

$\mathbb{P}_2(\mathbb{N}_4)$

* List all elems of

$Z = \quad \cup \ldots \cup$

$n = 3.$

* List $\mathbb{P}_3(\mathbb{N}_6)$

* List $Z = \mathbb{P}_0(\mathbb{N}_3)^2 \cup \mathbb{P}_1(\mathbb{N}_n)^2 \cup \ldots$

$\cup\ \mathbb{P}_n(\mathbb{N}$