Lecture 26 (Ch 4.2) Linear Approximations and differentials

Recall: If $f(x)$ is a function,

* $f'(a)$ is the $\left\{\begin{array}{l} \text{* slope} \\ \text{* inst. rate of change} \\ \text{* derivative} \end{array}\right.$ at a

* The tangent line at $x=a$ is

$$y = f'(a)(x-a) + f(a)$$

(The line with slope $f'(a)$ and passing through $(a, f(a))$).

The "tangent line" is also called the "linear approximation".

Example 4.5

* Find the linear approximation of

$f(x) = \sqrt{x}$ at $x = 9$.

* Use the approx to estimate $\sqrt{9.1}$

Sol'n $f(x) = \sqrt{x} = x^{1/2}$

$$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}}$$

$$f'(9) = \frac{1}{2 \cdot 9^{1/2}} = \frac{1}{6}$$

So the linear approx. at $x=9$, $y=3$

$$y = \frac{1}{6}(x-9) + 3.$$

$f(9) = \sqrt{9} = 3.$

See graph. $y = \frac{1}{6}(x-9)+3$

is an approximation to

$y = \sqrt{x}$

$y = \frac{1}{6}(x-9)$ is not the tangent line because at $x=9$, $y=0$.

$y = \frac{1}{6}(x-9)+3$ passes through $(9,3)$:

at $x=9$, $y = \frac{1}{6}(9-9)+3$

$= 0+3$

$= 3.$

* Estimate $\sqrt{9.1}$ using linear approx.

Plug in $x=9.1$ into

$$y = \frac{1}{6}(x-9) + 3.$$

$$= \frac{1}{6}(9.1-9)+3$$

$$= \frac{1}{6}(0.1)+3$$

$$= 3.0166666\ldots$$

See that the estimate is very good: $\sqrt{9.1} = 3.016620625\ldots$

Poll Let $f(x)=(1+x)^n$, $n \neq 0$

$f'(0) =$

a) n 35%

b) 0

c) 1

d) $n-1$

Answer a):

$$f'(x) = n(1+x)^{n-1} \cdot 1$$

$$f'(0) = n(1+0)^{n-1}$$

$$= n$$

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* $f(0) = (1+0)^n = 1.$
* $f'(0) = n$

Example 4.7

a) Find the linear approx for

$$f(x) = (1+x)^n \quad \text{at} \quad x=0$$

b) Use this to estimate $(1.01)^3$

Soln.

a) We already know $f(0)=1$

$f'(0)=n.$

So the linear approx is

$$y = n(x-0)+1$$

$$y = f'(a)(x-a)+f(a)$$

$a=0$

eqn for tangent line.

b) Approx to $f(x)=(1+x)^3$ is

$$y = 3x+1$$

To approximate $(1.01)^3$, i.e. $f(0.01)$, plug in $x=0.01$ into $y=3x+1$.

$$y = 3(0.01) + 1$$

$$= 0.03 + 1$$

$$= 1.03$$

Why $x = 0.01$ and not $x = 1.01$

Goal: $(1.01)^3$.

$$f(x) = (1 + x)^3$$

$$f(1.01) = (1 + 1.01)^3 = (2.01)^3$$

$$f(0.01) = (1 + 0.01)^3 = (1.01)^3.$$

Where did $3x+1$ come from?

In part a), we found approx to $f(x) = (1+x)^n$ is

$$y = nx + 1.$$

Therefore, approx to

$$f(x) = (1+x)^3$$ is

$$y = 3x + 1.$$

Again, plot

$$y = 3x+1$$

$$y = (1+x)^3.$$

Very close for x near 0.

Next time: 4.2, 4.3.

Questions (After Lecture).

Linear approx to

$$y = (1+x)^n \quad \text{at} \quad x = 2$$

$$\left.\begin{aligned} \frac{dy}{dx} &= n(1+x)^{n-1} \\ &= n(3)^{n-1} \end{aligned}\right\} \text{Slope.}$$

Also $y = 3^n$ at $x = 2$.

$= (1+2)^n$

So linear approx is

$$y = n3^{n-1}(x-2) + 3^4$$

Question: $3x+1$?

Assuming:

if $f(x) = (1+x)^n$ then approx

is

$$y = nx + 1$$

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if $f(x) = (1+x)^3$ then approx

is

$$y = 3x + 1$$