Lecture 24

Last time $\frac{d}{dx} \ln(x) = \frac{1}{x}$

Logarithmic Differentiation

if we want to differentiate, say

$$y = \frac{x(2x+1)^{1/2}}{e^x \sin^3(x)}$$

we could use product/chain rule/quotient rule.

On the other hand, we can take of these log laws:

$$* \log(a^b) = b\log(a)$$

$$* \log(a \cdot b) = \log(a) + \log(b)$$

$$* \log(a/b) = \log(a) - \log(b)$$

$$* \log(1) = 0$$

1. Take ln of both sides

$$\ln y = \ln\left(\frac{x(2x+1)^{1/2}}{e^x \sin^3(x)}\right)$$

2. Use log laws to expand the eqn.

$$\ln y = \ln\left(x(2x+1)^{1/2}\right) - \left(\ln\left(e^x \sin^3(x)\right)\right)$$

$$= \ln(x) + \ln\left((2x+1)^{1/2}\right) - \left(\ln(e^x) + \ln(\sin^3(x))\right)$$

$$\ln(y) = \ln(x) + \frac{1}{2}\ln(2x+1) - x - 3\ln(\sin(x))$$

3. Differentiate both sides (implicitly).

Apply $\frac{d}{dx}$ to both sides:

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} + \frac{1}{2}\,\frac{1}{2x+1}\cdot 2 - 1 - 3\,\frac{1}{\sin(x)}\cos(x).$$

$$\boxed{\ln(e^x) = x}$$

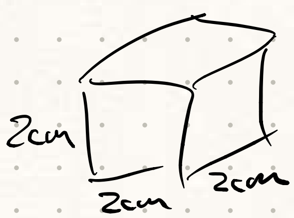
4. Solve for $\frac{dy}{dx}$, plug in y.

$$\frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos(x)}{\sin(x)}\right)$$

$$= \frac{x(2x+1)^{1/2}}{e^x \sin^3(x)}\left(\frac{1}{x} + \frac{1}{2x+1} - 1 - 3\frac{\cos(x)}{\sin(x)}\right)$$

Related Rates Ch 4.1

Blowing up balloon, putting in $2\ cm^3$ of air per second.



As the balloon gets bigger, the rate at which the radius changes, gets smaller.

(Even though, radius is getting bigger)

How to say this using the language we've developed.

Let $r(t)$ = radius of balloon at time t

[As $r(t)$ gets bigger, $r'(t)$ gets smaller.

We can use calculus to quantify this.

Let $V(t)$ = volume of balloon at time t.

Then "putting in $2cm^3$ of air / second"

means "Volume is increasing by 2 every second" i.e. $V'(t) = 2 \quad cm^3/s.$

not

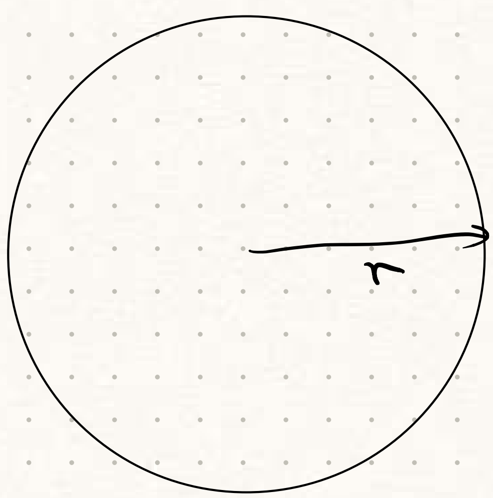
$$V'(t) = 2t$$

$$V'(0) = 0$$

$$V'(2) = 4$$

How to use this to find $r'(t)$.

Use relationship between volume and radius.



$$V = \frac{4\pi r^3}{3}$$

$$V(t) = \frac{4\pi \, r(t)^3}{3}.$$

Can use relationship between

V and r

to find relationship between

V' and r'.

By differentiating:

$$V(t) = \frac{4\pi r(t)^3}{3}$$

Diff both sides: (Apply $\frac{d}{dt}$)

$$V'(t) = \frac{4\pi}{3} \cdot 3r(t)^2 \cdot r'(t)$$

or

$$\frac{d}{dt}V = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt}$$

relationship between $\frac{dV}{dt}$, $\frac{dr}{dt}$

So

$$r'(t) = \frac{V'(t)}{4\pi r(t)^2}$$

$$= \frac{2}{4\pi r(t)^2}$$

$$r'(t) = \frac{1}{2\pi r(t)^2}$$

E.g When radius is 6cm, how quickly is the radius increasing?

Ans: $r'(t) = \dfrac{1}{2\pi\, 6^2} = \dfrac{1}{72\pi}$

$\approx 0.004\ldots$