Lecture 23 Ch 3.9

Last time we found that

if

$f(x) = e^x$ where $e = 2.718...$

then $f'(x) = e^x$

List of derivatives:

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* $\frac{d}{dx} x^n = nx^{n-1}$
* $\frac{d}{dx} \sin(x) = \cos(x)$
* $\frac{d}{dx} \cos(x) = -\sin(x)$
* $\frac{d}{dx} e^x = e^x$

Example: $\frac{d}{dx} \sin(x) \cdot e^x$

a) $e^x \cos(x)$ 20%

b) $e^x \sin(x)$ 10%

c) $e^x \sin(x) + e^x \cos(x)$ 60%

Answer c):

$$\frac{d}{dx} \sin(x) \cdot e^x = \left(\frac{d}{dx}\sin(x)\right) e^x$$

product rule

$$+ \sin(x) \frac{d}{dx}(e^x)$$

$$= \cos(x)\, e^x$$

$$+ \sin(x)\, e^x$$

Today Derivative of $\log_e(x)$

$= \ln(x)$

Recall $\ln(x)$ is inverse of e^x

i.e.

$$e^{\ln(x)} = x \qquad \ln(e^x) = x$$

Want to find $\frac{d}{dx} \ln(x)$.

One way:

use limit definition.

$$f'(x) = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \ldots$$

Recall implicit differentiation.

if $y^2 = x$,

differentiate both sides

a) $2y = 1$ 3%

b) $2y \frac{dy}{dx} = 1$ 65%

c) $2y \frac{dy}{dx} = 1 \cdot \frac{dx}{dx}$ 26%

Back to derivative of $\log_e(x)$
$\ln(x)$

Suppose $y = \ln(x)$

Then $e^y = e^{\ln(x)} = x$

$$e^y = x$$

Now differentiate both sides.

$$e^y \frac{dy}{dx} = 1$$

So $\frac{dy}{dx} = \frac{1}{e^y}$

$$= \frac{1}{x}.$$

Summary:

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Makes sense:

When x is large,

slope is almost 0.

E.g. if $x = 20$.

$$\frac{d}{dx}\ln(x) = \frac{1}{20} = 0.05$$

Example

Find $\frac{dy}{dx}$ if

1) $y = \ln(x^2) =$

a) $\frac{1}{x^2}$ 53%

b) $\frac{2}{x}$ 42%

c) $\frac{1}{x}$ 2%.

Answer: b)

Chain rule.

$$\frac{d}{dx}\left(\ln(x^2)\right) = \frac{1}{x^2} \cdot \frac{d}{dx} x^2$$

$$= \frac{1}{x^2} 2x$$

$$= \frac{2}{x}.$$

$$y = \ln(x^2)$$

$$= f(g(x)) \quad \text{where} \quad g(x) = x^2, \; f(x) = \ln(x)$$

So from chain rule:

$$y' = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{x^2} \cdot 2x.$$

$g'(x) = 2x$

$f'(x) = \frac{1}{x}$

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if $y = \sin(x^2)$

then $\frac{dy}{dx} = \cos(x^2) \cdot 2x.$

if $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \cdot 2x$$

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if $y = \sin(x)$

$$\frac{dy}{dx} = \cos(x)$$

$$\frac{d}{dx} \ln(x^2) = \frac{1}{x^2} \cdot \frac{d}{dx} x^2$$

$$= \frac{1}{x^2} \cdot 2x.$$

Logarithmic Differentiation.

Find $\frac{dy}{dx}$ if

1) $y = x^x$

or

2) $y = \frac{x(2x+1)^{1/2}}{e^x \sin^3(x)}$

For 1):

Take ln of both sides

$\ln(y) = \ln(x^x)$

$\ln(y) = x\ln(x)$

Differentiate both sides

$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\frac{d}{dx}x\right)\ln(x) + x\left(\frac{d}{dx}\ln(x)\right)$

$= \ln(x) + x\left(\frac{1}{x}\right)$

Log laws:

$\ln(a^b) = b\ln(a)$

$$\frac{1}{y}\frac{dy}{dx} = \ln(x) + 1$$

Solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left(\ln(x) + 1\right)$$

$$\boxed{\frac{dy}{dx} = x^x\left(\ln(x) + 1\right)}.$$

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2) $$y = \frac{x(2x+1)^{1/2}}{e^x \sin^3(x)}$$

1) Take ln of both sides
2) Use log laws to simplify.
   * $\log(a^b) = b\log(a)$
   * $\log(a \cdot b) = \log(a) + \log(b)$
   * $\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$.
3) Implicitly diff.