Lecture 22 Derivatives of exponential and logarithm functions.

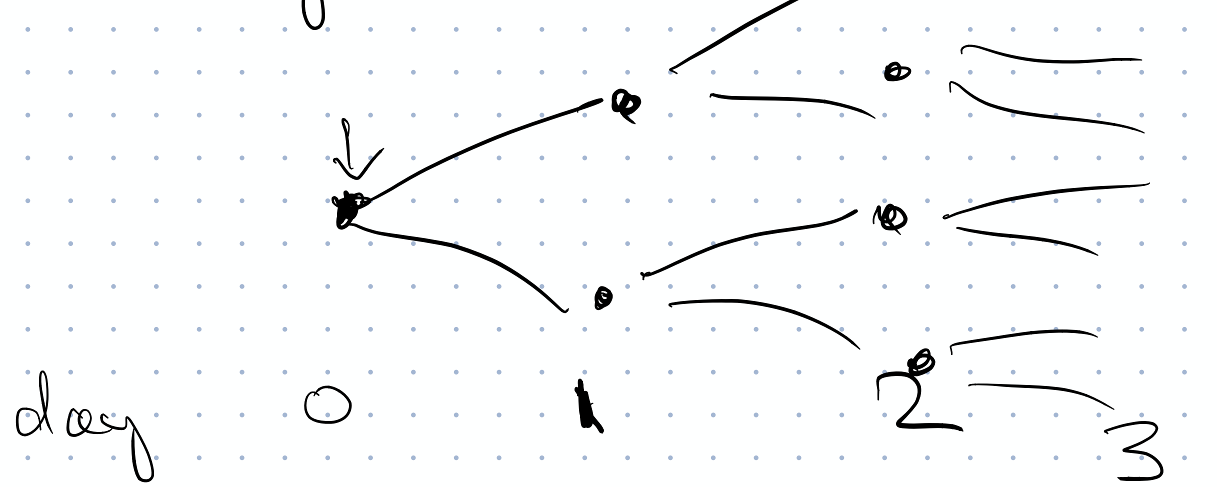
Problem: Suppose

* 1 person is infected on day 0
* Every day, each infected person infects 2 new people, and then recovers.

How many infected people are there on day 3?
day 30?
day x?

Solution:

day 3: 8.



Every, number of infected is doubling.

* day 3: $1 \times 2 \times 2 \times 2 = 2^3 = 8$.
* day 30: $\underbrace{2^{30}}_{\underbrace{2 \times 2 \times 2 \cdots \times 2}_{30 \text{ times}}} \simeq 1{,}000{,}000{,}000$
* if $f(x)$ is the number of infected on day x,

$$f(x) = 2^x.$$

This is an example of an exponential function.

Note: This is obviously a very simplistic model and inaccurate still a good starting point for other models.

Exponential Functions

For $b > 0$,

$$f(x) = b^x$$

is the exponential function with base b.

Observations about the exponential function:

* For b large, b^x is growing quickly
* For $b < 1$, b^x is decreasing
* For $b > 1$, b^x is increasing.

Basic facts about exponentials:

Recall:

* $b^{x+h} = b^x b^h$

(e.g.) $$\underbrace{2^{3+5}}_{\underbrace{2\times2\times2\cdots2}_{8\ \text{times}}} = \underbrace{2^3}_{2\times2\times2}\ \underbrace{2^5}_{2\times2\times2\times2\times2}$$

Today: Derivative of b^x

Let $f(x) = b^x$

Then

$$f'(x) = \lim_{h\to0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to0} \frac{b^{x+h}-b^x}{h}$$

$$= \lim_{h\to0} \frac{b^x b^h - b^x}{h}$$

$$= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h}$$

$$= b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

$$= b^x \underbrace{\lim_{h \to 0} \frac{b^h - b^0}{h}}_{f'(0)}$$

Limit law:

$$\lim_{h \to 0} 3\, g(h)$$

$$3 \lim_{h \to 0} g(h)$$

$$f'(x) = b^x \, f'(0).$$

E.g.

if $f(x) = 2^x$

$$f'(x) = 2^x \, f'(0)$$

Using graphing calculator, can estimate $f'(0) \approx 0.69\ldots$

$$f'(x) = 0.69\ldots \cdot 2^x$$

Doing some experiments, we see that

* $f'(0)$ is increasing as b increasing

* There's a special value of b where $f'(0)=1$

This is at

$$b \approx 2.718\ldots = e$$

Euler's number.

Summary:

For $f(x) = \overset{2.718\ldots}{e}{}^x$

$f'(0)=1$

$$f'(x) = e^x \cdot 1 = e^x$$

Example Find derivative of

1) $f(x) = 3e^x$

2) $f(x) = e^{3x}$

3) $f(x) = e^{x^2}$

4) $f(x) = 2^x$

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1) $f'(x) = 3 \frac{d}{dx} e^x = 3e^x$.

2) $f'(x) = e^{3x}\left(\frac{d}{dx} 3x\right) = e^{3x} \cdot 3$

chain rule

3) $f'(x) = e^{x^2}\left(\frac{d}{dx} x^2\right) = e^{x^2} \cdot 2x$.

4) Recall: $(a^b)^c = a^{bc}$ (i)

Also: $e^{\ln(a)} = a$ (ii)

So $f(x) = 2^x = \left(e^{\ln(2)}\right)^x$ by ii)

$$f(x) = 2^x = \left(e^{\ln(2)}\right)^x$$

$$= e^{\ln(2)\,x}$$ by i)

$\ln(2) \approx 0.69\ldots$

Same as ex. 2).

So

$$f'(x) = e^{\ln(2)x} \cdot \frac{d}{dx}\left(\ln(2)\cdot x\right)$$

$$= e^{\ln(2)x} \cdot \ln(2)$$

$$= 2^x \cdot \ln(2)$$

$$\approx 2^x \cdot 0.69\ldots$$

Check website for notes
+ recordings.

$f(x) = e^{3x}$

$f'(x) = ?$

Need to use chain rule.

Recall:

if $\quad f(x) = g(h(x))$

$$\underbrace{f'(x) = g'(h(x))\, h'(x)}_{\text{chain rule.}}$$

In our example,

$f(x) = g(h(x))$

$g(x) = e^x$

$h(x) = 3x$

So

$$f'(x) = g'(h(x))\, h'(x)$$
$$= e^{3x} \cdot 3.$$

$$f(x) = e^{\ln(2) \cdot x}$$

$$f'(x) = e^{\ln(2) \cdot x} \cdot \ln(2)$$

$$g(x) = e^{3x}$$

$$g'(x) = e^{3x} \cdot 3$$