# Conformal structures on stochastic subdivision rules

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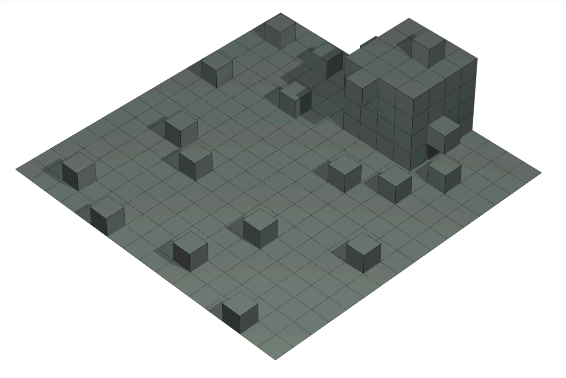
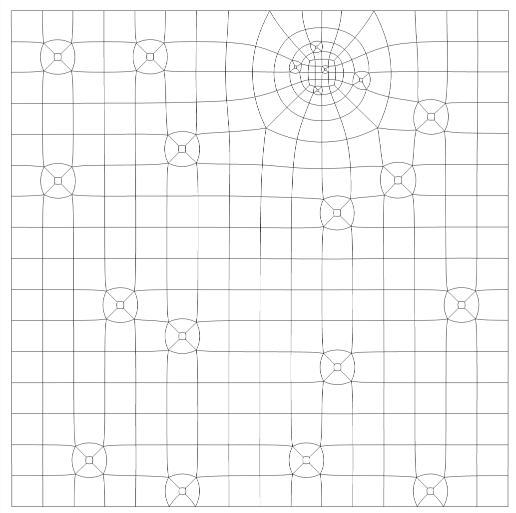
Edited version of talk given at
Random Geometry and Statistical Physics Workshop @ UPenn Oct 2022.

- Let $X_0$ be the unit square $[0,1]^2$
- To generate $X_{n+1}$, for each square of $X_n$,
    - Subdivide square into $L \times L$ squares of side length $L^{-1}$
    - Choose $K/4$ uniformly random non-corner squares and replace them with $5$ copies of $[0,L^{-1}]^2$ glued to form a 'cubical protrusion'
- This gives a random sequence $X_0, X_1, X_2, \ldots$ of Riemannian manifolds with conical singularities.



(K = 4,   L = 4,   n = 2)

See http://math.stonybrook.edu/~bplin/subdivision-rule/ for $K = 1 \cdot 4$ and $L = 4$.



**Proposition:** Almost surely, $X_n \to X_\infty$ as metric spaces in the Gromov-Hausdorff sense. $X_\infty$ is a fractal: $\dim(X_\infty) > 2$.

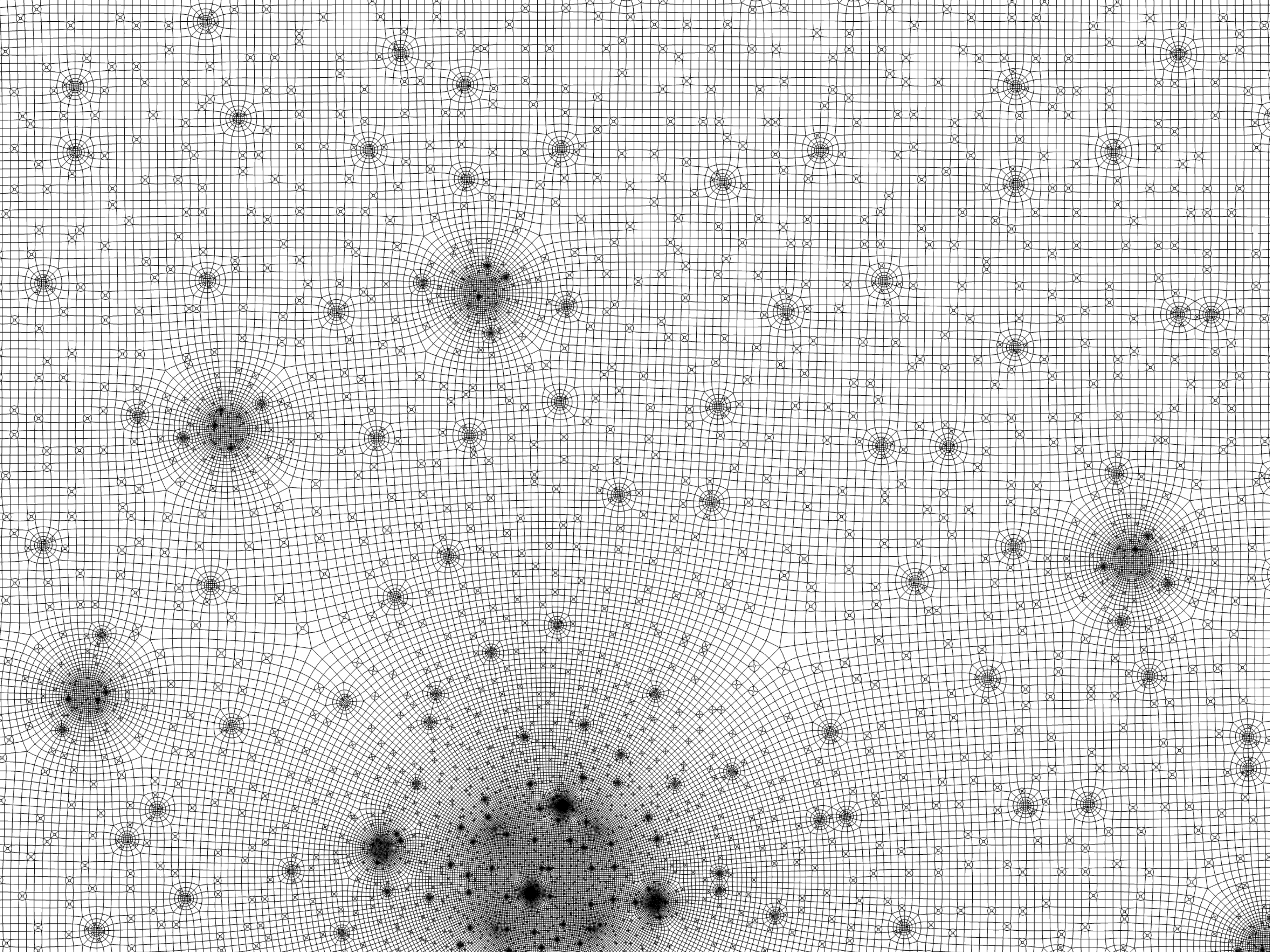
We are interested in the behavior of the conformal embeddings.

**Uniformization Theorem**

There is a unique $H_n > 0$ and a unique conformal map $\varphi_n : X_n \to [0,1] \times [0,H_n]$ that sends the corners of $X_n$ to the respective corners of the rectangle.

**Definition:**

- $\varphi_n : X_n \to [0,1] \times [0,H_n]$ is the *rectangular conformal embedding.*
- $\mathrm{mod}(X_n) := \max(H_n, H_n^{-1})$ is the *(absolute) modulus* of $X_n$ .

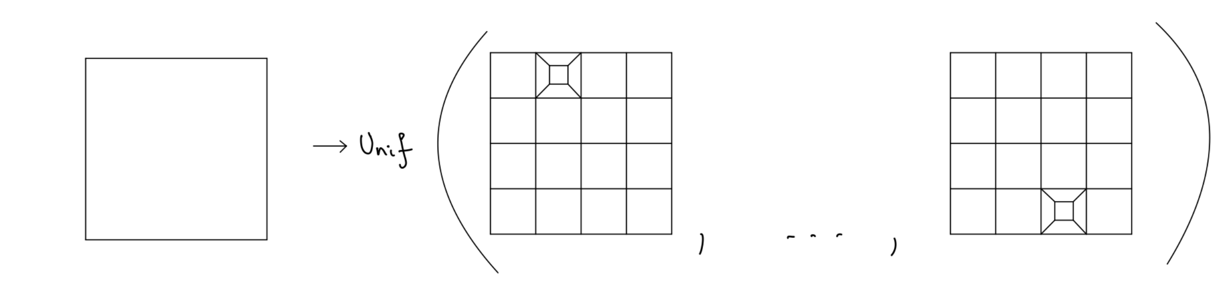


**Theorem (L):**

For $K > 0$ , for sufficiently large $L$,

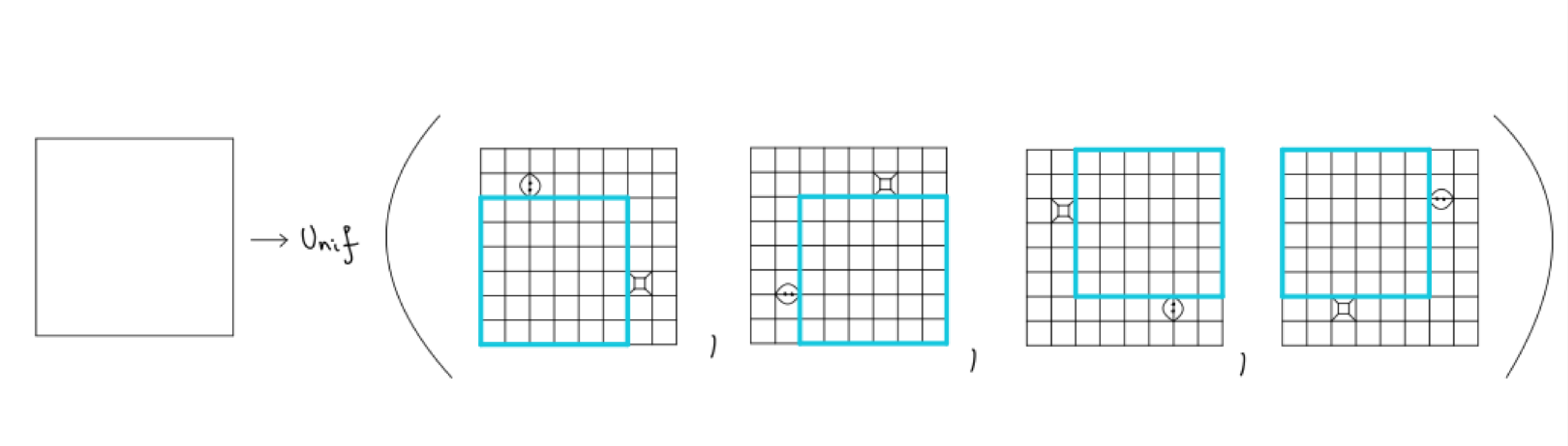
- $\sup_n \mathbb{E}\mathrm{mod}(X_n) < \infty$
- $\exists \alpha > 0 : \quad \sup_n \mathbb{E}e^{\alpha \mathrm{mod}(X_n)^2} < \infty$

Subdivision rule $\Sigma_{K,L}$:




(K = 4, L = 4)

Theorem holds in more generality: just need P large and rotationally invariant distribution.




(K = 4+2, L = 8, P = 6 )

**Theorem (L):**

For $K > 0$, for sufficiently large $L$,

- $\sup_n \mathbb{E}\mathrm{mod}(X_n) < \infty$
- $\exists \alpha > 0 : \quad \sup_n \mathbb{E} e^{\alpha \mathrm{mod}(F_n)^2} < \infty$

**Theorem (L):**

Suppose $\exists \alpha > 0 : \quad \sup_n \mathbb{E} e^{\alpha \mathrm{mod}(F_n)^2} < \infty$.

Let $\varphi_n : X_n \to [0,1] \times [0,H_n]$ be the rectangular conformal uniformization.

Almost surely,

- $\varphi_n$ converges subsequentially to a homeomorphism $X_\infty \to [0,1] \times [0,H]$
- Any two subsequential limits are equal modulo quasiconformal map:

  $\varphi = f \circ \tilde{\varphi}$ where $f$ is quasiconformal.

$F_n$ is the surface obtained by putting two i.i.d. copies of $X_n$ next to each other to form a "domino"

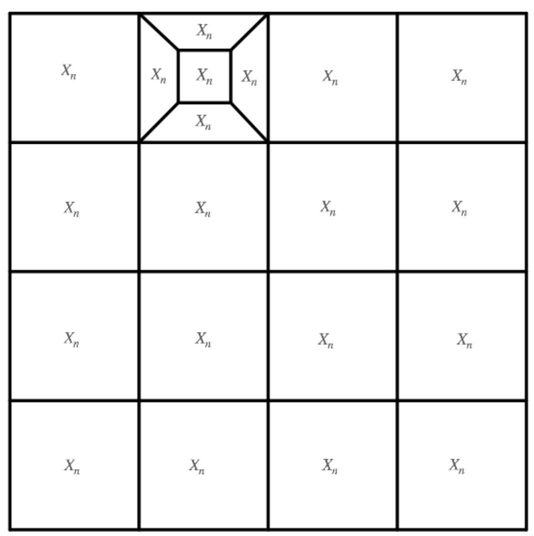
$f$ is $K$-quasiconformal if it maps infinitesimal circles to infinitesimal ellipses of eccentricity bounded by $K$.

The theorems endow the fractal space $X_\infty$ with a (quasi)conformal structure.

# Stochastic self similarity

- Our $X_n$ is a random sequence of metric measure spaces with boundary measure.
- The spaces are stochastically self similar:

$$X_{n+1} \stackrel{d}{=}$$



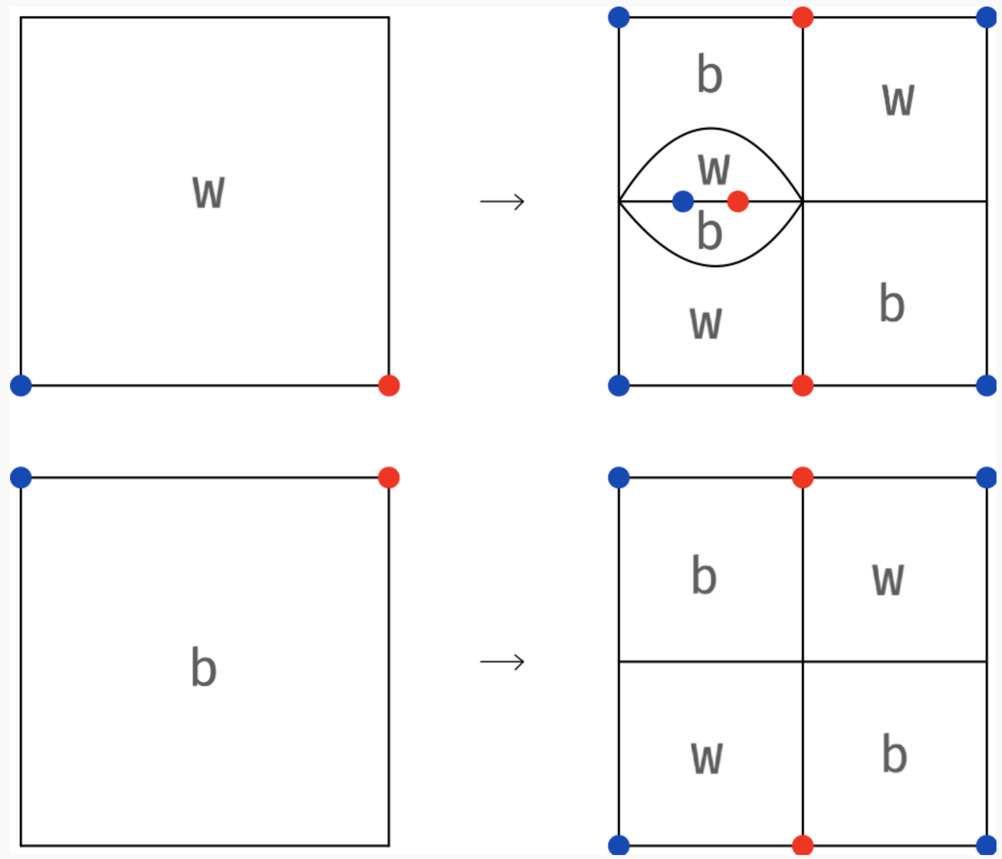
In other words, $X_{n+1} = \mathscr{R}(X_n)$

- Where $\mathscr{R} : \mathrm{Prob}(\mathfrak{X}) \to \mathrm{Prob}(\mathfrak{X})$ where

  $\mathrm{Prob}(\mathfrak{X}) =$ Probability distributions on conformal disks with boundary parameterization
- Our theorem implies that iterates of $\mathscr{R}$ have subsequential limits.
- Central limit theorem: can we show that $\mathscr{R}$ has an attractive fixed point?

- Similar identities hold for random planar maps, e.g. Mullin bijection for $\kappa = 8$, and mating of trees for LQG.
- Mating of trees implies that the corresponding $\mathscr{R}$ has a fixed point (LQG)
- fixed point sufficiently attractive $\Longrightarrow$ convergence of conformal embedding of RPM to LQG.

The conformal modulus of the following deterministic rule degenerates as $n \to \infty$ :



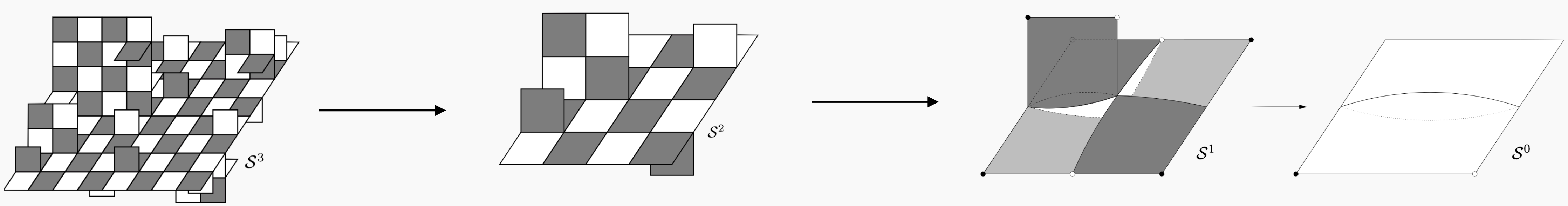
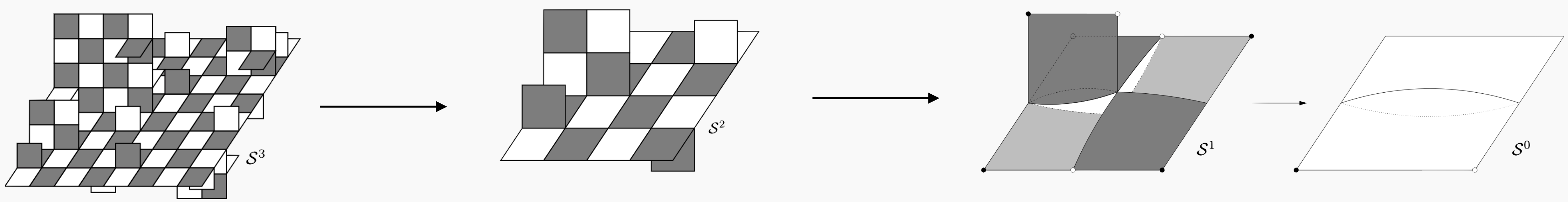




Image from Bonk-Meyer 2017

## Subdivision/Thurston iteration

Riemannian manifolds




$\downarrow f_3$ $\quad$ $\downarrow f_2$ $\quad$ $\downarrow f_1$ $\quad$ $\downarrow f_0$

Teichmüller space $(\hat{\mathbb{C}}, \gamma_3)$ $\quad$ $(\hat{\mathbb{C}}, \gamma_2)$ $\quad$ $(\hat{\mathbb{C}}, \gamma_1)$ $\quad$ $(\hat{\mathbb{C}}, \gamma_0)$

$\downarrow$ $\quad$ $\downarrow$ $\quad$ $\downarrow$ $\quad$ $\downarrow$

Moduli Space $z_3 \in \hat{\mathbb{C}} - \{0,1,\infty\}$ $\quad$ $z_2 \in \hat{\mathbb{C}} - \{0,1,\infty\}$ $\quad$ $z_1 \in \hat{\mathbb{C}} - \{0,1,\infty\}$ $\quad$ $z_0 \in \hat{\mathbb{C}} - \{0,1,\infty\}$

Thurston's topological characterization of rational maps

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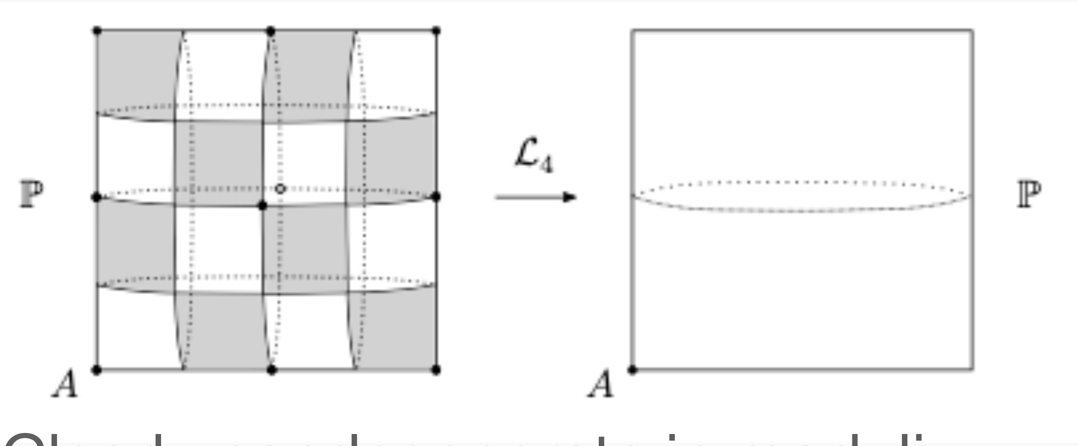
criterion for nondegeneracy of iterates in moduli space

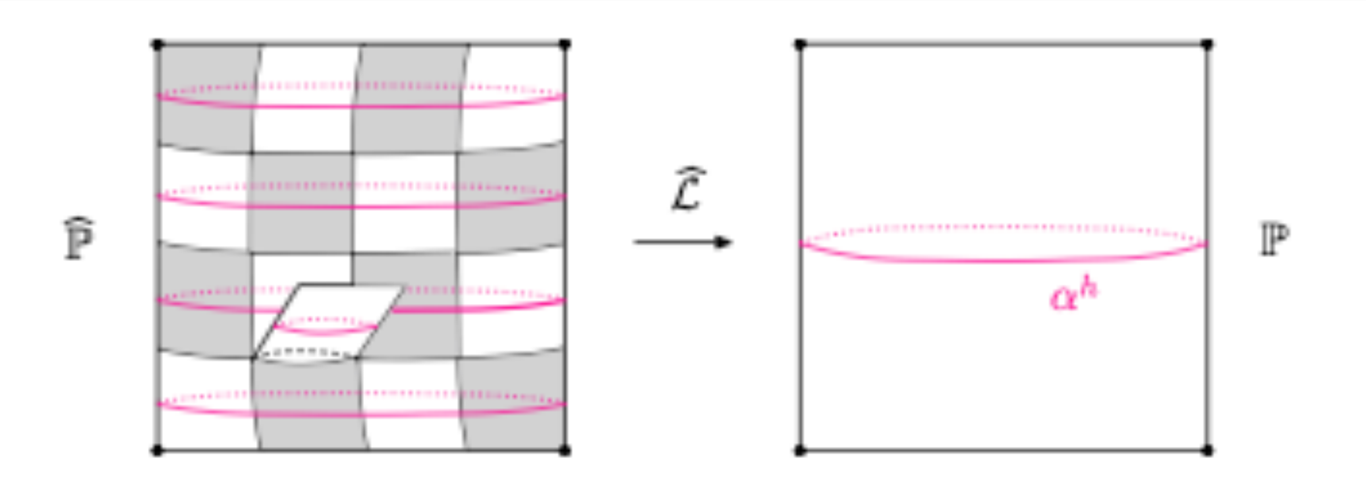
All images from Bonk-Hlushchanka-Iseli paper.

**Example 1.**



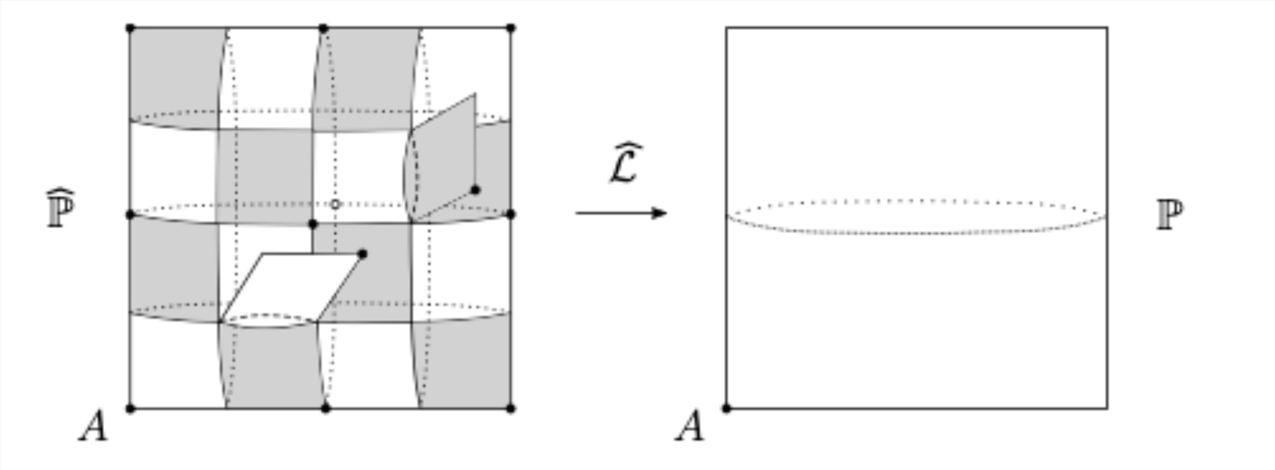

Clearly nondegenerate in moduli space because subdivision is perfect.

**Example 2.**




Degenerates in moduli space due to Thurston obstruction.

**Example 3.**




**Theorem:** Bonk-Hlushchanka-Iseli `21
As long as there are vertical *and* horizontal flaps, then moduli stay bounded under iteration.

**Proof:** Using Thurston's criterion.

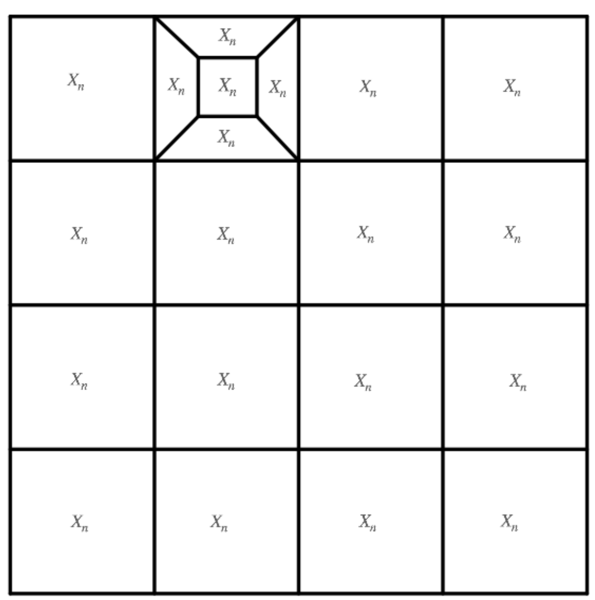
# Proof ideas

**Theorem (L):**

For $K > 0$ , for sufficiently large $L$,

- $\sup_n \mathbb{E}\mathrm{mod}(X_n) < \infty$

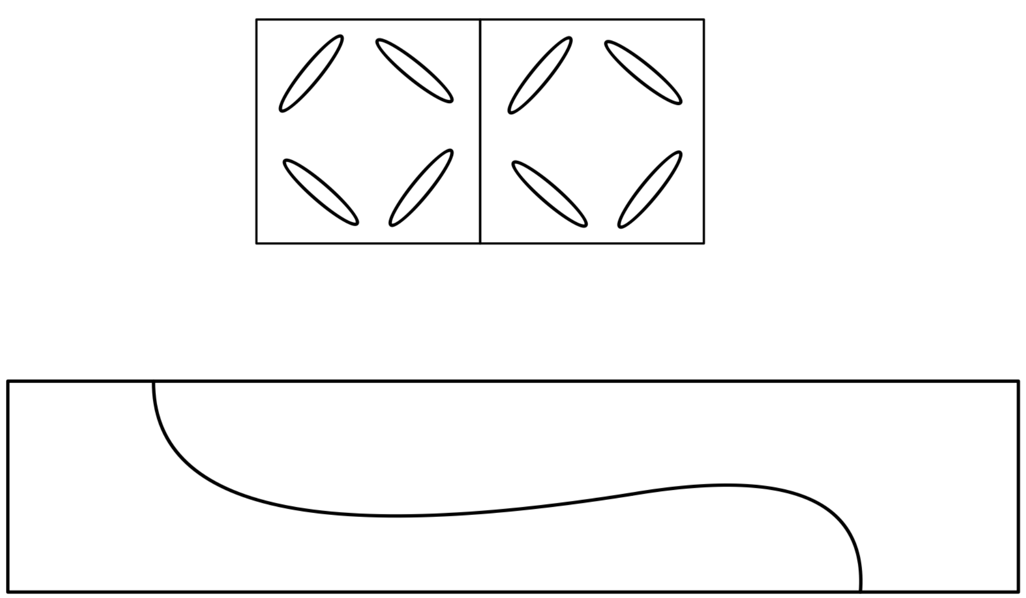
$X_{n+1} \overset{d}{=}$



- Recall $X_{n+1}$ is the welding of i.i.d. copies of $X_n$.
- View each $X_n$ as inducing a deformation of the complex structure.
- $\mathrm{mod}(X_n)$ measures the magnitude of the deformation, but it is a very coarse measure.
- In particular it does not work well with conformal welding:

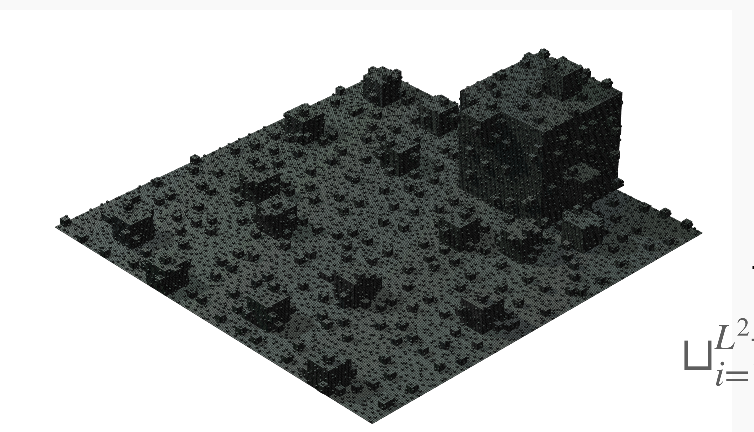
Modulus of the deformed squares are both 1 because the deformations are $\pi/2$ invariant.

But modulus of rectangle can be arbitrarily large.

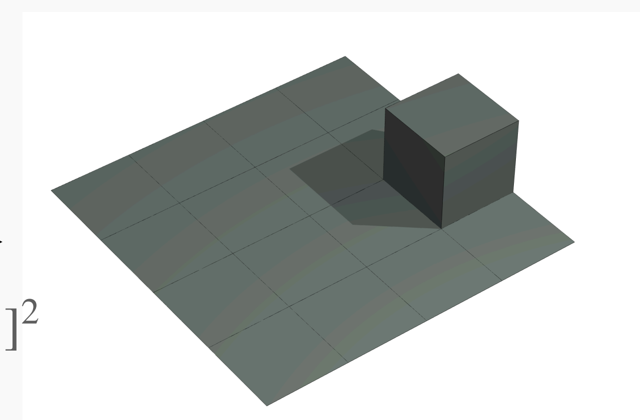


**Definition:** The *energy* $\mathscr{E}(X_n)$ of $X_n$ is the Dirichlet energy of the harmonic extension $X_n \to [0,1]^2$ of the boundary parameterization $\partial X_n \to \partial[0,1]^2$

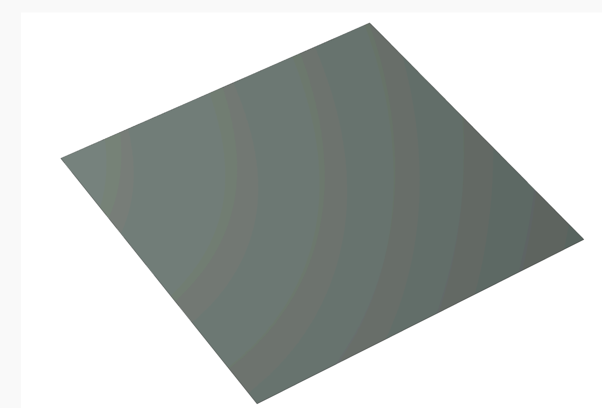
We have $\mathscr{E}(X_{n+1}) \leq \frac{1}{L^2} \sum_{i=1}^{L^2+4K} \mathscr{E}(X_n^{(i)})$ because we can concatenate maps:



$\sqcup_{i=1}^{L^2+4K} X_n^{(i)} \to \frac{1}{L^2}[0,1]^2$



1-Lip



So $\mathbb{E}\mathscr{E}(X_{n+1}) \leq \frac{L^2+4K}{L^2}\mathbb{E}\mathscr{E}(X_n)$.

**Theorem (L):**

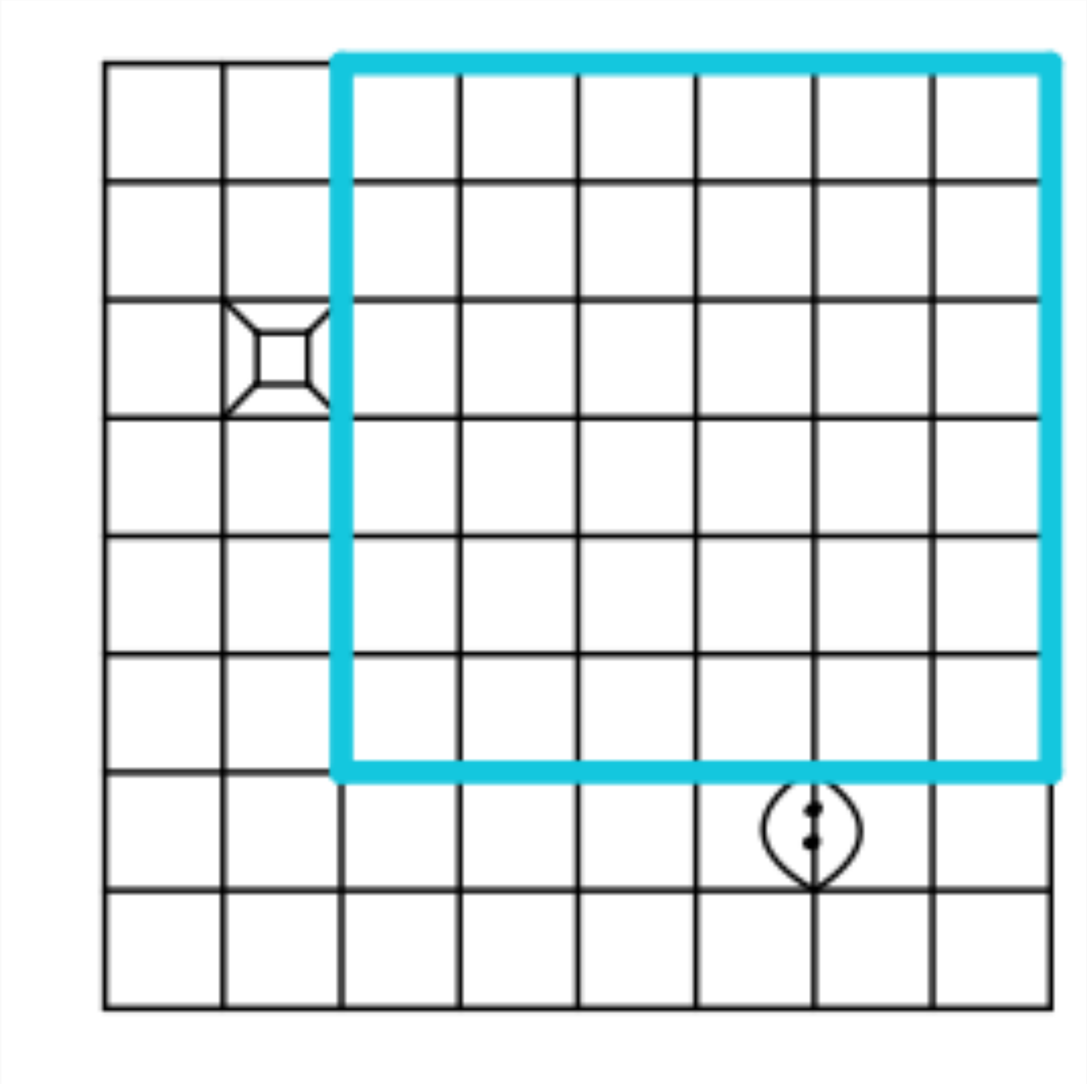
For $K > 0$ , for sufficiently large $L$,

- $\sup_n \mathbb{E}\mathscr{E}(X_n) < \infty$

So $\mathbb{E}\mathscr{E}(X_{n+1}) \leq \frac{L^2 + 4K}{L^2}\mathbb{E}\mathscr{E}(X_n)$.

This is still exponentially increasing growth.

However, we know from conformal deformations in a grid tend to cancel each other out: stochastic homogenization, random QC maps, random walk on random environment.



**Theorem:** (Astala-Rohde-Saksman-Tao, Ivrii-Markovic):

Let $\lambda$ be a $\mathbb{D}$ valued random variable, with "rotationally invariant" distribution.

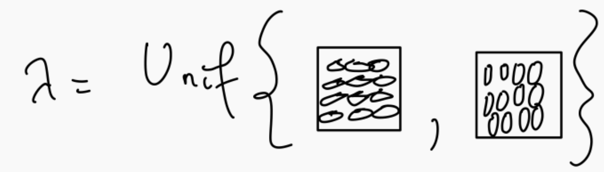
Let $\mu_n$ be the random Beltrami coefficient on $[0,1]^2$ obtained by putting
an i.i.d. copy of $\lambda$ on each $1/n \times 1/n$ subsquare.

Let $f_n : \mathbb{C} \to \mathbb{C}$ be the solution to Beltrami equation fixing 0,1, $\infty$ .

Then for all $\epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(\|f_n - \mathrm{Id}\| > \epsilon) = 0.$

$|\lambda| \leq c < 1$ from ARST
No restrictions on $\lambda$ from Ivrii-Markovic

Image of unit square under random quasiconformal map




$n \approx 35$

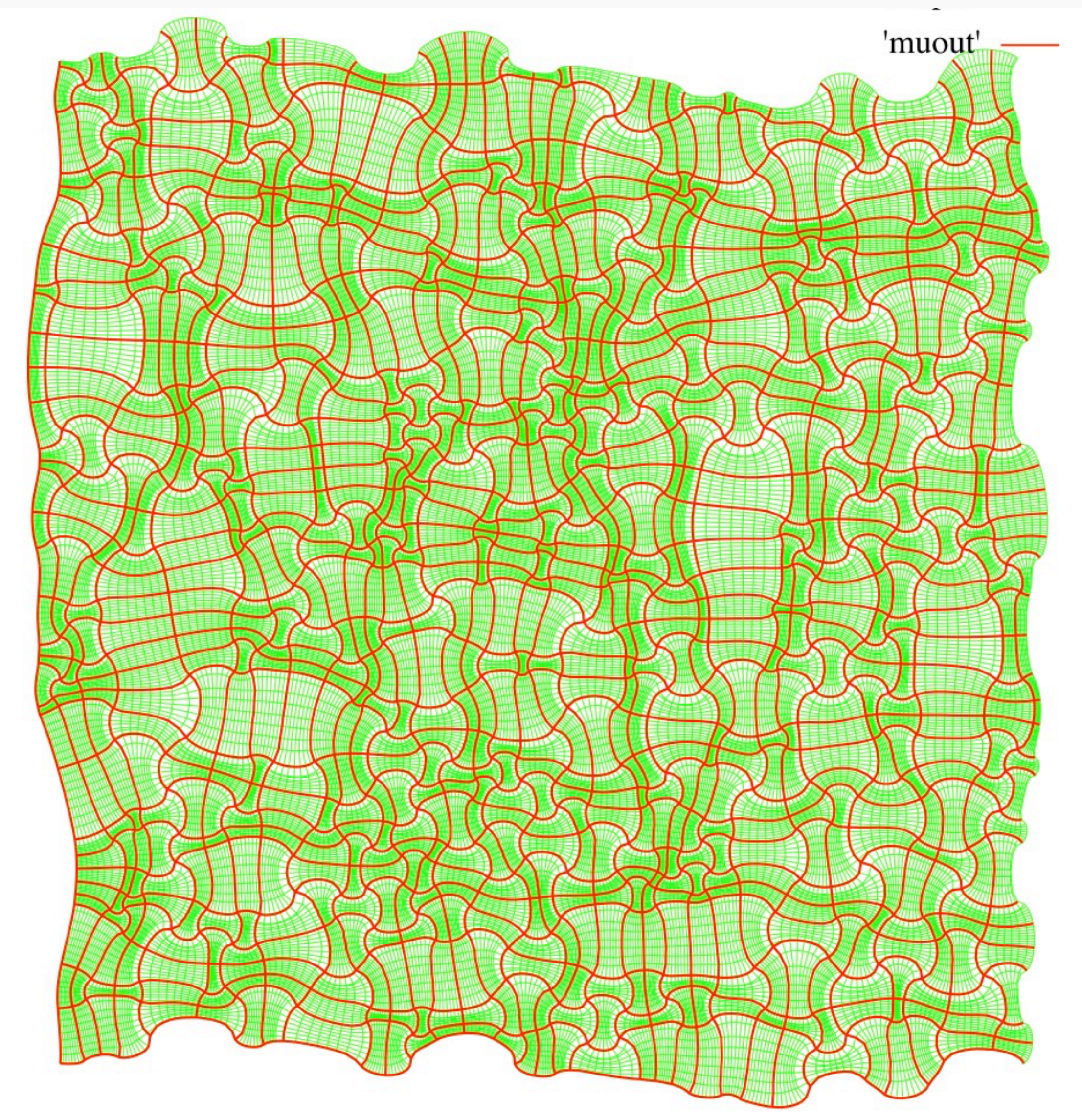



Image from Astala-Rohde-Saksman-Tao,
with credit to David White.

**Theorem (L):** $\forall \epsilon > 0,\ \forall M > 0,\ \exists c_0 > 0:$

Let $\mu$ be a random Beltrami coefficient on $[0,1]^2$ with rotationally invariant distribution.

Suppose $\mathbb{E}\mathscr{E}(\mu)^{1+\epsilon} \leq M$.

If $\mu_1, \ldots, \mu_4$ are i.i.d. samples of $\mu$ then

$$\mathbb{E}\mathscr{E}(\begin{bmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{bmatrix}) - 1 \leq E - c_0 T(E)$$

where $E = \mathbb{E}(\mathscr{E}(\mu) - 1)$ and $T(x) = \min(x, x^3)$.

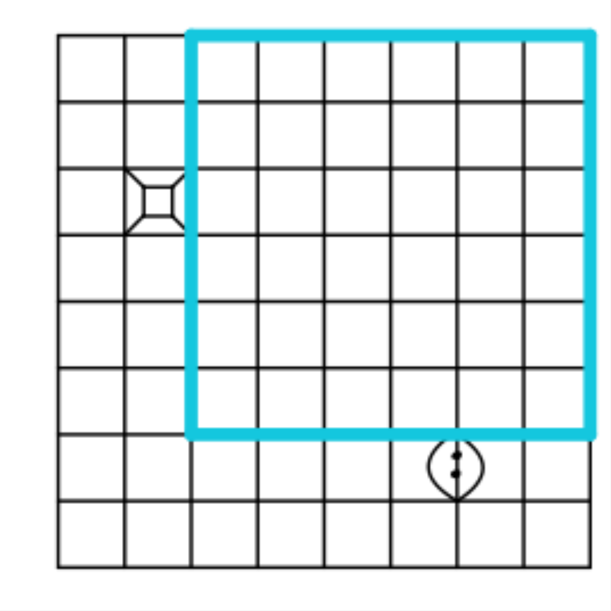
Using this, we can prove the desired contraction:

- Decompose $\mathscr{E}(X_{n+1})$ into sum of energy from large embedded subsquare and energy from the other squares.
- Obtain something like (with $\epsilon = 1$)

  $\mathbb{E}\mathscr{E}(X_{n+1}) \leq (1 - c_1)\mathbb{E}\mathscr{E}(X_n) + 1$ , *and*

  $\mathbb{E}(\mathscr{E}(X_{n+1})^2) \leq (1 - c_2)\mathbb{E}(\mathscr{E}(X_n)^2) + C_0(\mathbb{E}\mathscr{E}(X_n))^2$.

  This latter bound does not rely on any sort of cancellation.
- Boundedness of $\mathbb{E}\mathscr{E}(X_n)$ and $\mathbb{E}\mathscr{E}(X_n^2)$ follows.



The random cancellation theorem follows from the following deterministic statement.

**Lemma (L):** $\exists c_0 > 0$ :

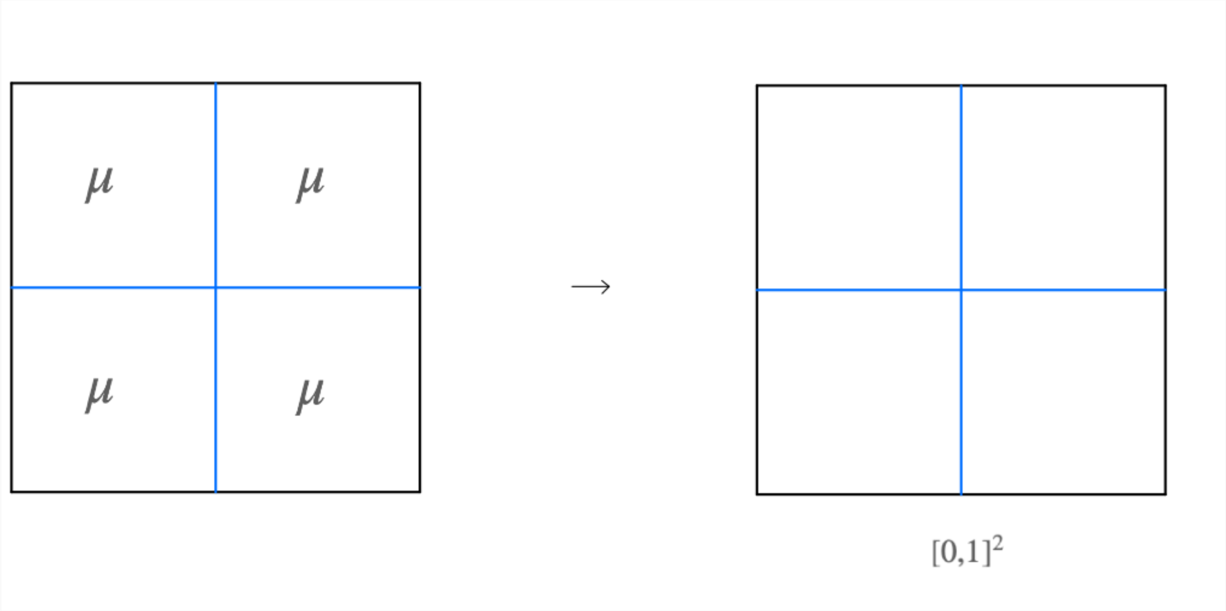
Suppose $\mu_1, \ldots, \mu_5$ are Beltrami coefficients on $[0,1]^2$.

Then there exists rotations $r_1, \ldots, r_4 \in \mathbb{Z}_4$ and an injective $\pi : \{1,2,3,4\} \to \{1,2,3,4,5\}$ such that

$$\mathscr{E}(\begin{bmatrix} r_1\mu_{\pi 1} & r_2\mu_{\pi 2} \\ r_3\mu_{\pi 3} & r_4\mu_{\pi 4} \end{bmatrix}) - 1 \leq \frac{1}{4}\sum_{i=k}^{4}(\mathscr{E}(\mu_{\pi k}) - 1) - c_0 T(\max_k \mathscr{E}(\mu_k) - 1)$$

where $T(x) = \min(x^{-1}, x^3)$.

Proof:




$$\mathscr{E}(\begin{bmatrix} r_1\mu_{\pi 1} & r_2\mu_{\pi 2} \\ r_3\mu_{\pi 3} & r_4\mu_{\pi 4} \end{bmatrix}) - 1 \leq \frac{1}{4}\sum_{i=k}^{4}(\mathscr{E}(\mu_{\pi k}) - 1)$$

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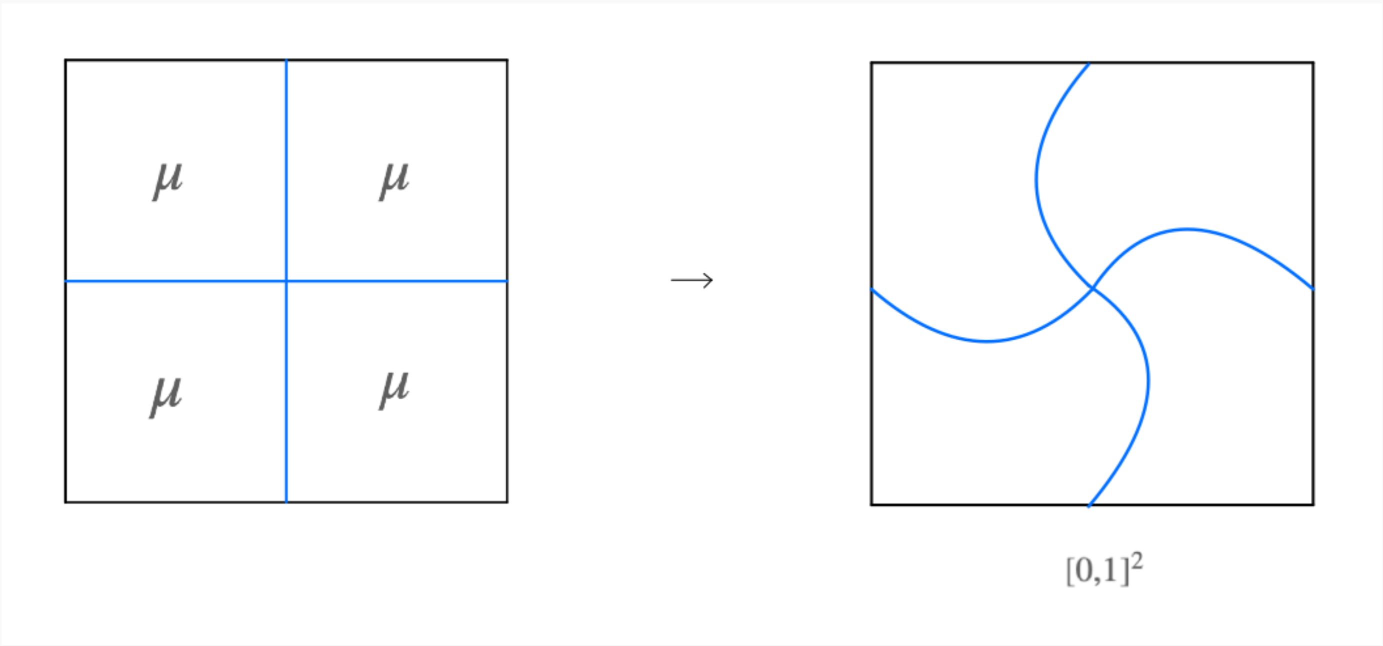


Reduce Dirichlet energy by perturbing in the right way

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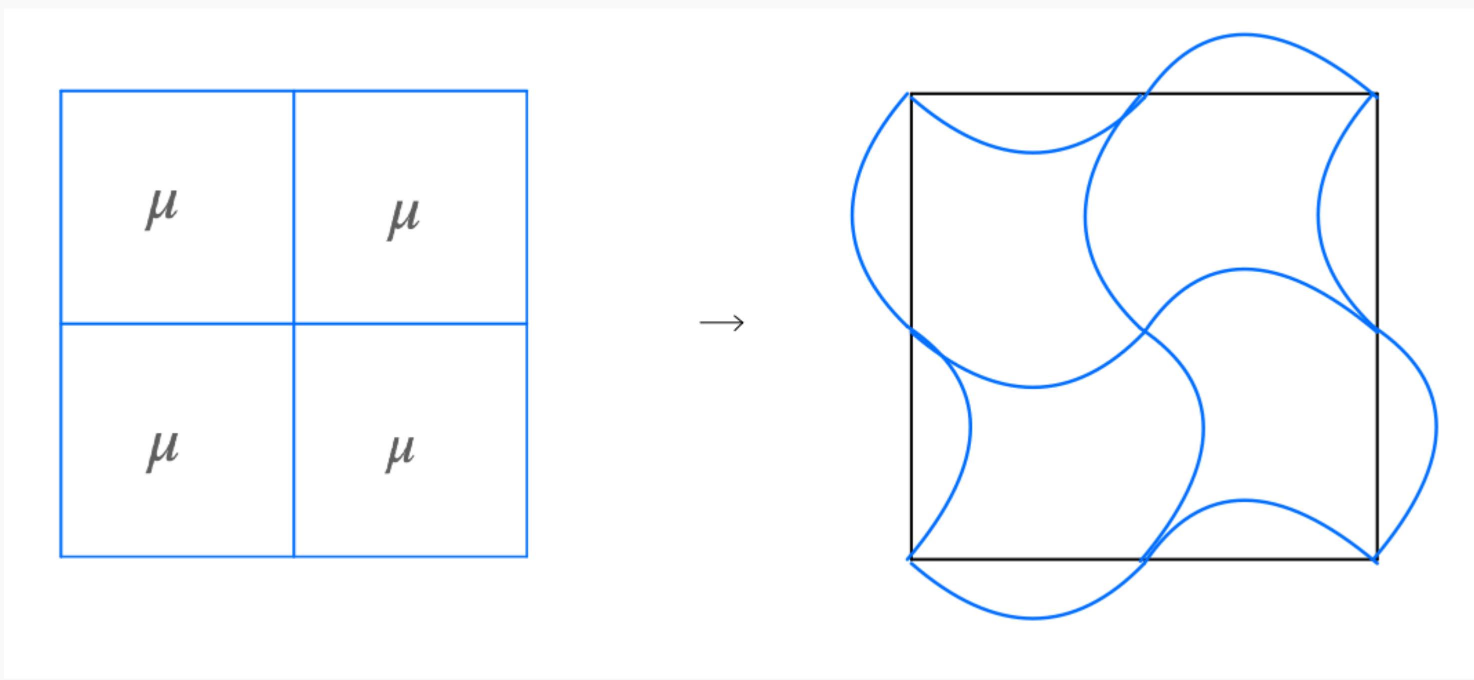
Proof:




By straightening the ellipse field,
there is a function with Dirichlet energy 1,
and with translation and rotational symmetries.

The random cancellation theorem follows from the following deterministic statement.

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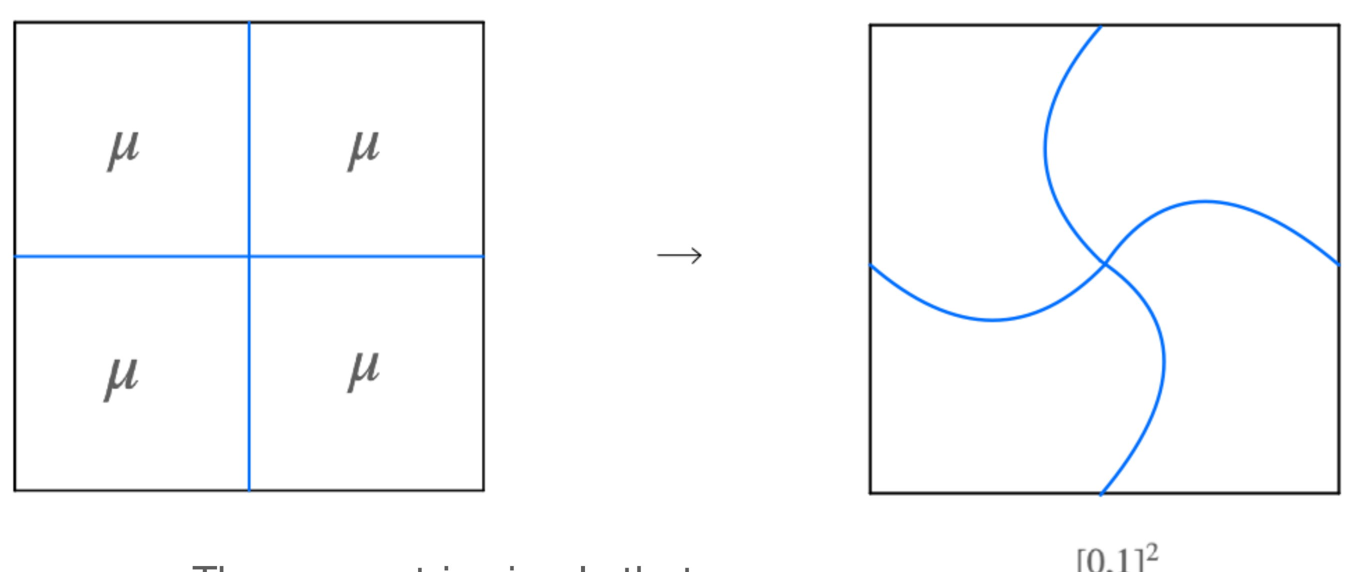
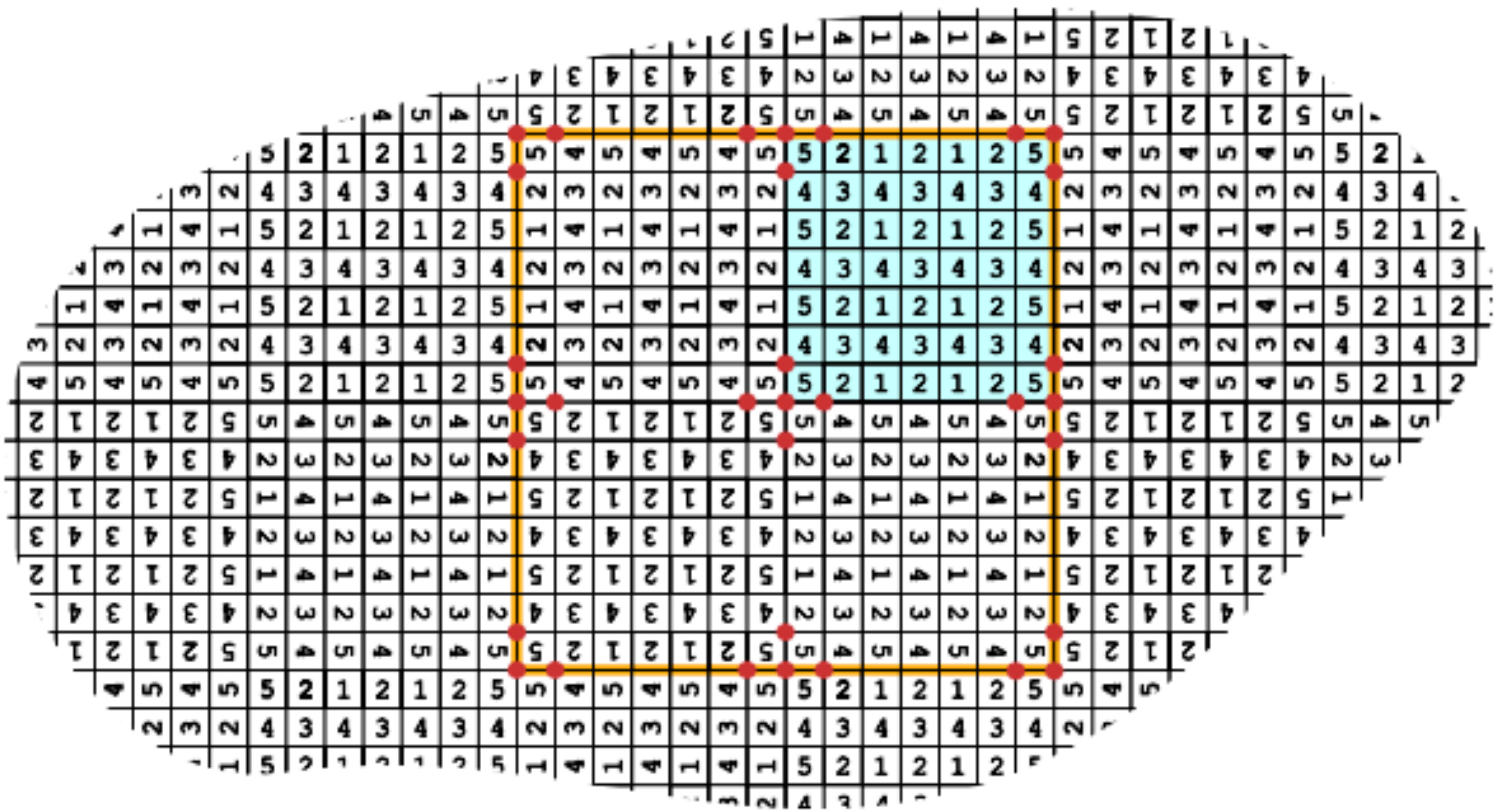



The symmetries imply that we can truncate the perturbation so that it doesn't change the boundary values.

$$\mathcal{E}(\begin{bmatrix} r_1\mu_{\pi 1} & r_2\mu_{\pi 2} \\ r_3\mu_{\pi 3} & r_4\mu_{\pi 4} \end{bmatrix}) - 1 \leq \frac{1}{4}\sum_{i=k}^{4} (\mathcal{E}(\mu_{\pi k}) - 1) - c_0 T(\max_k \mathcal{E}(\mu_k) - 1)$$

For the general case when $\mu_1, \ldots, \mu_5$ are distinct:

Glue Beltrami coefficients in the following symmetric way



Must be able to localize the “improvement” to one of the 2x2 squares

Thank you!