Quasisymmetry of semihyperbolic Julia equivalences

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Abstract

Every $\alpha \in (0,1)$ is associated to a Julia equivalence $\approx_\alpha$ on the circle. For certain α, these Julia equivalences provide a combinatorial model for the identifications made from the conformal map from the disk to the exterior of a connected tree-like quadratic Julia set. We define a notion of semihyperbolicity for the combinatorial parameter α, and we prove that the equivalences associated to such parameters satisfy a regularity condition which is analogous to the quasisymmetry condition for weldings of quasicircles. We also relate combinatorial semihyperbolicity to the notion of semihyperbolicity for polynomials introduced by Carleson, Jones and Yoccoz.

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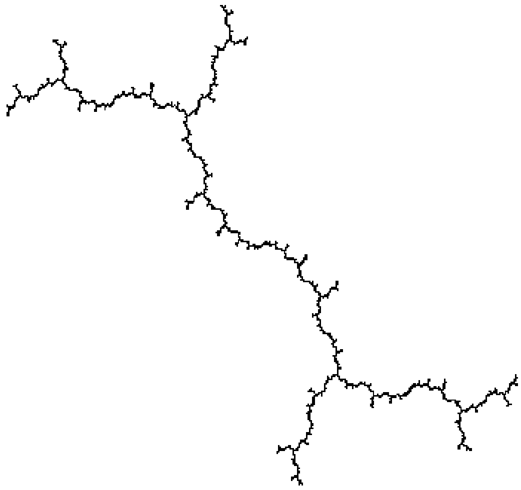
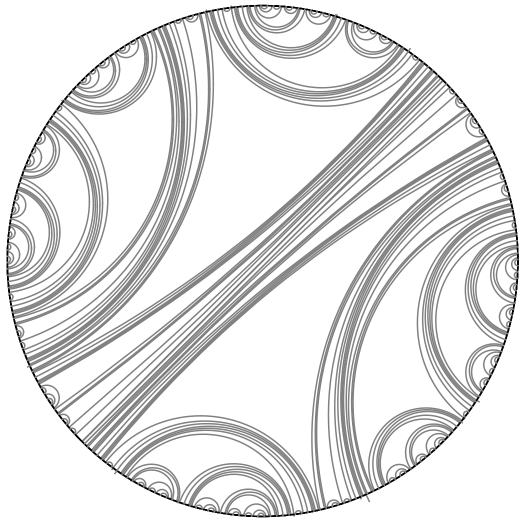
1 Introduction

Let $J \subset \mathbb{C}$ be a dendrite (compact, connected, locally connected, contains at least two points, and contains no simple closed curves) and let $\varphi : \mathbb{D}^* \to \hat{\mathbb{C}}\backslash J$ be a normalized conformal map from the exterior of the closed disk to the outside of J, extended continuously to the boundary. Julia sets of quadratic polynomials provide many examples of such dendrites. Then φ makes certain identifications on the boundary $\partial\mathbb{D}$ of $\mathbb{D}^*$. For instance if $J = [-1,1]$ then $\varphi(z) = \frac{1}{2}(z + z^{-1})$ and φ identifies each $z \in \partial\mathbb{D}$ with $\bar{z}$. See Figure 1 for a more interesting example.

Conversely, let $\sim$ be an equivalence relation on $\partial\mathbb{D} \cong \mathbb{T}$. We say that $\sim$ is *weldable* if it is induced by a conformal map to the complement of a dendrite. It is easy to see from topological considerations that $\sim$ must be a lamination (see the introduction to Section 2 for definitions) in order to be weldable, but this is not sufficient. Indeed, the term *conformal welding* often refers to the case of this problem when $\partial\mathbb{D}/\sim$ is homeomorphic to an arc and $\sim$ is given by, say, a homeomorphism h from the top half $\partial\mathbb{D}$ to the bottom half of $\partial\mathbb{D}$. Even in this setting, it is difficult in general to determine whether $\sim$ is weldable [Bis07]. It is



(a) The Julia set associated to the polynomial $z^2 + c$ where $c \approx .228 + 1.115i$ is a solution to $p_c^{\circ 4}(c) = p_c^{\circ 3}(c)$.



(b) The lamination $\sim_\alpha$ associated to $\alpha = 1/4$. The Riemann map from the outside of the disk to the outside of the Julia set of Figure 1a identifies the endpoints of chords in this picture.

Figure 1

known that if h is quasisymmetric, then $\sim$ is weldable and the resulting arc is a quasiarc. The converse is also true.

In [LR], it is suggested that the notion of a *Gehring tree* is the correct analog of a quasiarc in the dendritic setting. A Gehring tree is a dendrite $E \subset \mathbb{C}$ for which the complement $\mathbb{C}\backslash E$ is a *John domain*. A John domain is a domain for which any two points can be joined by a arc that does not pass too close to the boundary.

One of the main theorems of [LR] is a necessary and sufficient condition for a relation $\sim$ to have a Gehring tree welding.

Theorem 1.1 ([LR, Theorem 1])**.** *Suppose $\sim$ is a lamination and* $\mathrm{gs}(\sim)$ *holds uniformly over x and N. That is, there exists constants C, M, c, β, η such that for each $x \in \mathbb{T}$ and $N \geq 0$ there is $m \leq M$ such that* $\mathrm{gs}(\sim; x, N; C, m, c, \beta, \eta)$ *holds. Then $\sim$ is tree-weldable and the resulting dendrite J is unique up to Mobius transformation. Moreover, J is a Gehring tree.*

Conversely, if J is a Gehring tree and $\sim$ is the lamination induced by J, then $\mathrm{gs}(\sim)$ *holds uniformly over x and N.*

See the Section 3 for the definition of the 'gs' condition. Roughly speaking, $\sim$ has a good gluing at scale N and at $x \in \mathbb{T}$ if one can find a chain of a bounded 'chain' of intervals surrounding x, where each interval is glued to the next interval.

In this paper we show that the criterion can be applied to a certain class of combinatorially defined Julia equivalences.

For each $\alpha \in \mathbb{T}$ there is an associated Julia equivalence $\approx_\alpha$, which is defined a purely combinatorial fashion and models the welding relation arising from certain quadratic Julia sets. See Figure 1b and Section 2.2 for examples and definitions.

We show directly that if α is combinatorially semihyperbolic (see Section 3 for the definition), then the lamination $\approx_\alpha$ associated to α satisfies the good gluing condition of Theorem 1.1.

Theorem 1.2. *If $\alpha \in \mathbb{T}$ is combinatorially semihyperbolic then there exists uniformly good gluings at all $x \in \mathbb{T}$ and all scales $N > 0$.*

The notion of semihyperbolicity was introduced for polynomials in [CJY94], where it was shown (among other things) that a polynomial p is semihyperbolic if and only if either of the following equivalent conditions hold.

Theorem 1.3 ([CJY94, Theorem 1.1])**.** *The following conditions are equivalent.*

- *p has no parabolic perioidic points and $w \notin \overline{\bigcup_{t \geq 1} p^t(w)}$ for all points w such that $p'(w) = 0$ (critical points).*
- *The basin of attraction to ∞ is a John domain.*

When the Julia set J_c of a quadratic polynomial p_c is a dendrite, the lamination describing its welding is equal to $\approx_\alpha$ where $\alpha \in \mathbb{T}$ is the landing angle of the critical value c.

We show in that in this sitaution, combinatorially semihyperbolicity of α is equivalent to semihyperbolicity of p_c.

Theorem 1.4. *Suppose $c \in \mathbb{C}$ is a parameter for which J_c is a dendrite. Suppose c is semihyperbolic. Let $\alpha \in \mathbb{T}$ be the landing angle of the critical value, $\varphi(\alpha) = c$. Then α is combinatorially semihyperbolic.*

Conversely, if α is combinatorially semihyperbolic, there exists $c \in \mathbb{C}$ for which J_c is a dendrite and c is semihyperbolic, and the angle α lands at c and $\approx_\alpha = \sim_c$.

The proof of this Theorem is at the end of this paper, in Section 5. All the arguments preceding this section are purely combinatorial.

In future work, we would like to extend the results of this paper to cover the Collett-Eckmann (CE) [PRLS03, Smi00, GS98] quadratic polynomials. The work of [GS98] characterizates polymomials with Hölder Fatou components in terms of a dynamical condition on the critical points called Collett-Eckmann (CE) [PRLS03, Smi00, GS98].

The 'gs' condition in Theorem 1.1 also has an analogue for Hölder domains. Roughly speaking, if a lamination has good gluings at a positive density of scales, then the welding exists and the complement of the resulting tree is a Hölder domain. M We expect that the techniques in this paper can be used to show that laminations associated to CE quadratic polynomials satisfy the 'positive density of good gluings' condition.

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2 The minimal and dynamical laminations $\sim_\alpha$ and $\approx_\alpha$

Much of this material is in [Thu09] and [BK92]. Experts should skip to Section 3 and refer back as needed.

We provide a mostly self contained review of the minimal and dynamical α-equivalences $\sim_\alpha$ and $\approx_\alpha$ in Section 2.1 and 2.2. These equivalences are equal in our setting (see Theorem 2.4), and they provide a family of conformal welding problems. In Section 2.3 we introduce the notion of M-closeness which provides a simple characterization of the topology of quotient space $\mathbb{T}/\approx_\alpha$.

Even though it is known [BK92, Theorem 1] that $\sim_\alpha = \approx_\alpha$, our construction will use some more detailed results on the relationship between $\sim_\alpha$ and $\approx_\alpha$, and this is the content of Section 2.4.

In Section 3 we define precisely the good gluing condition that appears in Theorem 1.2.

Before moving on, we fix some notation. In this paper we make the identification $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$, and for $a, b \in \mathbb{T}$ we write (a, b) to mean the counterclockwise open arc from a to b, and similarly for $[a, b]$. We define the doubling map $h : \mathbb{T} \to \mathbb{T}$ by $h(x) = 2x \mod 1$. If $I = (a, b) \subset \mathbb{T}$ or $I = [a, b] \subset \mathbb{T}$ is an open or closed interval, let $|I| = b - a$ denote its length, normalized so that $|\mathbb{T}| = 1$.

An equivalence relation $\sim$ on $\mathbb{T}$ is *flat* if whenever $x \sim y$ and $z \sim w$, and the chord from x to y intersects the chord from z to w , then $x \sim y \sim z \sim w$. A *lamination* is a flat equivalence relation on $\mathbb{T}$.[1]

If $g \in A^*$ is a finite word on some alphabet set A, we write $|g|$ to denote the number of letters in g. If in addition $h \in A^* \cup A^\infty$ is a finite or infinite word, gh denotes the concatenation. If k is in a positive integer, g^k denotes the k-fold concatenation and g^∞ denotes the periodic infinite word $ggg\dots$. For integer $m \geq 0$, g_m denotes the mth letter of g, and $g|_m$ denotes the subword of g of the first m letters $g_0 g_1 g_2 \cdots g_{m-1} \in A^m$.

If $a, b, c, d \in \mathbb{T}$ are distinct, we say that $\{a, b\}$ *crosses* $\{c, d\}$ if the chord joining a to b intersects (in $\overline{\mathbb{D}}$) the chord from c to d. This is equivalent to saying that if U_1 and U_2 are the two components of $\mathbb{T} \backslash \{a, b\}$, then $\overline{U_1}$ and $\overline{U_2}$ each contain an element of $\{c, d\}$. We sometimes refer to two element subsets $\{a, b\} \subset \mathbb{T}$ as *chords*. If $a \sim b$ for some relation $\sim$ (depending on context), we may refer to $\{a, b\}$ as a *leaf*.

If $\sim$ is an equivalence relation on $\mathbb{T}$ and $x \in \mathbb{T}$, we write $[x]$ or $[x]_\sim$ to denote the equivalence class of x as a subset of $\mathbb{T}$.

To reduce clutter, we will drop subscripts, superscripts, and parentheses for function arguments when they are clear from context.

2.1 Invariant Laminations and the Minimal α-Equivalence $\sim_\alpha$

Fix $\alpha \in \mathbb{T} \backslash \{0\}$. We assume in the rest of the paper that α is not perioidic under iteration by h. The preimages of α under the angle doubling map h are $*_1 := \alpha/2$ and $*_2 := \alpha/2 + 1/2$. These two points divide the circle $\mathbb{T}$ into two semi-circular arcs, $(*_1, *_2)$ and $(*_2, *_1)$. Let L be the semicircle containing α and let R be the other semicircle.

For $x \neq \alpha$, there are two preimages of x under h. We let $\widetilde{L}x \in \{x/2, x/2 + 1/2\} \subset \mathbb{T}$ be the pre-image that lies in L. Similarly $\widetilde{R}x := \widetilde{L}x + 1/2$ is other pre-image of x, lying in R. Let $\sim$ be an equivalence relation on $\mathbb{T}$. Here are some proprerties that $\sim$ may have:

1. Forward invariant: $x \sim y \implies h(x) \sim h(y)$
2. Backward Invariant: For $x, y \neq \alpha$, we have $x \sim y \implies \widetilde{L}x \sim \widetilde{L}y$ and $\widetilde{R}x \sim \widetilde{R}y$. If $x \sim \alpha$ and $x \neq \alpha$, then $\widetilde{L}x \sim *_1$ and $\widetilde{R}x \sim *_1$.
3. Closed: If $x_n \to x$ and $y_n \to y$ and $x_n \sim y_n$ for all n then $x \sim y$.

Following [Thu09], any equivalence relation satisfying properties 1) and 2) above is said to be *invariant*. The *minimal α-equivalence* is the minimal closed invariant equivalence relation $\sim_\alpha$ for which $*_1 \sim_\alpha *_2$.

The following general observation about invariant relations will be useful throughout the rest of the article, and is easily proved by induction.

Lemma 2.1. *Fix $\alpha \in \mathbb{T}$ and suppose $\sim$ is a forward and backward invariant equivalence relation with respect to α. Then for every $x \in \mathbb{T}$ and $t \geq 0$, we have $h^t([x]_\sim) = [h^t(x)]_\sim$.*

Backwards invariance of $\sim_\alpha$ allows us to construct some chords of $\sim_\alpha$ concretely. Recall that $\widetilde{L}$ and $\widetilde{R}$ are the continuous inverse branches of h on $\mathbb{T} \backslash \{\alpha\}$. Every finite word $g \in \{L, R\}^*$ encodes a composition of such mappings, where we use the usual right to left ordering convention for function composition. Denote this mapping by $\widetilde{g}$.

[1] In the literature, for instance [Thu09], a *lamination* refers to a collection of noncrossing chords of the circle. The two usages are clearly related, but in this paper a *lamination* will always be an equivalence relation.

Since each function in the composition $\widetilde{g}$ is only well defined away from α, the domain of $\widetilde{g}$ is $\mathbb{T}\backslash A_g$ where A_g is some subset of the *postcritical set* $P_\alpha := \{h^t\alpha : 0 \le t \le |g|\}$. In fact, it is easy to see that

$$A_g = \{x \in \mathbb{T} : \widetilde{\sigma^t g}(x) = \alpha \text{ for some } 1 \le t \le |g|\} \tag{1}$$

where we recall that σ is the left shift operator on words. Notice that the derivative of $\widetilde{g}$ is $2^{-|g|}$ and $\widetilde{g}$ is linear on each component of $\mathbb{T}\backslash A_g$.

Since α is not periodic, $*_1, *_2$ are not in P_α. Therefore $\widetilde{g}*_1$ and $\widetilde{g}*_2$ are well defined for every choice of g. The following observation follows immediately from backwards invariance, and gives us a way of describing a dense subset of the relation $\sim_\alpha$.

Proposition 2.2. *For all $g \in \{L,R\}^*$, we have $\widetilde{g}*_1 \sim_\alpha \widetilde{g}*_2$.*

If $\widetilde{g}\{*_1, *_2\}$ is a leaf of the minimal equivalence, and $\widetilde{g}$ is finite, we say that $|g|$ is the *depth* of that leaf. The choice of $\widetilde{g}$ used to represent the leaf is unique because α is not periodic, so this is well defined.

2.2 Itineraries and the Dynamical α-Equivalence $\approx_\alpha$

In the course of our construction we will need the following alternative description of the lamination $\sim_\alpha$, in terms of itineraries. Every point in $\mathbb{T}$ lies in either L, R or $\{*_1, *_2\}$. For $x \in \mathbb{T}$, the *itinerary* $I^\alpha(x) \in \{L, R, \star\}^\infty$ is an infinite sequence on a three letter alphabet, which keeps track of which half of the circle the iterates of x lie in. It is defined as follows:

$$I^\alpha(x)_n = \begin{cases} L & \text{if } h^n x \in L \\ R & \text{if } h^n x \in R \\ \star & \text{if } h^n x \in \{*_1, *_2\}. \end{cases} \qquad \text{for } n \ge 0.$$

It follows immediately from the definitions that $h : \mathbb{T} \to \mathbb{T}$ is semiconjugate to the left shift $\sigma : \{L, R, \star\}^\infty \to \{L, R, \star\}^\infty$, which maps $\sigma : u_0u_1u_2 \cdots \mapsto u_1u_2 \ldots$. In other words $I^\alpha(hx) = \sigma I^\alpha(x)$. We also have

$$LI^\alpha(x) = I^\alpha(\widetilde{L}x), \qquad RI^\alpha(x) = I^\alpha(\widetilde{R}x), \qquad \text{for } x \in \mathbb{T}\backslash\{\alpha\}. \tag{2}$$

The itinerary $I^\alpha(\alpha)$ plays a special role and it is called the *kneading sequence* for α. We will assume that $I^\alpha(\alpha)$ is not periodic. This is stronger than assuming that α is not perioidic. This assumption is justified by our Definition 1 of combinatorial semihyperbolicity.

The *dynamical α-equivalence* $\approx_\alpha$ is the smallest equivalence relation such that points with the same itinerary are identified, where the $\star$ symbol is used as a wildcard when comparing two itineraries.

Formally, this is described as follows. We say that an infinite word $g \in \{L, R, \star\}^\infty$ is *precritical* if it can be written as $g = usI^\alpha(\alpha)$ where $u \in \{L, R\}^*$ is a finite word and $s \in \{L, R, \star\}$. Note that by nonperiodicity of $I^\alpha(\alpha)$, such a decomposition, if it exists, is unique. If g is precritical and $g = usI^\alpha(\alpha)$ as above, then define the infinite words $g^L = uLI^\alpha(\alpha)$, $g^R = uRI^\alpha(\alpha)$ and $g^\star = u \star I^\alpha(\alpha)$. Then $\approx_\alpha$ is defined as follows:

$$x \approx_\alpha y \iff \begin{cases} I^\alpha(x) = I^\alpha(y), & \text{or } I^\alpha(x), I^\alpha(y) \text{ are precritical and} \\ I^\alpha(x)^\star = I^\alpha(y)^\star. \end{cases} \tag{3}$$

Note that $I^\alpha(x)^\star = I^\alpha(y)^\star$ iff $I^\alpha(x)^L = I^\alpha(y)^L$ iff $I^\alpha(x)^R = I^\alpha(y)^R$ iff $I^\alpha(y) \in \{I^\alpha(x)^L, I^\alpha(x)^R, I^\alpha(x)^\star\}$. In particular, if $I^\alpha(x)$ is precritical and $y \in \mathbb{T}$, then $x \approx_\alpha y$ iff $I^\alpha(y) \in \{I^\alpha(x)^L, I^\alpha(x)^R, I^\alpha(x)^\star\}$. If $I^\alpha(x)$ is not precritical then the only way that y can be $\approx_\alpha$ equivalent to x is if $I^\alpha(y) = I^\alpha(x)$.

From this we see that $\approx_\alpha$ as defined in (3) is an equivalence relation.

From (2) we see that $\approx_\alpha$ is forward and backward invariant, in particular Lemma 2.1 applies to $\approx_\alpha$ and therefore $h^t[x]_{\approx_\alpha} = [h^t x]_{\approx_\alpha}$ for all $x \in \mathbb{T}$ and $t \ge 0$.

If $I^\alpha(x)$ is precritical then this is equivalent to saying that $h^t x \in [*_1]$ for some $t \geq 0$. By the above this is also equivalent to saying that $h^t[x] = [*_1]$.

We see that $I^\alpha(\alpha)$ is not precritical, because $I^\alpha(\alpha)$ is not periodic. As a consequence, $[\alpha]$ is not periodic, in other words $h^t[\alpha] = [h^t\alpha] \neq [\alpha]$ for $t \geq 1$.

This then implies that $h^t[*_1] \neq [*_1]$ for $t \geq 1$. We will use these observations repeatedly throughout the rest of this paper.

We will also need the following results.

Theorem 2.3 ([BK92, Proposition 6.2])**.** *Suppose $I^\alpha(\alpha)$ is nonperiodic. Then all equivalence classes of $\approx_\alpha$ are finite.*

Theorem 2.4 ([BK92, Theorem 1])**.** *Suppose $I^\alpha(\alpha)$ is nonperiodic. Then $\sim_\alpha = \approx_\alpha$.*

2.3 Convergence in $\mathbb{T}/\approx_\alpha$ in terms of itineraries

In this subsection we will develop a useful characterization (Proposition 2.6) of the convergent sequences in $\mathbb{T}/\approx_\alpha$ in terms of itineraries of points in the sequence.

The idea is that nearby points should have itineraries that agree on long initial subwords. However, the definition is a little complicated because $\star$ needs to be treated as a wildcard; a good example to keep in mind is the points $*_1 + \epsilon$ and $*_1 - \epsilon$ for $\epsilon > 0$ small. They are close in $\mathbb{T}/\approx_\alpha$, that is $[*_1 - \epsilon]$ and $[*_1 + \epsilon]$ converge to the same point in $\mathbb{T}/\approx_\alpha$ as $\epsilon \to 0$, but their itineraries differ at the first letter.

Motivated by this example, we say that two length M words $g_1, g_2 \in \{L, R, \star\}^M$ are M-close if $g_1 = g_2$ or there exists a finite word $u \in \bigcup_{k=0}^{M-1}\{L, R\}^j$ of length at most $M-1$ such that $g_1 = us_1v$ and $g_2 = us_2v$, where $s_1, s_2 \in \{L, R, \star\}$, and v is an initial subword of $I^\alpha(\alpha)$ of length $M - |u| - 1$.

We can extend this definition to words of length greater than M, including infinite words, by saying that two such words are M-close if their restrictions to the first M letters are M-close. We also say that two points $x, y \in \mathbb{T}$ are M-close if their itineraries are M-close. We use the notation $x \overset{M}{\asymp} y$ and $I^\alpha(x) \overset{M}{\asymp} I^\alpha(y)$ to denote M-closeness.

For each $x \in \mathbb{T}$, we define the M-neighborhood $B_M(x)$ around x to be the set of points in $\mathbb{T}$ which are M-close to x.

First we prove that if y is sufficiently close to x with respect to the standard topology on $\mathbb{T}$, then y is M-close to x. The nonperiodicity of $[*_1]$ is crucial here.

Lemma 2.5. *For each $M > 0$ and $x \in \mathbb{T}$, there exists an open neighborhood $U_M(x)$ containing $[x]_{\approx_\alpha}$ such that $y \in U_M(x) \implies x \overset{M}{\asymp} y$. If in addition $I^\alpha(x)$ is not precritical, we can strengthen the conclusion by replacing it with $y \in U_M(x) \implies I^\alpha(x)|_M = I^\alpha(y)|_M$.*

Proof. Recall that $I^\alpha(x)$ is not precritical if and only if $h^t[x] \neq [*_1]$ for $t \geq 0$. From this we see that $I^\alpha(x)$ is not precritical if and only if $h^t[x] = [h^t x]$ never intersects $\{*_1, *_2\}$ for $t \geq 0$.

First suppose that the sets $h^t[x]$ never intersect $\{*_1, *_2\}$ for $t \geq 0$. For each $x' \in [x]$ we can choose an open arc $I_{x'}$, containing x', small enough that $h^t I_{x'} \cap \{*_1, *_2\}$ is empty for $0 \leq t \leq M$. Now, for $0 \leq t \leq M$, we have that $h^t I_{x'}$ and $h^t x'$ lie in the same semicircle (either L or R). Therefore any $y \in I_{x'}$ has the same itinerary as x' for the first M letters. But every $x' \in [x]$ has the same itinerary as x for the first M letters because $I^\alpha(x)$ is not precritical. Therefore every $y \in U_M := \bigcup_{x' \in [x]} I_{x'}$ has the same itinerary as x for the first M letters. Moreover, the first M letters of the common itinerary all lie in $\{L, R\}$.

Now suppose $h^t[x] = [*_1]$ for some $0 \leq t \leq M$. This can happen for at most one value of t because $[*_1]$ is not periodic.

Let T be the unique time such that $h^T[x] = [*_1]$, then we can write $I^\alpha(x)|_M = usv$ where $u \in \{L, R\}^T$, $s \in \{L, R, \star\}$, and $v \in \{L, R\}^{M-T-1}$ is an initial subword of $I^\alpha(\alpha)$. For each $x' \in [x]$, let $I_{x'}$ be an open interval containing x' and small enough such that $h^t I_{x'}$ does not intersect $\{*_1, *_2\}$ for $0 \leq t \leq T-1$ and $T+1 \leq t \leq M$.

The same reasoning as in the previous case shows that if $y \in \bigcup_{x' \in [x]} I_{x'}$ then $I^\alpha(y) = I^a(x)$, except possibly at the index $t = T$. $\square$

The converse to the previous Lemma is also true; if x is M-close to y for large M then y is close to $[x]_{\approx_\alpha}$ in $\mathbb{T}$. This gives us the characterization of convergence we wanted.

Proposition 2.6. *Suppose x_n is a sequence on $\mathbb{T}$, and suppose $x \in \mathbb{T}$. Then $[x_n] \to [x]$ in the quotient topology of $\mathbb{T}/\approx_\alpha$ iff for all integer M, there exists N_0 such that $n > N_0$ implies $I^\alpha(x_n)$ is M-close to $I^\alpha(x)$.*

Before proving this proposition, we need a few lemmas.

The next lemma says that if z is close to y and $x \approx_\alpha z$ then x is close to y.

Lemma 2.7. *For each $M > 0$ there exists $K > 0$ such that the following holds. If $x \overset{M+K}{\asymp} z$ and $z \overset{M+K}{\asymp} y$ then $x \overset{M}{\asymp} z$.*

Proof. Suppose $x \overset{M+K}{\asymp} z$ and $z \overset{M+K}{\asymp} y$. Then

$$\begin{aligned} I^\alpha(x)|_{M+K} &= uav \\ I^\alpha(z)|_{M+K} &= ubv \qquad (4) \\ I^\alpha(z)|_{M+K} &= u'a'v' \qquad (5) \\ I^\alpha(y)|_{M+K} &= u'b'v' \end{aligned}$$

where $u, u' \in \{L, R\}^*$ are finite words, $a, b, a', b' \in \{L, R, \star\}$ and v, v' are initial subwords of $I^\alpha(\alpha)$. Suppose first that $|u| \geq M$. If $|u'| \geq M$ too then we are done, because $I^\alpha(x)|_M = u|_M = I^\alpha(z)|_M = u'|_M = I^\alpha(y)|_M$. On the other hand if $|u'| < M$, we have $I^\alpha(x)|_M = u|_M = I^\alpha(z)|_M = u'a'v''$ where v'' is an initial subword of v', and hence v'' is an initial subword of $I^\alpha(\alpha)$. Also $I^\alpha(y)|_M = u'b'v''$, so $x \overset{M}{\asymp} y$.

The case where $|u'| \geq M$ and $|u| < M$ is then taken care of by the symmetric argument, so now suppose $|u'| < M$ and $|u| < M$. This is the only case where we use the fact that K is large and $I^\alpha(\alpha)$ is nonperiodic. Since $I^\alpha(\alpha)$ is nonperiodic, every shift $\sigma^i I^\alpha(\alpha)$ must eventually disagree with $I^\alpha(\alpha)$ if we look deep enough into the sequence. We choose K to be the largest index we need to observe to find this disagreement, when we restrict to shifts of length less than M. That is,

$$K = \sup_{0 \leq i \leq M} \inf\{T : \sigma^i I^\alpha(\alpha)|_T \neq I^\alpha(\alpha)|_T\}.$$

It suffices to show that $|u| = |u'|$ because then the two different ways of writing $I^\alpha(z)$ forces $u = u'$ and hence $x \overset{M}{\asymp} y$. So suppose for contradiction that, say, $|u| < |u'|$ and write $u' = ucw$ for some $c \in \{L, R\}$ and some (possibly empty) word $w \in \{L, R\}^*$. Note that $0 \leq |w| < |u'| < M$.

Applying the shift $\sigma^{|u|+1}$ to (4) and (5) gives

$$\begin{aligned} \sigma^{|u|+1}(I^\alpha(z)|_{M+K}) &= v \\ \sigma^{|u|+1}(I^\alpha(z)|_{M+K}) &= wa'v' \end{aligned}$$

which shows $\sigma^{|w|+1} v = v'$. But recall that v and v' are initial subwords of $I^\alpha(\alpha)$, and the lengths of v and v' are at least K, so this means that we actually have $\sigma^{|w|+1} I^\alpha(\alpha)|_K = I^\alpha(\alpha)|_K$. This contradicts the definition of K. $\square$

This lemma says that the topology induced by the B_M is fine enough to distinguish $\approx_\alpha$ classes.

Lemma 2.8. *For each $x \in \mathbb{T}$ and $M > 0$ there exists $K > 0$ such that $\overline{B_{M+K}(x)} \subset B_M(x)$. Therefore*

$$[x]_{\approx_\alpha} = \bigcap_M B_M(x) = \bigcap_M \overline{B_M(x)}.$$

Proof. We begin by proving the first equality. If $x \approx_\alpha y$, it follows from the definitions that $x \overset{M}{\asymp} y$ for all M. On the other hand, suppose $x \overset{M}{\asymp} y$ for all M. Let T be the smallest number that $I^\alpha(x)|_{T+1} \neq I^\alpha(y)|_{T+1}$, if $T = \infty$ then we are done so suppose $T < \infty$. Let $u = I^\alpha(x)|_T = I^\alpha(y)|_T$. Then for each $M > T$ we have $x|_M = us_1v$ and $y|_M = us_2v$ where $s_1, s_2 \in \{L, R, \star\}^\infty$ and v is an initial subword of $I^\alpha(\alpha)$. Letting $M \to \infty$ shows that $x = us_1I^\alpha(\alpha)$ and $y = us_2I^\alpha(\alpha)$, which means $x \approx_\alpha y$.

Now we turn to the second equality. Fix $M > 0$ and $x \in \mathbb{T}$ and suppose $K > 0$ is large. Suppose $y_i \in B_{M+K}(x)$ is a sequence converging to $y \in \overline{B_{M+K}(x)}$.

Lemma 2.5 implies that for sufficiently large i, we have $y_i \overset{M+K}{\asymp} y$. Taking $z = y_i$ in Lemma 2.7 implies that $x \overset{M}{\asymp} y$. Thus $\overline{B_{M+K}(x)} \subset B_M(x)$, and this proves the second equality in the statement of the lemma. ☐

Proposition 2.6 now follows from the next lemma.

Lemma 2.9. *Suppose $x \in \mathbb{T}$, and let $U \subset \mathbb{T}$ be an open set containing $[x]$. For sufficiently large M, we have $B_M(x) \subset U$.*

Proof. To prove the claim, suppose for contradiction that there is a sequence $y_{m_j} \in \mathbb{T}\backslash U$ such that $y_{m_j} \in B_{m_j}(x)$ and $m_j \to \infty$, then by compactness we can assume that $y_{m_j} \to y \in \mathbb{T}\backslash U$. Because $\overline{B_M(x)}$ is decreasing in M, we have for each M that the tail of the sequence (y_{m_j}) is contained in $\overline{B_M(x)}$. Therefore $y \in \bigcap_M \overline{B_M(x)}$ and by Lemma 2.8 we get $y \in [x]$, which contradicts the fact that U contains $[x]$. ☐

Proof of Proposition 2.6. Assume $[x_n] \to [x]$ in $\mathbb{T}/\approx_\alpha$. Let $M \geq 0$ be arbitrary. By Lemma 2.7 we can choose $K > 0$ large enough that for all $z, y, x \in \mathbb{T}$, we have that $z \approx_\alpha y$ and $y \overset{M+K}{\asymp} x$ implies $z \overset{M}{\asymp} x$.

By Lemma 2.5 we can let U be an open neighborhood of $[x]$ in $\mathbb{T}$ such that $y \in U \implies y \overset{M+K}{\asymp} x$. By definition of the quotient topology, $[U] \subset \mathbb{T}/\approx_\alpha$ is an open neighborhood of $[x]$ in $\mathbb{T}/\approx_\alpha$. The assumption implies that $[x_n] \in [U]$ for sufficiently large n. This means that $x_n \approx_\alpha y$ for some $y \in U$. But $y \overset{M+K}{\asymp} x$, so $x_n \overset{M}{\asymp} x$. Since M was arbitrary, this proves the 'only if' direction.

For the other direction, let W be an arbitrary open neighborhood of $[x]$ in $\mathbb{T}/\approx_\alpha$, and let U be its preimage under the quotient map $x \mapsto [x]$. Then U is an open set containing $[x] \subset \mathbb{T}$. By Lemma 2.9, we have for sufficiently large M that $B_M(x) \subset U$, and this completes the proof. ☐

2.4 Cylinder Sets and Boundary Leaves

The material in this section appears can also be found in [BK92, Section 4].

If $g \in \{L, R\}^*$ a finite word, define the *(open) cylinder* $C(g) \subset \mathbb{T}$ to be the set of points whose initial itinerary is equal g, that is $C(g) = \{x \in \mathbb{T} : I^\alpha(x)|_{|g|} = g\}$. These sets are closely related to the M-neighbourhoods introduced in the previous section, indeed every M-neighbourhood is a union of finitely many cylinder sets.

One important part of our proof of Theorem 1.2 is in finding leaves in the relation $\sim_\alpha$ which have one endpoint close to a given point x. We use this idea twice, in Lemma 4.2 and also in the Lemma 4.5. We do this by considering cylinder sets containing x.

1. By the results of the previous section, in particular Lemma 2.8, we see that $C(I^\alpha(x)|_n)$ does actually converge to $[x]$ as $n \to \infty$. Actually this only makes sense for x that are not precritical, since we have not defined the cylinder sets for words that contain the symbol $\star$. Proposition 2.10 deals with this issue.
2. On the other hand, every boundary chord of a cylinder $C(g)$ is actually a leaf of the lamination (Proposition 2.11).

The combination of these results allows us to construct the desired approximations for every x. The rest of this section contains the proofs of these two statements, and the approximation result is summarized in Proposition 2.12.

Recall from section 2.2 that an itinerary $g \in \{L, R, \star\}^\infty$ is said to be precritical if it can be written in the form $g = usI^\alpha(\alpha)$ where $u \in \{L, R\}^*$ and $s \in \{L, R, \star\}$. If g is precritical, $g^L = uLI^\alpha(\alpha)$ and $g^R = uRI^\alpha(\alpha)$ are the words obtained from g by replacing the symbol $\star$ with L and R respectively.

Proposition 2.10. *Suppose $x \in \mathbb{T}$ and let $g = I^\alpha(x) \in \{L, R, \star\}^\infty$. First suppose g is not precritical. Then*

$$[x]_{\approx_\alpha} = \bigcap_{n\geq 1} \overline{C(g|_n)} = \bigcap_{n\geq 1} C(g|_n).$$

On the other hand, if g is precritical, then

$$[x]_{\approx_\alpha} = \bigcap_{n\geq 1} \overline{C(g^L|_n)} \cup \bigcap_{n\geq 1} \overline{C(g^R|_n)}. \tag{6}$$

Moreover, in this case,

$$\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)} = \widetilde{u}\{*_1, *_2\},$$

where $u \in \{L, R\}^*$ is defined implicitly via $g^\star = usI^\alpha(\alpha)$ for some $s \in \{L, R, \star\}$.

Proof. Consider first the case where g is not precritical. It is clear that $\overline{C(g|_n)} \subset \overline{B_n(x)}$. So by Lemma 2.8, the intersection $\bigcap_{n\geq 1} \overline{C(g|_n)}$ is contained inside $[x]_{\approx_\alpha}$ On the other hand, if $y \in [x]_{\approx_\alpha}$, this means that $I^\alpha(y) = I^\alpha(x)$. So $y \in C(g|_n)$ for all n. We have shown that $\bigcap_{n\geq 1} \overline{C(g|_n)} \subset [x]_{\approx_\alpha} \subset \bigcap_{n\geq 1} C(g|_n)$, so we are done.

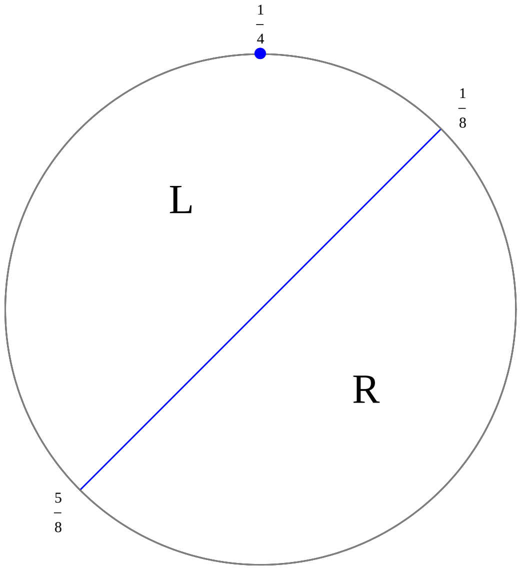
Now consider the case where g is precritical and write $g = usI^\alpha(\alpha)$. Again we have for all $n \geq 1$ that $C(g^L|_n) \subset B_n(x)$ and $C(g^R|_n) \subset B_n(x)$, therefore $\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cup \bigcap_{n\geq 1} \overline{C(g^R|_n)} \subset [x]_{\approx_\alpha}$.

For the other direction, suppose $y \in [x]_{\approx_\alpha}$. There are three cases to consider.

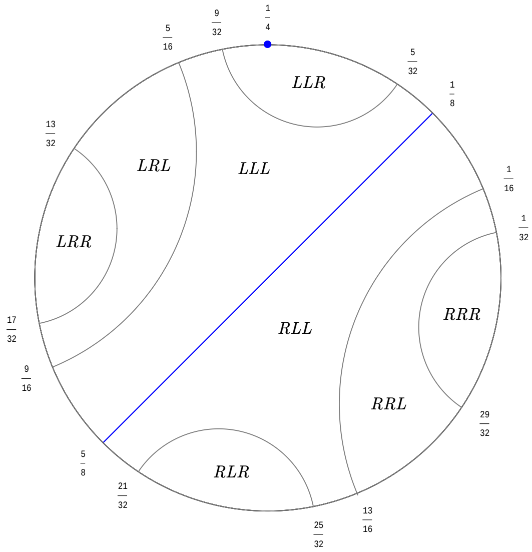
- If $I^\alpha(y) = uLI^\alpha(\alpha) = g^L$ then $y \in C(g^L|_M)$ for all M.
- If $I^\alpha(y) = uRI^\alpha(\alpha) = g^R$ then $y \in C(g^R|_M)$ for all M.
- It remains to check the case $I^\alpha(y) = u \star I^\alpha(\alpha)$. If this is the case, then $y \in \{\widetilde{u}*_1, \widetilde{u}*_2\}$. Assume that $y = \widetilde{u}*_1$, the other case $y = \widetilde{u}*_2$ is similar. We claim that $y = \widetilde{u}*_1 \in \overline{C(g^L|_M)}$ for all M. To see this, observe that $*_1 + \epsilon$ is in L, for all small $\epsilon > 0$. Also $h(*_1 + \epsilon)$ is close to α, so for sufficiently small ϵ we have that $I^\alpha(*_1 + \epsilon)|_M = LI^\alpha(\alpha)|_M$, see Lemma 2.5. Therefore we have by the conjugacy (2) that $\widetilde{u}(*_1 + \epsilon) \in C(u(LI^\alpha(\alpha)|_M))$, and $\lim_{\epsilon\to 0} \widetilde{u}(*_1 + \epsilon) = \widetilde{u}*_1 = y$. Here we have that $\widetilde{u}$ is continuous at $*_1$ because $*_1$ is not in the postcritical set P_α (see (1) and the surrounding dicussion). This proves the claim, and completes the proof of (6).

In the last item above we showed that $\widetilde{u}*_1, \widetilde{u}*_2 \in \overline{C(g^L|_M)}$ for all M. Similar arguments show that $\widetilde{u}*_1, \widetilde{u}*_2 \in \overline{C(g^R|_M)}$ for all M, and this proves that

$$\bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)} \supset \widetilde{u}\{*_1, *_2\}.$$



(a) The left and right halves L and R of S^1. Here $\alpha = 1/4$ and so $I^\alpha(\alpha) = LLRRR\ldots$.



(b) The collection of cylinders $C(g)$ for $g \in \{L, R\}^3$ partitions $\mathbb{T}$ up to a finite set. The boundary leaves of $C(LLL)$ are $\{*_1 = 1/8, *_2 = 5/8\}$, $\widetilde{L}\{*_1, *_2\} = \{5/16, 9/16\}$ and $\widetilde{LL}\{*_1, *_2\} = \{5/32, 9/32\}$, as we expect from Proposition 2.11.

For the reverse inclusion, suppose $y \in \bigcap_{n\geq 1} \overline{C(g^L|_n)} \cap \bigcap_{n\geq 1} \overline{C(g^R|_n)}$. Then $y \in \overline{C(g^L|_{|u|+1})} = \overline{C(uL)}$, therefore $h^{|u|}y \in \overline{h^{|u|}C(uL)} = \overline{C(L)} = \overline{L}$ by continuity of h. Similarly $h^{|u|}y \in \overline{R}$. This shows that $h^{|u|}y \in \overline{L} \cap \overline{R} = \{*_1, *_2\}$.

On the other hand $y \in \overline{C(g^L|_{|u|})} = \overline{C(u)}$, so $y \in \widetilde{u}\{*_1, *_2\}$ as desired. □

By (2), the cylinder $C(g)$ can also be described as the image of $\mathbb{T}\backslash A_g$ under $\widetilde{g}$ (recall the definition of A_g in (1)), that is $C(g) = \widetilde{g}(\mathbb{T}\backslash A_g)$. Since $\widetilde{g}$ is continuous on each component of $\mathbb{T}\backslash A_g$, this shows that $C(g)$ is a finite union of disjoint open intervals. Induction on the length of g shows that the closure of these intervals is disjoint too (nonperiodicity of α is needed here).

By keeping track of when a cylinder contains α, one sees that boundary chords of cylinders are always leaves in the minimal equivalence (see Proposition 2.2).

Proposition 2.11. *Suppose $g \in \{L, R\}^N$ is a finite word of length N. Then the boundary chords of $C(g)$ are all of the form $\widetilde{g|_t}\{*_1, *_2\}$ where $0 \leq t \leq N-1$. Moreover, $\widetilde{g|_t}\{*_1, *_2\}$ is a boundary chord iff $\sigma^{t+1}g$ is an initial subword of $I^\alpha(\alpha)$.*

Proof. We proceed by induction on $N = |g|$. For $N = 1$, the result is clear because $\{*_1, *_2\}$ is the boundary leaf of $L\mathbb{T}$ and $R\mathbb{T}$. Now suppose $sg \in \{L, R\}^{N+1}$ where $s \in \{L, R\}$ and $g \in \{L, R\}^N$. By the induction hypothesis, the boundary leaves of $C(g)$ are precisely the leaves of the form $\widetilde{g|_t}\{*_1, *_2\}$ where $1 \leq t \leq N$ is an integer such that $\sigma^{t+1}g$ is an initial subword of $I^\alpha(\alpha)$.

The images of the boundary leaves of $C(g)$ under $\widetilde{s}$ are always boundary leaves of $\widetilde{s}(C(g)) = C(sg)$. By the induction hypothesis, all leaves arising in this way are of the form $\widetilde{s} \circ \widetilde{g|_t}\{*_1, *_2\}$ where $1 \leq t \leq N$ is an integer such that $\sigma^{t+1}g = I^\alpha(\alpha)|_{N-t-1}$. Note that $\widetilde{s} \circ \widetilde{g|_t} = \widetilde{(sg)|_{t+1}}$, and $\sigma^{(t+1)+1}sg$ is an initial subword of $I^\alpha(\alpha)$.

If $\alpha \notin C(g)$ then these are the only boundary leaves of $C(sg)$. On the other hand if $\alpha \in C(g)$ then $\{*_1, *_2\}$ is a new boundary leaf of $C(sg) = \widetilde{s}(C(g))$. This new boundary leaf is equal to $\sigma^0\{*_1, *_2\}$, and indeed

$\sigma^{0+1}(sg) = g$ is an initial subword of $I^\alpha(\alpha)$ because $\alpha \in C(g)$ means that $I^\alpha(\alpha)|_n = g$.

This completes the induction. $\square$

As promised, combining the previous two propositions shows that every equivalence class in $\approx_\alpha$ be approximated by boundary leaves. In fact, in Lemma 4.2, we will need to construct several *distinct* approximations to a certain chord, so it is important that the approximations can be chosen to be strict approximations.

Proposition 2.12. *Let $g \in \{L, R\}^\infty$ be an infinite word. Then $C(g) := \bigcap_n \overline{C(g|_n)} = \{x_1, \ldots, x_m\}$ is finite. Assume the $x_1, \ldots, x_m$ are arranged in a counterclockwise order. Then for each i, and each $\epsilon > 0$ there is an integer n such that $\widetilde{g|_n}\{*_1, *_2\}$ is ϵ-close to $\{x_i, x_{i+1}\}$ in the Hausdorff sense. The approximations may be taken to be strict in the sense that $\widetilde{g|_n}*_1$ and $\widetilde{g|_n}*_2\}$ are not equal to any of the x_i.*

Proof. By Proposition 2.10, all the points of $\bigcap_n \overline{C(g|_n)}$ belong to the same $\approx_\alpha$ class, so by Theorem 2.3, $\bigcap_n \overline{C(g|_n)}$ is finite, and we can write $\bigcap_n \overline{C(g|_n)} = \{x_1, \ldots, x_m\}$.

Let $U = \cup_i (x_i - \epsilon, x_i + \epsilon)$ be the union of ϵ-balls around each point in $C(g)$. Assume ϵ is small enough that these balls are disjoint. By Lemmas 2.9 and 2.8, we have for sufficiently large M that $U \supset \overline{B_M(x)} \supset \overline{C(g|_M)} \supset \{x_1, \ldots, x_m\}$.

Fix M sufficiently large as above, and let (a, b) be a component of $\mathbb{T} \backslash \{x_1, \ldots, x_m\}$. Let z be a point of $\mathbb{T} \backslash U$ inside (a, b), and let I be the component of $\mathbb{T} \backslash \overline{C(g|_M)}$ containing z. By construction, the endpoints of I are within distance ϵ of x_i and x_{i+1}, and by Proposition 2.11, the endpoints of I are of the form $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ where u is an initial subword of $g|_M$.

Now we need to show that $\widetilde{u}*_1$ and $\widetilde{u}*_2$ are not equal to any of the x_i (at least if ϵ is sufficiently small. There are two cases to consider, depending on whether x_i and x_{i+1} are both in $\{*_1, *_2\}$ or not.

If either x_i or x_{i+1} are not in $\{*_1, *_2\}$, then wlog $x_i \notin \{*_1, *_2\}$. By flatness, the approximating leaf $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ does not intersect the leaf $\{x_i, x_{i+1}\}$ unless $\widetilde{u}*_1 \sim_\alpha x_i$. But this cannot occur because applying $h^{|u|+1}$ to both sides yields $[\alpha] = h^{|u|}[\alpha]$, so since $[\alpha]$ is nonperiodic we must have $|u| = 0$. But this is a contradiction because $\{\widetilde{u}*_1, \widetilde{u}*_2\} = \{*_1, *_2\}$ lies a bounded distance away from $\{x_i, x_{i+1}\}$ (since we assumed that $x_i \notin \{*_1, *_2\}$.) Therefore $\{\widetilde{u}*_1, \widetilde{u}*_2\}$ does not intersect $\{x_i, x_{i+1}\}$.

Now we consider the case where x_i and x_{i+1} are both in $\{*_1, *_2\}$. This implies that $|[*_1]| = 2$ and it implies that $g = LI^\alpha(\alpha)$ or $g = RI^\alpha(\alpha)$. Assume first that $g = LI^\alpha(\alpha)$, the other case is similar.

We will go through the construction again. As above, fix M large enough that $\overline{C(g|_M)}$ is contained in an ϵ-neighborhood of $\{*_1, *_2\}$. Assume that ϵ is small enough that this neighbourhood does not contain α.

Let I be the component of $\mathbb{T} \backslash \overline{C(g|_M)}$ that contains α. Then I is contained inside $L = (*_1, *_2)$.

Consider the boundary chord l joining the endpoints of I, which by Proposition 2.11 is of the form $\widetilde{u}\{*_1, *_2\}$ where u is an initial subword of $g|_M$. Note that I is not equal to L because $C(g|_M) \cap L = C(g|_M)$ has positive total length, so $\widetilde{u}$ is not the identity and u is not the empty word. In particular, the boundary chord $l = \widetilde{u}\{*_1, *_2\}$ is a strict approximation (close, but not equal) of $\{*_1, *_2\}$. $\square$

3 Combinatorial Semihyperbolicity and the good gluing condition

Our definition of combinatorial semihyperbolicity is clearly inspired by the characterization 1.3 and the characterization of the topology of $\mathbb{T}/\approx_\alpha$ (Proposition 2.6). See also the proof of Theorem 1.4.

Definition 1. *We say that $\alpha \in \mathbb{T}$ is* combinatorially semihyperbolic *if there exists $M_\alpha > 0$ integer such that for $t \geq 1$, $I^\alpha(h^t\alpha)|_{M_\alpha} \neq I^\alpha(\alpha)|_{M_\alpha}$.*

In the rest of this subsection we explain the definition of the quasisymmetric gluing condition in the hypothesis of Theorem 1.1. See also [LR].

First we recall the quasisymmetry condition in the classical conformal welding setting. If $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, we say that we say that f is *K-quasisymmetric* [Hei01, Chapter 10] if $\operatorname{diam} f(I) \leq K \operatorname{diam} f(J)$ whenever I and J are adjacent subintervals of $\mathbb{R}$ of the same length. The same definition holds for homeomorphisms $f : \partial\mathbb{D} \to \partial\mathbb{D}$ and also homoeomorphisms $f : I_1 \to I_2$ where I_1, I_2 are intervals. All linear maps are quasisymmetric with $K = 1$, and in the rest of paper we will actually only encounter linear maps.

A closed set $A \subset \mathbb{T}$ is *uniformly perfect* [Pom79] if there exists $c > 1$ such that no annulus $B(x, cr)\backslash B(x, r)$ seperates elements of A. An important example is the middle thirds Cantor set (embedded into $\mathbb{T}$ in the obvious way). In our construction, all our uniformly perfect sets will in fact be generalized Cantor sets.

Given a pair of open intervals (I^-, I^+) in $\mathbb{T}$, and points $x_1, x_2 \in \mathbb{T}$, we say that the pair (I^-, I^+) is *adjacent* to $\{x_1, x_2\}$ if I^- has one endpoint equal to x_1 and I^+ has one endpoint has one endpoint equal to x_2, or I^+ has one endpoint equal to x_1 and I^- has one endpoint equal to x_2.

Given m pairs of open intervals (I_j^-, I_j^+) for $j = 1, \ldots, m$, we say that they form a *cyclic chain* (of *degree* m) if:

1. The $2m$ intervals are mutually disjoint.
2. There exists m points $x_1, \ldots, x_m$ in $\mathbb{T}$ such that each component of $\mathbb{T}\backslash\{x_1, \ldots, x_m\}$ contains a single pair (I_j^-, I_j^+).
3. If U is a component of $\mathbb{T}\backslash\{x_1, \ldots, x_m\}$, and (I_j^-, I_j^+) is the pair that U contains, then (I_j^-, I_j^+) is adjacent to the two endpoints of U.

We say that a cyclic chain *contains* the point $x \in \mathbb{T}$ if one of the x_i are equal to x.

Definition 2 (Good Scales)**.** *Let $\sim$ be a lamination and let I^- and I^+ be open subintervals of $\mathbb{T}$. Suppose $\beta, \eta \in [0, 1]$ and $c \in (1, \infty)$.*

We say that I^- is (c, β, η)-glued to I^+ if there exists uniformly c-perfect subsets $A^- \subset I^-$ and $A^+ \subset I^+$ such that

- $\operatorname{diam}(A^\pm) \geq \beta \operatorname{diam}(I^\pm)$.
- *There exists an η-quasisymmetric bijection $\varphi : A^- \to A^+$ such that for $x \in A^-$ we have $x \sim \varphi(x)$.*

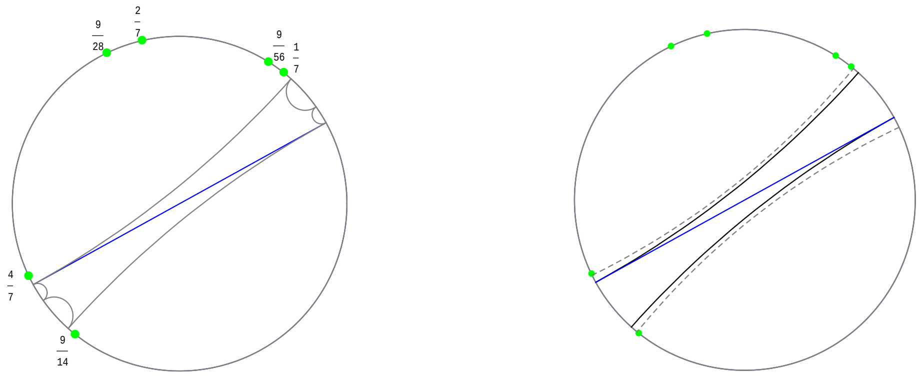
In this situation we say that the gluing is quasisymmetrically supported *on A^- and A^+.*

Now let $N \geq 1$ be integer. For $x \in \mathbb{T}$, a real number $C \geq 1$, and integer $m \geq 1$, we say that the condition $\mathrm{gs}(\sim; x, N, m; C, c, \beta, \eta)$ *holds, or that x is (C, c, β, η)-glued at scale N with degree m, if: there exists a cyclic chain (I_j^-, I_j^+) of degree m containing x such that for each $j = 1, \ldots, m$, we have*

- $C^{-1}2^{-N} \leq \operatorname{diam}(I_j^\pm) \leq C2^{-N}$.
- *I_j^- is (c, β, η)-glued to I_j^+.*

In our construction, it will be more convenient to construct *almost cyclic chains* instead of cyclic chains. Given m pairs of open intervals (I_j^-, I_j^+) for $j = 1, \ldots, m$, we say that they form a ϵ-almost cyclic chain (of *degree* m) if:

1. The $2m$ intervals are mutually disjoint.
2. There exists m closed intervals $\mathbf{x}_1, \ldots, \mathbf{x}_m$ such that the length of each interval is bounded by ϵ, and each component of $\mathbb{T}\backslash(\mathbf{x}_1 \cup \cdots \mathbf{x}_m)$ contains a single pair (I_j^-, I_j^+).
3. If U is a component of $\mathbb{T}\backslash(\mathbf{x}_1 \cup \cdots \mathbf{x}_m)$, and (I_j^-, I_j^+) is the pair contained in U, then (I_j^-, I_j^+) is adjacent to the two endpoints of U.



(a) The postcritical set P_α is in green. The diameter $D = \{*_1, *_2\}$ is in blue. The hyperbolic geodesics denote leaves in $\sim_\alpha$. Here $I' = [9/14, 1/7]$ and $I'' = [4/7, 9/14]$. Notice that $[*_1] \subset I' \cup I''$.

(b) Periodic leaves with endpoints in I' and I'' are constructed (Lemma 4.2).

Figure 3: The steps involved in the proof of Theorem 1.2, in the case $\alpha = 9/56$. Continued in Figure 4.

We say that an almost cyclic chain contains the point $x \in \mathbb{T}$ if one of the $\mathbf{x}_i$ contains x.

If $C \geq 1$ and $\mathcal{C}$ is a $C2^{-N}$-almost cyclic chain that contains x, where each interval in $\mathcal{C}$ has length between $C^{-1}2^{-N}$ and $C2^{-N}$, then we can expand each interval in $\mathcal{C}$ to get a cyclic chain $\mathcal{C}'$ of the same degree that still contains x. We can expand the intervals in such a way that the intervals of $\mathcal{C}'$ still have length at most $2C2^{-N}$. Therefore if the pairs of $\mathcal{C}$ are glued then the the pairs in $\mathcal{C}'$ are still glued with some of the constants differing by a factor of 2.

Let us define the gs$'$ condition to be exactly the same as the gs condition except in Definition 2 we replace 'cyclic chain' by '$C2^{-N}$-almost cyclic chain'. The above discussion shows that $\mathrm{gs}'(\sim; x, N; C, m, c, \beta, \eta) \implies \mathrm{gs}(\sim; x, N; 2C, m, c, \beta/2, \eta)$.

4 Gluing at Every Scale

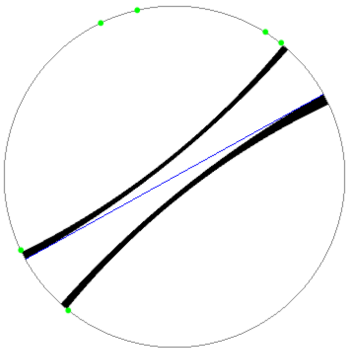
In this section we prove Theorem 1.2, that is for α combinatorially semihyperbolic, we show that the equivalence relation $\sim_\alpha$ has the desired gluing at every scale.

The construction is sketched in Figure 3 and 4. The idea is to first construct a gluing between intervals around $*_1$ and $*_2$ at scale $N = 1$, this is Proposition 4.3. To get the cantor set A for the gluing between $*_1$ and $*_2$, we will use periodic leaves near $*_1$ and $*_2$ to generate an iterated function system. The existence of such periodic leaves is shown in Lemma 4.2. See Figure 3a. After this gluing at the large scale is constructed, we use backwards iteration to get the cantor set around any point at any scale.

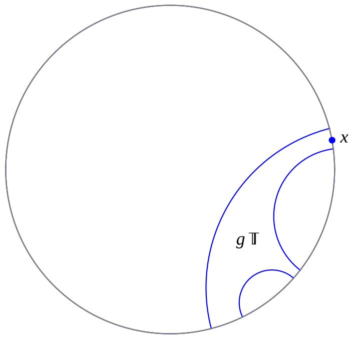
The construction will rely on the following fact that the class of the critical point $[*_1]$ is contained in the union of exactly two components I' and I'' of $\mathbb{T}\backslash\overline{P_\alpha}$, where we recall that $P_\alpha = \{h^t\alpha : t \geq 0\}$. In particular this means that I' and I'' are connected in $\mathbb{T}\backslash A_g$ for all $g \in \{L, R\}^*$. It follows that all compositions $\widetilde{g}$ of $\widetilde{L}$s and $\widetilde{R}$s are well defined on I' and I'', and hence the images $\widetilde{g}(I')$ and $\widetilde{g}(I'')$ are connected intervals. See Figure 3a. This guarantees that if we can construct the good gluing across the neighbourhoods I',I'', then we can pull them back to different scales and locations via the inverse branches of h^n.

Proposition 4.1. *Let I' and I'' be the components of $\mathbb{T}' := \mathbb{T}\backslash\overline{P_\alpha}$ containing $*_1$ and $*_2$ respectively. Then I' and I'' are distinct, and $[*_1] \subset I' \cup I''$.*

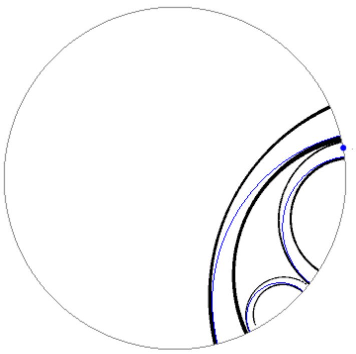
Note that the combinatorial semihyperbolicity assumption on α, together with Lemma 2.5, implies that $*_1, *_2 \notin \overline{P_\alpha}$. To see this, suppose for contradiction that $h^{t_n}(\alpha) \to *_i$ for some sequence $t_n \to \infty$. From



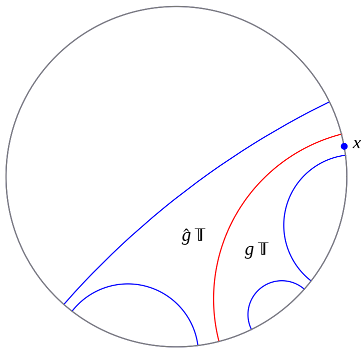
(a) We use the periodic leaves near to the diameter to construct an IFS, resulting in a gluing between the intervals I_1 and I_4 and between I_2 and I_3. (Proposition 4.3).



(b) We construct the gluing at $x \in \mathbb{T}$ and scale $N \geq 0$ by pulling back the gluing from the previous step. The cylinder of depth N containing x is bounded by leaves of the form $\widetilde{g|_{t_j}}D$ for some integers $t_j \leq N$, where $g = I^\alpha(x)$...



(c) ...so, as long as t_j is not too small, $\widetilde{g|_{t_j}}I_1$ and $\widetilde{g|_{t_j}}I_4$ (and $\widetilde{g|_{t_j}}I_2$ and $\widetilde{g|_{t_j}}I_3$) are glued at scale N. Note that even though there are gluings inside the cylinder, we do not use them in this construction.



(d) Sometimes one of the boundary leaves (the red one) is of the form $g^{N-t}D$ where t is too large, so that pulling back the gluing from I_1 to I_4 and I_2 and I_3 results in a gluing at too large a scale. In this case we use the leaves of the gap $\hat{g}\mathbb{T}$ instead of the red leaf.

Figure 4: The steps involved in the proof of Theorem 1.2, in the case $\alpha = 9/56$. Continued from Figure 3.

Lemma 2.5 this implies for sufficiently large n that $\sigma^{t_n} I^\alpha(\alpha) = I^\alpha(h^{t_n}\alpha) \overset{M_\alpha+1}{\asymp} I^\alpha(*_i) = \star I^\alpha(\alpha)$. Therefore $\sigma^{t_n+1} I^\alpha(\alpha) \overset{M_\alpha}{\asymp} I^\alpha(\alpha)$ and this contradicts combinatorial semihyperbolicity.

Therefore the components I' and I'' do exist.

Proof. Since the kneading sequence $I^\alpha(\alpha)$ is not periodic, it is not $LLL\ldots$. Therefore there exists a postcritical point $h^t\alpha \in R$. Then α and $h^t\alpha$ are in different components of $\mathbb{T}\backslash\{*_1, *_2\}$, which means $*_1$ and $*_2$ are in different components of $\mathbb{T}\backslash\{\alpha, h^t\alpha\}$, which means $*_1$ and $*_2$ are in different components of $\mathbb{T}\backslash P_\alpha$. Thus I' and I'' are distinct.

Now we turn to the second statement, which is equivalent to saying that all the postcritical points P_α lie in two components of $\mathbb{T}\backslash[*_1]$.

The idea is that if a postcritical point $h^t *_1$ lies in a component of $\mathbb{T}\backslash[*_1]$, then by flatness the whole class $h^t[*_1]$ of that point must lie in that same component. But since h is expanding on $\mathbb{T}$, any component of $\mathbb{T}\backslash[*_1]$ that contains an iterate $h^t[*_1]$ of the critical class must necessarily be 'large'. The desired result follows if we can show that only two components are 'large'.

More precisely, we will now show that if J is a component of $\mathbb{T}\backslash[*_1]$ and J contains a postcritical point $h^t\alpha$, $t \geq 1$, then the length of J satisfies $|J| > 1/4$.

For $x, y \in \mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ let $|x - y| \in [0, 1/2]$ be the distance between x and y on $\mathbb{T}$, or in other words the (normalized) length of the shortest arc joining x to y. Then the distance between any pair of points $x, y \in \mathbb{T}$ changes under the action of h according to the tent mapping:

$$|hx - hy| = \begin{cases} 2|x-y| & \text{if } |x-y| \leq 1/4 \\ 1 - 2|x-y| & \text{if } |x-y| > 1/4. \end{cases} \tag{7}$$

Let $*'$ and $*''$ be adjacent points of $[*_1]$, bounding some component J of $\mathbb{T}\backslash[*_1]$. Let $d = |\, *' - *''\,| = |J|$, and suppose $d \leq 1/4$. Then we have $|h *' - h *''\,| = 2d$. By (7), iterating h on the leaf $\{*', *''\}$ will yield longer and longer leaves until the length is greater than $1/4$. (The length of a leaf is defined to be the distance between its endpoints). After that point the length of the leaf may shrink.

However, the longest leaf in the lamination has length at most $1/2 - d$. This is because by flatness any leaf in the lamination must have both endpoints in the same component of $\mathbb{T}\backslash[*_1]$. The points $*' \approx_\alpha *'' \approx_\alpha *' + 1/2 \approx_\alpha *'' + 1/2$ are all in $[*_1]$ (the easiest way to see this is by using the definition of $\approx_\alpha$ and considering itineraries), and the largest component of $\mathbb{T}\backslash\{*', *'', *' + 1/2, *'' + 1/2\}$ has length $1/2 - d$, so the largest component of $\mathbb{T}\backslash[*_1]$ has length at most $1/2 - d$.

So from (7), the iterates $h^t\{*', *''\}$ of the leaf never get shorter than $2d = 2|J|$. In particular, for $t \geq 1$, $h^t *'$ and $h^t *''$ can never both be in J. By flatness of the lamination, neither of them can ever be in J, otherwise we would have $h^t *' \sim *'$. Thus, by flatness again, if $|J| \leq 1/4$, $h^t[*'] = h^t[*_1]$ is never contained in J, as desired.

Now let us use this fact to derive the desired result. Let J be the largest component of $\mathbb{T}\backslash[*_1]$. Then $|J| > 1/4$, otherwise there would be no postcritical points. Since $[*_1] = [*_1]_{\approx_\alpha}$ is invariant under $x \mapsto x + 1/2$ (again, by considering $\approx_\alpha$), the interval $J + 1/2$ is a component of $\mathbb{T}\backslash[*_1]$ and also has length equal to $|J| > 1/4$. This shows that components of length greater than $1/4$ occur in pairs. Since $(|J| + |J + 1/2|) + (1/4 + 1/4) > 1$, there can only be one pair of components of $\mathbb{T}\backslash[*_1]$ with length greater $1/4$, namely J and $J + 1/2$. Therefore the postcritical points all lie in $J \cup (J + 1/2)$, and so we are done. $\square$

A *periodic leaf* is a leaf such that both endpoints have periodic itineraries. We now show that we can find a pair of periodic leaves spanning the intervals I' and I''. These leaves will be used to generate an iterated function system, giving many leaves between I' and I''.

Lemma 4.2 (Existence of periodic leaves across the circle)**.** *There exists infinitely many distinct periodic leaves l' with both endpoints in L with one endpoint in I' and the other endpoint in I''. The same statement holds for R in place of L.*

Proof. We will only construct such leaves in L, the argument for getting leaves in R is exactly the same. First we prove that we can find a leaf l in L with one endpoint in I' and the other endpoint in I''. This leaf will be of the form $\widetilde{u}\{*_1, *_2\}$ for some $u \in \{L, R\}^*$. If u is sufficiently contracting then this tells us that $\widetilde{u}^{\circ 2}$ maps I' into I' and I'' into I''. We then use the contraction principle to find fixed points a', a'' of $\widetilde{u}^{\circ 2}$ in I' and I'' respectively. By definition, a' and a'' will be periodic with periodic itinerary equal to u^∞. In particular $a' \approx_\alpha a''$. Now we provide the details.

1. Let $g = I^\alpha(*_1) = \star I^\alpha(\alpha)$. By Proposition 2.10, we have
$$[*_1] = \bigcap_n \overline{C(g^L|_n)} \cup \bigcap_n \overline{C(g^R|_n)} = \bigcap_n \overline{C(LI^\alpha(\alpha))} \cup \bigcap_n \overline{C(RI^\alpha(\alpha))}.$$
Since, by definition, $C(L\cdots)$ and $C(R\cdots)$ are contained in L and R respectively, we conclude that $[*_1] \cap L \subset \bigcap_n \overline{C(g^L|_n)}$. Taking the closure of both sides yields $[*_1] \cap \overline{L} \subset \bigcap_n \overline{C(g^L|_n)}$. But the reverse inclusion holds too, so we get
$$[*_1] \cap \overline{L} = \bigcap_n \overline{C(g^L|_n)}. \tag{8}$$
Let $[*_1] \cap \overline{L} = \{y_1, \ldots, y_m\}$ where the y_i are assumed to be indexed in counterclockwise order with $y_1 = *_1$ and $y_m = *_2$. Let i be the maximal index such that $y_i \in I'$. Then $y_{i+1} \in I''$ by Proposition 4.1.

   By Theorem 2.12 and (8), the leaf $\{y_i, y_{i+1}\}$ is the limit of leaves of the form $\{\widetilde{g_n}*_1, \widetilde{g_n}*_2\}$ for a sequence of finite words $g_n \in \{L, R\}^*$, all these words are initial subwords of the word $g^L = LI^\alpha(\alpha)$, and $\limsup_n |g_n| = \infty$ because the approximations are strict. For simplicity we will pass to a subsequence for which $|g_n| \to \infty$.

2. Now for all n, $\widetilde{g_n}$ is a well defined contraction on the open sets I' and I''. For sufficiently large n, we have that $\widetilde{g_n}\{*_1, *_2\} \in I' \cup I''$. Therefore for sufficiently large n, we have that $\widetilde{g_n}$ maps $*_1$ into I' and $*_2$ into I'', or $*_1$ into I'' and $*_2$ into I'.

   Since $\lim_n |g_n| = \infty$, we have $\lim_n 2^{-|g_n|} = \lim_n |g_n'| = 0$, so for sufficiently large n we have that $\widetilde{g_n}$ maps I' into I' and I'' into I'', or I' into I'' and I'' into I'.

   For these n, we have that $\widetilde{g_n}^{\circ 2} = \widetilde{g_n} \circ \widetilde{g_n}$ maps I' into I' and I'' into I''.

   Also, $\widetilde{g_n}^2$ has constant derivative $2^{-2|g_n|} < 1$, so by the contraction principle $\widetilde{g_n}^{\circ 2}|_{I'}$ and $\widetilde{g_n}^{\circ 2}|_{I''}$ have fixed points a', a'' in I', I'' respectively. Using (2) we see that
$$I^\alpha(a') = I^\alpha(\widetilde{g_n}^{\circ 2} a') = g_n g_n I^\alpha(a'),$$
and hence a' has periodic itinerary $I^\alpha(a') = I^\alpha(a'') = g_n^\infty \in \{L, R\}^\infty$. The same argument shows that $I^\alpha(a'') = g_n^\infty$ too. Thus $\{a', a''\}$ is a periodic leaf with one endpoint in I' and the other endpoint in I'', and $\{a', a''\} \subset L$.

Step 2) shows that every sufficiently large n yields a periodic leaf with endpoints in I' and I''.

For our construction below we will need two different periodic leaves, to form an iterated function system. We will now show that we can choose $n' \neq n$ such that the applying the construction in Step 2) to $g_{n'}$ and g_n yields different periodic leaves. We do this by choosing n' and n so that $g_n^\infty \neq g_{n'}^\infty$. Then the periodic leaves that result from applying step 3) to g_n and $g_{n'}$ will be different because they will have different itineraries and hence will not intersect.

Suppose first that $g := LI^\alpha(\alpha)$ is eventually periodic (recall that g cannot be actually periodic), which means there exists some preperiod $t \geq 1$ such that $I^\alpha(h^t\alpha)$ is periodic of some period $K \geq 1$. Choose M

large enough that M is greater than the preperiod t of $I^\alpha(\alpha)$, and large enough that the first M letters of $LI^\alpha(\alpha)$ are enough to 'certify' that $g = LI^\alpha(\alpha)$ is not periodic of period K. That is, choose $M \geq t$ large enough that

$$LI^\alpha(\alpha)|_M \text{ is not of the form } w^\infty|_M \text{ for any } w \in \{L, R, \star\}^K.$$

Let n be large enough that $|g_n| \geq M$, and let n' be large enough that $|g_{n'}| - |g_n| \geq M$. Write $g'_n = g_n u$ where $|u| = |g_{n'}| - |g_n| \geq M$ (recall that g_n and $g_{n'}$ are initial subwords of $g = LI^\alpha(\alpha)$). Suppose for contradiction that $g_n^\infty = g_{n'}^\infty$, then applying the shift $\sigma^{|g_n|}$ to both sides yields $g_n^\infty = u g_{n'}^\infty$.

Since g_n is an initial subword of g of length at least M, and u has length at least M, the equality $g_n^\infty = u g_{n'}^\infty$ implies $g|_M = u|_M$. However, u is an initial subword of something that is periodic of period K (namely, $\sigma^{|g_n|} g$), so this contradicts the definition of M. Therefore $g_n^\infty \neq g_{n'}^\infty$.

Now we consider the case where $g = LI^\alpha(\alpha)$ is not eventually periodic. Fix n, and choose M large enough that the first M letters of $\sigma^{|g_n|} g$ are enough to 'certify that $\sigma^{|g_n|} g$ is not periodic of period $|g_n|$. That is, choose M large enough that

$$(\sigma^{|g_n|} g)|_M \text{ is not of the form } w^\infty|_M \text{ for any } w \in \{L, R, \star\}^{|g_n|}\}.$$

Then choose n' large enough that $|g_{n'}| - |g_n| \geq M$. Write $g'_n = g_n u$ where $|u| = |g_{n'}| - |g_n| \geq M$ (recall that g_n and $g_{n'}$ are initial subwords of $g = LI^\alpha(\alpha)$). Suppose for contradiction that $g_n^\infty = g_{n'}^\infty$, then applying the shift $\sigma^{|g_n|}$ to both sides yields $g_n^\infty = u g_{n'}^\infty$.

Restricting to the first M letters yields the contradiction $g_n^\infty|_M = u|_M = (\sigma^{|g_n|} g)|_M$, where the last equality holds because u is an initial subword of $\sigma^{|g_n|} g$. Therefore $g_n^\infty \neq g_{n'}^\infty$. $\square$

The points $*_1$ and $*_2$ cut I' and I'' respectively into two open subintervals each, giving a total of four intervals $J_1^-, J_2^-, J_2^+, J_1^+$ (assumed to be in counterclockwise order). Here we choose the indexing on the $\{J_j\}$ such that $*_1$ is the counterclockwise endpoint J_1^- and the clockwise endpoint of J_2^-, and $*_2$ is the counterclockwise endpoint of J_2^+ and the clockwise endpoint of J_1^+. Thus $J_1^- \cup J_2^- = I'$ and $J_1^+ \cup J_2^+ = I''$, and $J_1^- \cup J_1^+ \subset R$ while $J_2^- \cup J_2^+ \subset L$.

Let l be a periodic leaf in $\sim_\alpha$ of period p, where the period is defined as the smallest integer p such that h^p fixes both endpoints of l. Observe that the iterates of each endpoint never lie in $[*_1]$, as this would contradict nonperiodicity of $[*_1]$. In other words neither itinerary is precritical, therefore by (3) , the itinerary of both points are equal, and we will use $I^\alpha(l)$ to denote this common itinerary.

The common itinerary of both points is periodic of period p, so we can write $I^\alpha(l) = w^\infty$ where $w = I^\alpha(l)|_p$. Observe that $\widetilde{w}$ is a contraction that fixes the endpoints of l, indeed $\widetilde{w}$ is just the inverse branch of h^p that fixes the endpoints of l. If we let $w = I^\alpha(l)|_{2p}$ then $\widetilde{w}$ is orientation preserving and still fixes the endpoints of l.

The contractions arising from the periodic leaves we constructed in Lemma 4.2 generate an iterated function system, giving us a Cantor set on which a gluing between I' and I'' is supported.

Proposition 4.3 (Existence of Cantor set around main leaf)**.** *J_1^- is glued to J_1^+ and J_2^- is glued to J_2^+. In other words, $*_1$ is $(C, 2, c, \beta, \eta)$-glued at scale 0, for some constants C, c, β, η.*

Proof. First we show that J_1^- is glued to J_1^+. From Lemma 4.2 we get a pair of distinct periodic leaves l, l' each with one endpoint in J_1^- and the other endpoint in J_1^+. Let a^-, a^+ be the endpoints of l in J_1^- and J_1^+ respectively, and let a'^-, a'^+ be the endpoints of l' in J_1^- and J_1^+ respectively. Let H_1^- be the closed subinterval of J_1^- with endpoints $\{a^-, a'^-\}$ and H_1^+ be the closed subinterval of J_1^+ with endpoints $\{a'^+, a^+\}$. Let $\varphi : H_1^- \to H_1^+$ be the linear map that takes a^- to a^+ and a'^- to a'^+.

As in the above discussion, let $w = I^\alpha(l)|_{2p}$ and $w' = I^\alpha(l')|_{2p'}$ where p and p' are the periods of l and l' respectively. Recall that $\widetilde{w}$ fixes a^- and a^+ while $\widetilde{w'}$ fixes a'^- and a'^+, and $\widetilde{w}$ and $\widetilde{w'}$ are orientation preserving.

Consider the restrictions $\widetilde{w}|_{H_1^-}$ and $\widetilde{w}|_{H_1^+}$ of $\widetilde{w}$ to H_1^- and H_1^+ respectively. They are conjugate:

$$\widetilde{w}|_{H_1^+} \circ \varphi = \varphi \circ \widetilde{w}|_{H_1^-}. \tag{9}$$

To see this, note that both sides of the equation are linear mappings with the same derivative, and they are equal when evaluated at a^-, and they are both of the same orientation class.

Similarly, $\widetilde{w'}|_{H_1^+} \circ \varphi = \varphi \circ \widetilde{w'}|_{H_1^-}$ holds for the linear maps induced by the other periodic leaf, l'.

Consider the linear iterated function systems generated by the contractions $\widetilde{w}$ and $\widetilde{w'}$ on each of the intervals H_1^- and H_1^+. Let A_1^- and A_1^+ be the respective limit sets. That is, A_1^- is the closure of the orbit of the two endpoints $\partial H_1^- = \{a^-, a'^-\}$ of H_1^- under arbitrary finite compositions of $\widetilde{w}|_{H_1^-}$ and $\widetilde{w'}|_{H_1^-}$, and similarly for A_1^+. Since $\widetilde{w}, \widetilde{w'}$ are linear contractions on an interval, the limit set is a Cantor set and it is an easy exercise to show that the limit sets are uniformly perfect.

We claim that $x \sim \varphi(x)$ for all $x \in A_1^-$. This is true by definition when $x = a^-$ or $x = a'^-$. On the other hand, invariance of $\sim$ and the conjugacy (9) yields, for $x \in H_1^-$:

$$x \sim \varphi(x) \implies \{ \quad \widetilde{w}x \sim \widetilde{w}\varphi(x) = \varphi(\widetilde{w}x) \quad \text{and} \quad \widetilde{w'}x \sim \widetilde{w'}\varphi(x) = \varphi(\widetilde{w'}x). \quad \}$$

By induction, we get $x \sim \varphi(x)$ for all images of a^- and a^+ under arbitrary finite compositions of $\widetilde{w}$ and $\widetilde{w'}$. By topological closedness of $\sim$ we get $x \sim \varphi(x)$ for all $x \in A_1^-$.

Thus we have constructed a gluing between J_1^- and J_1^+ where φ is not only quasisymmetric but linear, and A_1^-, A_1^+ are not only uniformly perfect, but are linear Cantor sets. A similar argument shows that J_2^- and J_2^+ are glued together on some cantor sets A_2^- and A_2^+. Thus $*_1$ is $(C, 2, c, \beta, 1)$-glued at scale 0, where the constants will depend on the relative sizes of $I_{1,2}^{\pm}$ and $J_{1,2}^{\pm}$ and also on the derivatives of $\tilde{w}$ and $\tilde{w'}$. □

To get the gluing at scale $r = 2^{-N}$ and $x \in \mathbb{T}$, we will pull back the gluing we just constructed. For this purpose it is useful to introduce the notion of a *circular chain.*

Definition 3 (Circular chain)**.** *For $m \geq 2$, if $l_1, \dots, l_m$ are mutually non-intersecting chords in the lamination $\sim$, let* Gap(l) *be the component of $\overline{\mathbb{D}} \setminus \cup_i l_i$ that contains the convex hull of the $2m$ endpoints of the l_i. We say that $l_1, \dots, l_m$ form an ϵ*-circular chain *around $x \in \mathbb{T}$ if*

- *All the chords $l_1, \dots, l_m$ lie on the boundary of* Conv(l)*.*
- *All the components of* Conv$(l) \cap \mathbb{T}$ *have length bounded above by ϵ.*
- Gap$(l) \cap \mathbb{T}$ *contains x.*

For $m = 1$, we say that l_1 forms a ϵ-circular chain around x if both endpoints of l_1 are within ϵ of x, and if x is contained in the smaller of the two components of $\overline{\mathbb{D}} \setminus \{l_1\}$.

Recall that M_α is the quantitative parameter in the definition of combinatorial semihyperbolicity.

For each $x \in \mathbb{T}$ and each scale $r = 2^{-N}$, we will construct a $\asymp 2^{-N}$ circular chain around x by using boundary leaves of cylinder sets. By Proposition 2.11, these boundary leaves are all pullbacks of the main leaf $(*_1, *_2)$ under the doubling map h.

Thus for each leaf l_i of the circular chain, we can pull back the gluing around the main leaf (Proposition 4.3) to a gluing around l_i.

To make this work we need to ensure that we can construct the desired circular chain with a bounded number of boundary leaves.

Lemma 4.4. *Suppose $g \in \{L, R\}^N$ is a finite word of length N. There can be at most one integer $t < N - M_\alpha$ such that $\widetilde{g|_t}\{*_1, *_2\}$ is a boundary leaf. In particular the number of boundary leaves of $C(g)$ is bounded by $M_\alpha + 1$.*

Proof. Suppose for contradiction $t, t' \leq N - M_\alpha - 1$ are distinct integers such that $\widetilde{g|_t}\{*_1, *_2\}$ and $\widetilde{g|_{t'}}\{*_1, *_2\}$ are boundary leaves, and assume without loss of generality that $t < t'$. By Proposition 2.11, we have $g = g|_t sv = g|_{t'} s'v'$ where $s, s' \in \{L, R, \star\}$ and v, v' are initial subwords of $k_\alpha(\alpha)$. Applying the shift σ^{t+1} yields $v = (\sigma^{t+1} g|_{t'}) s'v'$. This shows that $\sigma^T v = v'$ where $T > 0$. Since $t' \leq N - M_\alpha - 1$, we have that $|v'| \geq M_\alpha$, so the last equality contradicts combinatorial semihyperbolicity. ☐

With this lemma, we can now prove the existence of circular chains around x.

Lemma 4.5 (Existence of chains around x)**.** *There exists $C > 0$ such that the following holds. For each $x \in \mathbb{T}$ and each $N \geq 0$, there exists m finite words $u(1), \ldots, u(m) \in \{L, R\}^*$ such that*

- $m \leq 2M_\alpha$
- *The lengths of the words satisfy $N - M_\alpha \leq |u(i)| \leq N$*
- *The leaves $l_i := \widetilde{u(i)}\{*_1, *_2\}$ form a $C2^{-N}$-circular chain around x in the sense of Definition 3.*

Proof. There are several cases to consider. Suppose $x \in \mathbb{T}$ and $N \geq 0$. Let $g = I^\alpha(x)$ be the itinerary of x. There are four cases to consider.

1. $g = u \star I^\alpha(\alpha)$ for some finite word $u \in \{L, R\}^*$, and $N - M_\alpha \leq |u| \leq N - 1$. The first part of this condition is equivalent to saying that $h^t x \in \{*_1, *_2\}$ for some t, the second part says that $N - M_\alpha \leq t \leq N - 1$.
2. The symbol $\star$ does not occur in $g|_N$, and if $0 \leq t \leq N - 1$ is an integer such that $\sigma^{t+1}(g|_N)$ is an initial subword of $I^\alpha(\alpha)$, then $t \geq N - M_\alpha$.
3. The symbol $\star$ does not occur in $g|_N$ and there exists an integer $0 \leq t < N - M_\alpha$ such that $\sigma^{t+1}(g|_N)$ is an initial subword of $I^\alpha(\alpha)$.
4. $g = u \star I^\alpha(\alpha)$ for some finite word $u \in \{L, R\}^*$, and $|u| < N - M_\alpha$.

In case 1), we can simply take $m = 2$ and $u(1) = u(2) = u$. Now we turn to the second case. The desired leaves will be the boundary leaves of a certain cylinder containing x. Let $g|_N \in \{L, R\}^\infty$ be the first N letters in the itinerary of x. The cylinder $C(g|_N)$ is a union of open intervals with disjoint closure, with total length 2^{-N}, that contains x. By Proposition 2.11, the boundary leaves of the cylinder are of the form $\widetilde{u}\{*_1, *_2\}$, and they clearly form a circular chain around x. It remains to verify that these boundary leaves satisfy the first two conclusions of Lemma. Let m be the number of boundary leaves of the cylinder $C(g|_N)$. By Lemma 4.4, $m \leq M_\alpha + 1$. By Proposition 2.11, the assumption of case 2) implies that all the boundary leaves have depth at least $N - M_\alpha$, so we are done.

The idea for case 3) is similar. We would like to use the boundary leaves of $C(g|_N)$, but the problem is that not all the boundary leaves are deep enough: there exists j' such that $t_{j'} < N - M_\alpha$. See Figure 4d. However, Lemma 4.4 guarantees that there is only one j' for which $\sigma^{t_1+1}(g|_N)$ is an initial subword of $I^\alpha(\alpha)$ and $t_{j'} < N - M_\alpha$. Wlog assume $j' = 1$. It suffices to find another cylinder $C(\hat{g}|_N)$ such that $C(\hat{g}|_N)$ and $C(g|_N)$ share the troublesome shallow boundary leaf $\widetilde{g|_{t_1}}\{*_1, *_2\}$. The boundary leaves of the closed union $\overline{C(g|_N) \cup C(\hat{g}|_N)}$ will all be deep since the shallow leaf is in the interior and is no longer on the boundary.

If $g|_N = g_1 g_2 \ldots g_N$, let $\hat{g}|_N = g_1 g_2 \ldots g_{t_1} \hat{g_{t_1+1}} \ldots g_N$ where $\hat{g_{t_1+1}} = L$ if $g_{t_1+1} = R$ and vice versa. Then $C(\hat{g}|_N)$ is a cylinder that also has $\widetilde{g|_{t_1}}\{*_1, *_2\}$ as a boundary leaf. This completes the proof for case 3).

For case 4), we consider the modified itinerary g^L which is equal to $g = I^\alpha(x)$ except that the symbol $\star$ is replaced by the symbol L. The modified itinerary $g^L = uLI^\alpha(\alpha)$ satisfies the hypotheses of case 3) with $t = |u|$, and $C(g^L|_N)$ contains x by Proposition 2.10, so we can apply the construction of case 3) to g^L to obtained the desired circular chain. ☐

Finally we show how to use the gluings from I_1^- to I_1^+ and I_2^- to I_2^+ to create gluings on all scales.

Proof of Theorem 1.2. Suppose $x \in \mathbb{T}$ and $N \geq 0$. Let $g = I^\alpha(x)$ be the itinerary of x. Let $u(1), \ldots, u(m)$ be the finite words provided by Lemma 4.5, so that the leaves $l_j := \widetilde{u(j)}\{*_1, *_2\}$ form a circular chain around x. Let $U = \mathrm{Gap}(l) \cap \partial\mathbb{D}$ be the *interior* of this chain.

For each $j = 1, \ldots, m$, the mapping $\widetilde{u(j)}$ maps the two intervals I', I'' (containing $*_1, *_2$) to a pair of neighborhoods $\widetilde{u(j)}I'$ and $\widetilde{u(j)}I''$. Here it is crucial that I' and I'' do not contain any postcritical points.

By backward invariance and linearity of the $\tilde{u}$, the pair of good gluings between the intervals I_j^- and I_j^+ constructed in Proposition 4.3 are mapped via $\widetilde{u(j)}$ to a pair of good gluings adjacent to $\widetilde{u(j)}*_1$ and $\widetilde{u(j)}*_2$. One of these gluings will lie in the *exterior* $\mathbb{T} \backslash U$ of the circular chain. The derivative of $\widetilde{u(j)}$ is between 2^{-N} and 2^{-N+M_α}, so the gluing is at scale N.

The collection of these gluings for $j = 1, \ldots, m$ provides the good gluing at scale N at x. $\square$

5 Combinatorial Semihyperbolicity and concrete semihyperbolicity

So far in this paper we have worked only with 'abstract' or 'combinatorial' laminations. In this section we relate our work to concrete Julia sets.

For $c \in \mathbb{C}$ let $p_c(z) = z^2 + c$ and let K_c be its filled Julia set. If K_c is connected there is a unique Riemann map $\varphi_c : \mathbb{D}^* \to \mathbb{C} \backslash K_c$ that fixes ∞ and $\varphi'(\infty) > 0$. If, in addition, K_c is locally connected, then φ_c extends continuously to the boundary $\partial\mathbb{D} \cong \mathbb{T}$. Let $\gamma : \mathbb{T} \to J_c$ be the restriction of this extension to the boundary, where $J_c = \partial K_c$ is the Julia set of p_c. We call γ the *Caratheodory loop*. The Caratheodory loop induces an equivalence relation $\sim_c$ on $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$ where points are identified if they have the same image under γ.

Any equivalence relation induced by a homeomorphism from the outside of the disk $\mathbb{D}^*$ to the complement of a compact set $K \subset \mathbb{C}$ is flat, ([Thu09, Proposition II.3.2]).

The key observation which relates to $\sim_c$ to $\approx_\alpha$ is that φ_c semiconjugates p_c to the map $z \mapsto z^2$ on $\mathbb{D}^*$, this is because $\varphi_c \circ p_c \circ \varphi_c^{-1} : \mathbb{D}^* \to \mathbb{D}^*$ is proper of degree 2 and fixes ∞, so $\varphi_c \circ p_c \circ \varphi_c^{-1}(z) = \lambda z^2$ for some $|\lambda| = 1$. But $\lambda = 1$, by considering what happens to large z. It follows that γ semiconjugates $p_c|_{J_c}$ to the doubling map $h : \mathbb{T} \to \mathbb{T}$:

$$p_c \circ \gamma = \gamma \circ h. \tag{10}$$

The semiconjugacy implies that $\sim_c$ is closed and invariant, and furthermore that if $\gamma(\alpha) = c$, then $*_1 = *_2$. We immediately get that $\sim_c$ contains the minimal α-equivalence defined in 2.1. It can also be shown that $\sim_c \subset \approx_\alpha$. When $I^\alpha(\alpha)$ is not perioidic, it can be shown that $\approx_\alpha$ is equal to the minimal α-equivalence, and it follows that $\sim_c = \approx_\alpha$ [BK92, Theorem 1].

Recall that $c \in \mathbb{C}$ is a *semihyperbolic* parameter if p_c has no parabolic periodic points and if c is not in the closure of its forward orbit [CJY94, Theorem 1.1]. This dynamical condition is equivalent to a condition on the geometry of the Fatou set: c is semihyperbolic if and only if the basin of attraction to ∞ is a John domain.

Our characterization of the topology of $\mathbb{T}/\approx_\alpha$ in terms of itineraries shows that our notion of combinatorial semihyperbolicity is equivalent to the notion of semihyperbolicity described above.

Proof of Theorem 1.4. For the first direction, we have from [BK92, Theorem 1] that $\mathbb{T}/\approx_\alpha$ is homeomorphic to J_c via the map $\gamma : \mathbb{T} \to J_c$. Since c is semihyperbolic we have $c \notin \overline{\bigcup_{t\geq 1} p_c^t(c)}$. Because $\gamma \circ h = p_c \circ \gamma$ on $\mathbb{T}$, we have that $[\alpha] \notin \overline{\bigcup_{t\geq 1} h^t([\alpha])}$. It follows from the characterization of the topology of $\mathbb{T}/\approx_\alpha$ in Proposition 2.6 that α is combinatorially semihyperbolic.

For the other direction, if α is combinatorially semihyperbolic, then our construction Theorem 1.2 together with [LR, Theorem 1] implies that there exists a conformally removable compact set $J \subset \hat{\mathbb{C}}$ such that conformal map $\varphi : \hat{\mathbb{C}} \backslash \mathbb{D} \to \hat{\mathbb{C}} \backslash J$ solves the welding problem $\approx_\alpha$, meaning $\varphi(e^{i\pi x}) = \varphi(e^{i\pi y}) \iff x \approx_\alpha y$. Suppose we chosen J so that φ satisfies the normalizations $\varphi(*_1) = 0$, $\varphi(\infty) = \infty$, and $\varphi(z) = z + o(1)$ as $z \to \infty$.

Now we show that J is the Julia set of $z \mapsto z^2 + c$.

Let $\tilde{h}(z) = z^2$ be the extension of the angle doubling map to the exterior of the unit disk, $\tilde{h} : \hat{\mathbb{C}} \backslash \mathbb{D} \to \hat{\mathbb{C}} \backslash \mathbb{D}$, and let $p(z) = \varphi \circ \tilde{h} \circ \varphi^{-1}$ be its conjugate on $\hat{\mathbb{C}} \backslash J_c$. Then p is holomorphic on $\hat{\mathbb{C}} \backslash J$, and the invariance properties of $\approx_\alpha$ imply that p extends continuously to $\hat{\mathbb{C}}$ to a topological degree two branched cover with two critical values, at ∞ and at $c := \varphi(\alpha)$.

We would like to conclude from removability of J that p is holomorphic on $\hat{\mathbb{C}}$ and is hence a polynomial. However, p is not a homemorphism so we cannot apply conformal removability directly. Instead, we will consider a lift of p.

The map $p_c : z \mapsto z^2 + c$ is a double sheeted cover of $\mathbb{C} \backslash \{c\}$ by $\mathbb{C} \backslash \{0\}$. On the other hand, $p : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{c\}$ is also a double sheeted cover. The covers are therefore equivalent, meaning that there is a homeomorphism $\pi : \mathbb{C} \backslash \{0\} \to \mathbb{C} \backslash \{0\}$ such that $p = p_c \circ \pi$. We have $\lim_{z \to \infty} p(z) = \infty = \lim_{z \to \infty} p_c(z)$, so $\lim_{z \to c} p_c(z) = 0$ and so π extends to a homeomorphism $\mathbb{C} \to \mathbb{C}$ mapping 0 to 0 and c to c.

On the other hand, the holomorphicity of p on $\mathbb{C} \backslash J$, and holomorphicity of p_c, implies that π is holomorphic on $\mathbb{C} \backslash J$. By removability of J we conclude that $\pi : \mathbb{C} \to \mathbb{C}$ is the identity.

We have thus shown that $p_c = p$ and it is clear that J is the Julia set of p. We have found a polynomial p_c for which $\mathbb{T} / \approx_\alpha \cong J_c$. It remains to show that c is semihyperbolic.

Since γ is a conjugacy between $(\mathbb{T} / \approx_\alpha)$ and (J_c, p_c), it suffices to show that $[\alpha] \notin \overline{\bigcup_{t \geq 1} h^t([\alpha])}$. By Proposition 2.6, we need to show that for all $t \geq 1$, $\sigma^t I^\alpha(\alpha)$ is not $2M_\alpha + 1$ close to $I^\alpha(\alpha)$. Suppose $I_\alpha(h^t \alpha)$ and $I_\alpha(\alpha)$ are $2M_\alpha + 1$ close for some t, then we can write $I_\alpha(\alpha)|_{2M_\alpha+1} = usI_\alpha(\alpha)|_{2M_\alpha+1}$ and $I_\alpha(h^t\alpha)|_{2M_\alpha+1} = us'I_\alpha(\alpha)|_{2M_\alpha+1}$ for some finite word $u \in \{L, R\}^*$ and some $s, s' \in \{L, R\}$. If $|u| > M_\alpha$ then this shows that $I_\alpha(\alpha)|_{M_\alpha} = u|_{M_\alpha} = k_\alpha(h^t\alpha)|_{M_\alpha}$, and this is a contradiction. On the other hand if $|u| \leq M_\alpha$, then by applying the shift $\sigma^{|u|+1}$ to the equality $I_\alpha(\alpha)|_{2M_\alpha+1} = usI_\alpha(\alpha)|_{2M_\alpha+1}$ shows that $k_\alpha(h^{|u|+1}\alpha)|_{2M_\alpha - |u|} = I_\alpha(\alpha)|_{2M_\alpha - |u|}$. Since $2M_\alpha - |u| \geq M_\alpha$ this again is a contradiction. $\square$

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