p-convexity, p-plurisubharmonicity and the Levi Problem

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ABSTRACT. Three results in p-convex geometry are established. First is the analogue of the Levi problem in several complex variables: namely, local p-convexity implies global p-convexity. The second asserts that the support of a minimal p-dimensional current is contained in the union of the p-hull of the boundary with the "core" of the space. Lastly, the extreme rays in the convex cone of p-positive matrices are characterized. This is a basic result with many applications.

CONTENTS

1. INTRODUCTION

On any Riemannian n-manifold there are intrinsic notions of p-plurisubharmonicity and p-convexity for integers p between 1 and n. They interpolate between convexity ($p = 1$) and subharmonicity ($p = n$), with $p = n - 1$ being an important case. They also arise naturally in many situations, and their study goes back to H. Wu [Wu]. The object of this paper is to prove three new results in p-geometry.

Indiana University Mathematics Journal ©, Vol. 62, No. 1 (2013)

The central algebraic concept is that of *p-positivity* for a quadratic form Q on a finite-dimensional inner product space V. By definition, Q is p-positive if the trace of its restriction to every p-dimensional subspace $W \subset V$ satisfies $\operatorname{tr}\{Q|_W\} \geq 0$. This is equivalent to the condition that $\lambda_1 + \cdots + \lambda_p \geq 0$ where $\lambda_1 \leq \cdots \leq \lambda_n$ are the ordered eigenvalues of Q. The set of such Q will be denoted $\mathcal{P}_p(V)$. On any Riemannian manifold X, a function $u \in C^2(X)$ is *p-plurisubharmonic* if its Riemannian Hessian is p-positive. An oriented hypersurface in X is *p-convex* if its second fundamental form is p-positive. The Riemann curvature R of X is *p-positive* if for each tangent vector v, the quadratic form $\langle R_{v,\cdot}\cdot, v\rangle$ is p-positive.

The smooth p-plurisubhamonic functions are "pluri"-subharmonic in the following sense.

Theorem 2.13. *A function $u \in C^2(X)$ is p-plurisubharmonic if and only if its restriction to every p-dimensional minimal submanifold is subharmonic in the induced metric.*

The notion of p-plurisubhamonicity can be generalized to arbitrary upper semicontinuous $[-\infty, \infty)$-valued functions using standard viscosity test functions (cf. [CIL], [C]). For $p = 1, n$, this recaptures the classical notions of general convex and subharmonic functions on a Riemannian manifold X. This family of upper semicontinuous p-plurisubharmonic functions, denoted $\mathrm{PSH}_p(X)$, has many of the useful properties of subharmonic functions (see Theorem 2.6 in [HL_5]). Moreover, the Restriction Theorem 2.12 has a nontrivial extension to general, upper semicontinuous p-plurisubharmonic functions (see [HL_6]).

The smooth p-plurisubharmonic functions can be used to introduce a notion of p-convexity as follows. Given a compact subset $K \subset X$, define the *p-convex hull* of K to be the set $\hat{K}$ of points $x \in X$ such that $u(x) \leq \sup_K u$ for all smooth p-plurisubharmonic functions u on X. Then X is said to be *p-convex* if

$$K \Subset X \Longrightarrow \hat{K} \Subset X.$$

The following result was proven in [HL_7]:

> *A Riemannian manifold X is p-convex if and only if X admits a smooth p-plurisubharmonic proper exhaustion function.*

A domain $\Omega \subset X$ is said to be *locally p-convex* if each point $x \in \partial\Omega$ has a neighborhood U such that $\Omega \cap U$ is p-convex. Note that p-convex domains are locally p-convex (see (3.1)). The following converse is an analogue of the Levi Problem in complex analysis, and is one of the three new results of this paper.

Theorem 3.7. *Let $\Omega \Subset \mathbb{R}^n$ be a domain with smooth boundary. If Ω is locally p-convex, then Ω is p-convex.*

There is also a notion of p-convexity for the boundary. Let II denote the second fundamental form of the boundary $\partial\Omega$ with respect to the interior normal. Then the boundary $\partial\Omega$ is *p-convex* if II_x is p-positive at each point $x \in \partial\Omega$.

Theorem 3.9. *Let $\Omega \Subset \mathbb{R}^n$ be a domain with smooth boundary. If Ω is locally p-convex, then $\partial\Omega$ is p-convex.*

From Theorem 3.10, one thus concludes that, for such domains Ω,

$$\Omega \text{ is } p\text{-convex} \iff \Omega \text{ is locally } p\text{-convex} \iff \partial\Omega \text{ is } p\text{-convex.}$$

A quadratic form A on an inner product space V is said to be *strictly p-positive* if $\operatorname{tr}\{A|_W\} > 0$ for all p-planes $W \subset V$. This gives notions of strict p-plurisubharmonicity, strict p-convexity, and so forth. In Section 4, a number of results concerning strictly p-convex domains and strictly p-convex boundaries are discussed. A key concept here is that of the *core* of X, a subset which governs the existence of strictly p-plurisubharmonic functions and proper exhaustions (see Remark 4.4).

The core contains all compact p-dimensional minimal submanifolds without boundary in X. This result is extended to include noncompact minimal submanifolds and currents. A p-dimensional rectifiable current $T \in \mathcal{R}_p(X)$ on X is called *minimal* if the first variation of the mass of T is zero with respect to deformations supported away from its boundary ∂T (see Definition 4.8).vspace2pt

Corollary 4.11. *Suppose $T \in \mathcal{R}_p(X)$ is a minimal current, and let u be any smooth p-plurisubharmonic function which vanishes on a neighborhood of* $\operatorname{supp}(\partial T)$. *Then*

$$\operatorname{tr}_{\vec{T}}(\operatorname{Hess} u) \equiv 0 \quad \text{on } \operatorname{supp}(T).$$

If $T = [M]$ is the current associated to a connected p-dimension minimal submanifold, and if the p-plurisubharmonic function u and its gradient both vanish at points of ∂M, then

$$u\,|_M \equiv 0 \quad \text{or, if } \partial M = \emptyset,\ u\,|_M \equiv \text{ constant.}$$

Our second result is the following.

Theorem 4.12. *Let $K \subset X$ be a compact subset and suppose $T \in \mathcal{R}_p(X)$ is a minimal current such that* $\operatorname{supp}(\partial T) \subset K$. *Then* $\operatorname{supp}(T) \subset \widehat{K} \cup \operatorname{Core}(X)$.

This leads to the notion of a *minimal surface hull* of a compact set $K \subset X$, namely, the union of the supports of all minimal currents $T \in \mathcal{R}_p(X)$ whose boundaries are supported in K. Theorem 4.12 says that this hull is contained in $\widehat{K} \cup \operatorname{Core}(X)$.

Much of this discussion carries over to minimal (not necessarily rectifiable) p-currents.

Our third new result (see Section 5) describes the extreme rays in the convex cone $\mathcal{P}_p(V)$, defined for each real number $1 \le p \le n$ by

$$(1.1) \quad \mathcal{P}_p(\mathbb{R}^n) \stackrel{\text{def}}{=} \{A \in \operatorname{Sym}^2(\mathbb{R}^n) : \lambda_1(A) + \dots \\ + \lambda_{[p]}(A) + (p - [p])\lambda_{[p]+1}(A) \ge 0\},$$

where $[p]$ denotes the greatest integer $\le p$ (cf. Remark 2.9). The endpoint cases can be excluded from the discussion since $\mathcal{P}_n(V)$ is a half-space (and hence has no extreme rays,) while it is well known that the extreme rays in $\mathcal{P}(V) = \mathcal{P}_1(V) = \{A \geq 0\}$ are generated by the orthogonal projections onto lines. These rays remain extreme in $\mathcal{P}_p(V)$ for $1 \leq p < n-1$. Theorem 5.1c states that, for $1 < p < n$, the only other extreme rays are generated by the elements of $\mathrm{Sym}^2(V)$ with one negative eigenvalue $-(p-1)$ and all other eigenvalues 1.

This technical result is more important than it may seem at first glance. These generators are exactly (up to a positive scale) the second derivatives of the *Riesz kernel* $K_p(X)$, which is defined by:

$$K_p(X) = \begin{cases} |x|^{2-p} & \text{if } 1 \leq p < 2 \\ \log|x| & \text{if } p = 2, \\ -\dfrac{1}{|x|^{p-2}} & \text{if } 2 < p \leq n. \end{cases} \tag{1.2}$$

Consequently, an equivalent formulation of Theorem 5.1c is the following.

Theorem 5.1a **($1 < p < n$).** *Suppose $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a convex cone subequation. The Riesz kernel K_p is F-subharmonic if and only if $\mathcal{P}_p(\mathbb{R}^n) \subset F$.*

This result has many applications, partly because it holds for all *real* numbers p between 1 and n. In addition, we note the following:

Many of the results from p-convex analysis hold for any real p, $1 \leq p \leq n$.

Specifically, since $\mathcal{P}_p(\mathbb{R}^n) \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a convex cone, all the results of [HL$_4$] apply.

Finally, we note that the basic notions of p-plurisubharmonicity and p-convexity also make sense with the Grassmann bundle G(p,TX) replaced by a closed subset $\mathbb{G} \subset G(p, TX)$. There are surprisingly many results which hold in the general context of a "$\mathbb{G}$-geometry". These are discussed in a companion paper [HL$_7$].

2. PLURISUBHARMONICITY

Euclidean space. Suppose V is an n-dimensional real inner product space, and fix an integer p, with $1 \leq p \leq n$. Let $\mathrm{Sym}^2(V)$ denote the space of symmetric endomorphisms of V. Using the inner product, this space is identified with the space of quadratic forms on V. The notion of p-plurisubharmonicity for a smooth function u on V is defined by requiring that its Hessian (i.e., second derivative $D^2_x u$) belong to a certain subset $\mathcal{P}_p(V) \subset \mathrm{Sym}^2(V)$. To better understand this subset, we offer several (equivalent) definitions.

Definition 2.1. Suppose $A \in \mathrm{Sym}^2(V)$. Then $A \in \mathcal{P}_p(V)$, or A is p-*positive*, if the following equivalent conditions hold:

(1) $\mathrm{tr}_W A \geq 0$ for all $W \in G(p, V)$,
(2) $\lambda_1(A) + \cdots + \lambda_p(A) \geq 0$,
(3) $D_A \geq 0$,

where

(1) $G(p, V)$ denotes the set of p-dimensional subspaces of V; and for $W \in G(p, V)$, the *W-trace* of A (denoted $\operatorname{tr}_W A$) is the trace of the restriction $A|_W$ of A to W;
(2) $\lambda_1(A) \le \cdots \le \lambda_n(A)$ are the ordered eigenvalues of A, so condition (2) says the sum of the p smallest eigenvalues is ≥ 0;
(3) $D_A : \Lambda^p V \to \Lambda^p V$ is the linear action of A as a derivation on the space $\Lambda^p V$ of p-vectors; that is, on simple p-vectors, one has

$$\begin{aligned} D_A(v_1 \wedge \cdots \wedge v_p) &= (Av_1) \wedge v_2 \wedge \cdots \wedge v_p \\ &\quad + v_1 \wedge (Av_2) \wedge \cdots \wedge v_p + v_1 \wedge v_2 \wedge \cdots \wedge (Av_p). \end{aligned}$$

The inner product on V induces an inner product on $\Lambda^p V$, and we have $D_A \in \operatorname{Sym}^2(\Lambda^p V)$; and so the notions of non-negativity, $D_A \ge 0$, and positive definiteness, $D_A > 0$, make sense for D_A.

The proof that conditions (1), (2), and (3) are equivalent will be given below.

Definition 2.2 (p-plurisubharmonicity). A smooth function u defined on an open subset $X \subset \mathbb{R}^n$ is said to be *p-plurisubharmonic* if $D_x^2 u \in \mathcal{P}_p(\mathbb{R}^n)$ for each point $x \in X$.

The next result justifies the terminology.

Proposition 2.3. *A function $u \in C^\infty(X)$ is p-plurisubharmonic if and only if the restriction $u|_{W\cap X}$ is subharmonic for all affine p-planes $W \subset \mathbb{R}^n$. (Here, "subharmonic" means that $\Delta_W(u|_{W\cap X}) \ge 0$ where Δ_W is the Euclidean Laplacian on the affine subspace W).*

Proof. This is obvious from condition (2), since with $v = u|_{W\cap X}$, we have $\operatorname{tr}_W D^2 u = \Delta_W v$ on $W \cap X$. ◻

Remark 2.4. The endpoint cases are classical.

i. ($p = 1$) *Convex Functions:* Note that $A \in \mathcal{P}_1 \iff \lambda_{\min}(A) \ge 0 \iff A \ge 0$, so that u is 1-plurisubharmonic if and only if u is convex.
ii. ($p = n$) *Classical Subharmonic Functions:* Note that $A \in \mathcal{P}_n \iff \operatorname{tr} A \ge 0$, so that u is n-plurisubharmonic if and only if $\Delta u \ge 0$, that is, u is classically subharmonic.

    Consequently, the simplest new case is when $p = 2$ in $\mathbb{R}^3$ where u is 2-plurisubharmonic if and only if the restriction of u to each affine plane in $\mathbb{R}^3$ is classically subharmonic. One generalization of this case has an interesting characterization.
iii. ($p = n - 1$): If $p = n - 1$, then $* : \Lambda^1 V \to \Lambda^{n-1} V$ is an isomorphism. This induces an isomorphism $\operatorname{Sym}^2(\Lambda^{n-1}V) \to \operatorname{Sym}^2(\Lambda^1 V)$ sending $D_A \mapsto (\operatorname{tr} A)I - A$. Therefore $u \in C^\infty(X)$ is $(n-1)$-plurisubharmonic if and only if

$$(\Delta u)I - \operatorname{Hess} u \ge 0.$$

Note.

(a) It is obvious from condition (2) that $\mathcal{P}_p(V) \subset \mathcal{P}_{p+1}(V)$; or, equivalently, if u is p-plurisubharmonic, then u is $(p+1)$-plurisubharmonic. In particular, each p-plurisubharmonic function is classically subharmonic, and every convex function is p-plurisubharmonic for all p.

(b) The set $\mathcal{P}_p(V)$ is a closed convex cone with vertex at the origin.

The proof of the equivalence of conditions (1), (2), and (3) in Definition 2.1 requires some elementary facts. Note that each p-plane $W \subset V$ determines a line $L(W) \subset \Lambda^p V$, namely the line through $v_1 \wedge \cdots \wedge v_p$ where $v_1, \dots, v_p$ is any basis for W. If $e_1, \dots, e_n$ is an orthonormal basis of V, we set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p}$ for $I = (i_1, \dots, i_p)$ with $i_1 < i_2 < \cdots < i_p$.

Lemma 2.5. *Given $A \in \mathrm{Sym}^2(V)$, consider $D_A \in \mathrm{Sym}^2(\Lambda^p V)$. Then we have*

(a) *For all $W \in G(p, V)$,*

$$\operatorname{tr}_W A = \operatorname{tr}_{L(W)} D_A. \tag{2.1}$$

(b) *If A has eigenvectors $e_1, \dots, e_n$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$, then D_A has eigenvectors e_I with corresponding eigenvalues*

$$\lambda_I = \lambda_{i_1} + \cdots + \lambda_{i_p}. \tag{2.2}$$

Proof. For (a), note that, if $e_1, \dots, e_p$ is an orthonormal basis of W, then

$$\begin{aligned}
\operatorname{tr}_{L(W)} D_A &= \langle D_A(e_1 \wedge \cdots \wedge e_p), e_1 \wedge \cdots \wedge e_p \rangle \\
&= \sum_{j=1}^{n} \langle e_1 \wedge \cdots \wedge Ae_j \wedge \cdots \wedge e_p, e_1 \wedge \cdots \wedge e_p \rangle \\
&= \sum_{j=1}^{n} \langle Ae_j, e_j \rangle = \operatorname{tr}_W A.
\end{aligned}$$

For (b), compute $D_A e_I = \lambda_I e_I$. □

Corollary 2.6. *Suppose $A \in \mathrm{Sym}^2(V)$ has ordered eigenvalues $\lambda_1(A) \le \cdots \le \lambda_n(A)$. Then*

$$\inf_{W \in G(p,W)} \operatorname{tr}_W A = \lambda_1(A) + \cdots + \lambda_p(A) = \lambda_{\min}(D_A), \tag{2.3}$$

the smallest eigenvalue of D_A.

Proof. Since D_A has eigenvalues λ_I by part (b), the smallest is $\lambda_1(A) + \cdots + \lambda_p(A) = \operatorname{tr}_{L(\overline{W})} D_A$, where $\overline{W} = \operatorname{span}\{e_1, \dots, e_p\}$. Now the smallest eigenvalue of D_A equals the infimum of $\operatorname{tr}_L D_A$ over all lines in $\Lambda_p V$, so in this case it is also the infimum over the restricted set of lines of the form $L(W)$ with $W \in G(p, V)$. By part (a) in Lemma 2.5, this proves (2.3). □

The equivalence of conditions (1), (2), and (3) in Definition 2.1 is immediate from Corollary 2.6.

Definition 2.7 (p-harmonicity). A smooth function u defined on an open subset $X \subset \mathbb{R}^n$ is *p-harmonic* if $D_x^2 u \in \partial \mathcal{P}_p$ for all $x \in X$, or equivalently, if $\lambda_{\min}(D_{D_x^2 u}) = \lambda_1(D_x^2 u) + \cdots + \lambda_p(D_x^2 u) = 0$ for all $x \in X$.

Example 2.8 (Radial harmonics).

i. ($p = 1$) The function $|x|$ is 1-harmonic on $\mathbb{R}^n \setminus \{0\}$.
ii. ($p = 2$) The function $\log|x|$ is 2-harmonic on $\mathbb{R}^n \setminus \{0\}$.
iii. ($3 \le p \le n$) The function $-1/|x|^{p-2}$ is p-harmonic on $\mathbb{R}^n \setminus \{0\}$.

Proof. Given a non-zero vector $x \in \mathbb{R}^n$, let $P_x(1/|x|^2)x \circ x$ denote orthogonal projection onto the line through x. One calculates that

$$D^2|x| = \frac{1}{|x|}(I - P_x), \tag{2.4}$$

$$D^2 \log|x| = \frac{1}{|x|^2}(I - 2P_x), \tag{2.5}$$

$$D^2\left(-\frac{1}{|x|^{p-2}}\right) = \frac{(p-2)}{|x|^p}(I - pP_x). \tag{2.6}$$

Note that in all cases the function $u(x)$ defined in Example 2.8 has second derivative $D^2 u$, which is a positive scalar multiple of $H \equiv I - pP_x$, and that H has one negative eigenvalue $-(p-1)$; the other eigenvalues are 1. By Lemma 2.5 (b), this implies that the eigenvalues of D_H are 0 and p, and in particular, $\lambda_{\min}(D_H) = 0$. ∎

Remark 2.9 (Non-integer p). The subset (subequation) $\mathcal{P}_p(V)$ can be defined for any real number p between 1 and n in such a way that many of the results in this paper continue to hold for noninteger values of p. Let $\bar{p} = [p]$ denote the greatest integer in p. Then we define $A \in \mathrm{Sym}^2(V)$ to be *p-positive*, or $A \in \mathcal{P}_p(V)$, if

$$\lambda_1(A) + \cdots + \lambda_{\bar{p}}(A) + (p - \bar{p})\lambda_{\bar{p}+1} \ge 0, \tag{2.7}$$

where, as before, $\lambda_1(A) \le \cdots \le \lambda_n(A)$ denote the ordered eigenvalues of A. To see that $\mathcal{P}_p(V)$ is a convex cone, one shows that it is the polar of the set of $P_{e_1} + \cdots + P_{e_{\bar{p}}} + (p - \bar{p})P_{e_{\bar{p}+1}}$, where $e_1, \ldots, e_n$ are orthonormal.

The motivation for this definition of $\mathcal{P}_p$ is provided by the next remark and Theorem 5.1. These are the only two other places in this paper where noninteger values of p are discussed. In the other places (such as Definition 3.1), the gaps are left to the reader.

Remark 2.10 (The Riesz kernel). The family of functions defined in Example 2.8 naturally extends by (1.2) to all real numbers p between 1 and n, and we have the following result.

Lemma 2.11. *For each real number p with $1 \le p \le n$,*

$$K_p(x) \text{ is } p\text{-harmonic on } \mathbb{R}^n \setminus \{0\} \text{ and } p\text{-plurisubharmonic on } \mathbb{R}^n.$$

Proof. Up to a positive scalar multiple, $D_x^2 K_p$ equals $H = I - pP_x$. As noted above, $D_H \ge 0$ and $\lambda_{\min}(D_H) = 0$. ◻

Riemannian manifolds. Suppose X is an n-dimensional Riemannian manifold. Then the Euclidean notions above carry over, with $V = T_xX$ and the ordinary Hessian of a smooth function replaced by the *Riemannian Hessian*. For $u \in C^2(X)$, this is a well-defined section of the bundle $\mathrm{Sym}^2(TX)$ given on tangent vector fields V, W by

$$(\mathrm{Hess}\, u)(V, W) = VWu - (\nabla_V W)u, \tag{2.8}$$

where ∇ denotes the Levi-Civita connection. Acting as a derivation, it determines a well-defined section $D_{\mathrm{Hess}\, u}$ of $\mathrm{Sym}^2(\Lambda^p TX)$ for each p, $1 \le p \le n$.

Definition 2.12 (p-plurisubharmonicity). A smooth function u on X is said to be *p-plurisubharmonic* if $\mathrm{Hess}_x\, u$ is p-positive at each point $x \in X$ (see Definition 2.1).

The appropriate geometric objects for restriction are the p-dimensional minimal (stationary) submanifolds of X. In the Euclidean case, this enlarges the family of affine p-planes used in Proposition 2.3 when $1 < p < n$.

Theorem 2.13. *A function $u \in C^2(X)$ is p-plurisubharmonic if and only if the restriction of u to every p-dimensional minimal submanifold is subharmonic.*

Proof. Suppose $M \subset X$ is any p-dimensional submanifold, and let H_M denote its mean curvature vector field. Then (see Proposition 2.10 in [HL$_2$])

$$\Delta_M(u|_M) = \mathrm{tr}_{TM}\, \mathrm{Hess}\, u - H_M u. \tag{2.9}$$

In particular, if M is minimal, then

$$\Delta_M(u|_M) = \mathrm{tr}_{TM}\, \mathrm{Hess}\, u. \tag{2.10}$$

It is an elementary fact (see Lemma 3.13) that for every point $x \in X$ and every p-plane $W \subset T_xX$, there exists a minimal submanifold M with $T_xM = W$. This is enough to conclude Theorem 2.13 from (2.10). ◻

3. Convexity, Boundary Convexity, and Local Convexity

Riemannian manifolds. Let $\mathrm{PSH}_p^\infty(X)$ denote the smooth p-plurisubharmonic functions on a Riemannian manifold X.

Definition 3.1. Given a compact subset $K \subset X$, the *p-convex hull of* K is the set

$$\widehat{K} \equiv \{x \in X : u(x) \le \sup_K u \quad \text{for all } u \in \mathrm{PSH}_p^\infty(X)\}$$

Proposition 3.2. *If $M \subset X$ is a compact connected p-dimensional minimal submanifold with boundary $\partial M \ne \emptyset$, then $M \subset \widehat{\partial M}$.*

Proof. Apply Theorem 2.13 and the maximum principle for subharmonic functions on M. ◻

Definition 3.3. We say that X is *p-convex* if for all compact sets $K \subset X$, the hull $\widehat{K}$ is also compact.

Theorem 3.4. *Suppose X is a Riemannian manifold. Then the following properties are equivalent:*

(1) *X is p-convex;*
(2) *X admits a smooth p-plurisubharmonic proper exhaustion function.*

Proof. See Theorem 4.4 in [HL_7] for the proof. It is exactly the same proof as the one given for Theorem 4.3 in [HL_2]. ◻

Condition (2) can be weakened to a local condition at ∞ in the one-point compactification $\bar{X} = X \cup \{\infty\}$. This follows from the next lemma.

Lemma 3.5. *Suppose $X - K$ admits a smooth p-plurisubharmonic function v with $\lim_{x\to\infty} v(x) = \infty$ where K is compact. Then X admits a smooth p-plurisubharmonic proper exhaustion function which agrees with v near ∞.*

Proof. This is a special case of Lemma 4.6 in [HL_7]. ◻

Euclidean space. We now show that the p-convexity of a compact domain with smooth boundary in Euclidean space is a local condition on the domain near the boundary. This result is to some degree analogous to the Levi Problem in complex analysis, and is one of the three new results of this paper.

Definition 3.6. A domain $\Omega \subset \mathbb{R}^n$ is *locally p-convex* if each point $x \in \partial\Omega$ has a neighborhood U in $\mathbb{R}^n$ such that $\Omega \cap U$ is p-convex.

Each ball in $\mathbb{R}^n$ is p-convex, and the intersection of two p-convex domains is again p-convex. Therefore

(3.1) If Ω is p-convex, then Ω is locally p-convex.

Our main result is the converse.

Theorem 3.7. *Suppose that Ω is a compact domain with smooth boundary. If Ω is locally p-convex, then Ω is p-convex.*

Intermediate between local and global convexity is the notion of boundary convexity. Suppose now that $\partial\Omega$ is smooth. Denote by $II = II_{\partial\Omega}$ the second fundamental form of the boundary with respect to the *inward pointing* normal n. This is a symmetric bilinear form on each tangent space $T_x\,\partial\Omega$ defined by

$$II_{\partial\Omega}(v,w) = -\langle \nabla_v n, w\rangle = \langle n, \nabla_v W\rangle,$$

where W is any vector field tangent to $\partial\Omega$ with $W_x = w$.

Definition 3.8. The boundary $\partial\Omega$ is p-convex at a point x if $\mathrm{tr}_W\{II_{\partial\Omega}\} \geq 0$ for all tangential p-planes $W \subset T_x(\partial\Omega)$ at x.

Theorem 3.7 is the compilation of the following two results.

Theorem 3.9. *If the domain Ω is locally p-convex, then its boundary $\partial\Omega$ is p-convex.*

Theorem 3.10. *If the boundary $\partial\Omega$ is p-convex, then the domain Ω is p-convex.*

Before proving these two theorems, we make some remarks on boundary convexity.

Remark 3.11 (Local defining functions). Suppose ρ is a smooth function on a neighborhood B of a point $x \in \partial\Omega$ with $\partial\Omega \cap B = \{\rho = 0\}$ and $\Omega \cap B = \{\rho < 0\}$. If $\mathrm{d}\rho$ is nonzero on $\partial\Omega \cap B$, then ρ is called a *local defining function for* $\partial\Omega$. It has the property that

$$D_x^2\rho = |\nabla\rho(x)| II_x \tag{3.2}$$

on $\partial\Omega \cap B$. To see this, suppose that e is a vector field tangent to $\partial\Omega$ along $\partial\Omega$, and note that $II(e,e) = \langle n, \nabla_e e\rangle = -(1/|\nabla\rho|)\langle\nabla\rho, \nabla_e e\rangle$ and $-\langle\nabla\rho, \nabla_e e\rangle = -(\nabla_e e)(\rho) = e(e\rho) - (\nabla_e e)(\rho) = (D^2\rho)(e,e)$. As a consequence, we have that $\partial\Omega$ is p-convex at a point x if and only if

$$\mathrm{tr}_W\, D_x^2\rho \geq 0 \quad \text{for all } p\text{-planes } W \text{ tangent to } \partial\Omega \text{ at } x \tag{3.3}$$

where ρ is a local defining function for $\partial\Omega$. Moreover, (3.3) is independent of the choice of the local defining function.

Remark 3.12 (Principal curvatures). Let $\kappa_1 \leq \cdots \leq \kappa_{n-1}$ denote the ordered eigenvalues of II_x. Then we have that

$$\partial\Omega \text{ is } p\text{-convex at } x \iff \kappa_1 + \cdots + \kappa_p \geq 0. \tag{3.4}$$

Proof. Apply Corollary 2.6 to $A \equiv II$ with $V \equiv T_x\,\partial\Omega$. ☐

We now give the proof of Theorem 3.9, that local p-convexity implies boundary p-convexity.

Lemma 3.13. *If $\partial\Omega$ is not p-convex at a point $x \in \partial\Omega$, then there exists an embedded minimal p-dimensional submanifold M through the point x with*

$$M \setminus \{x\} \subset \Omega \quad \text{in a neighborhood of } x. \tag{3.5}$$

Proof of Theorem 3.9. Assume that $\partial\Omega$ is not p-convex at a point $x \in \partial\Omega$. Let B denote the ε-ball about x. It suffices to show that $\Omega \cap B$ is not p-convex. This is done by constructing a "tin can" inside B using Lemma 3.13. We can assume that M is a compact manifold with boundary and $M \subset B$.

Let $M_t \equiv M + t\nu$ denote the translate of M by $t\nu$ where ν is the outward-pointing unit normal to $\partial\Omega$ at x. Choose $r > 0$ sufficiently small that each $M_t \subset \Omega$ for $-r \le t < 0$. Let K denote the "empty tin can" consisting of the "bottom" M_{-r} and the "label" $\bigcup_{-r\le t\le 0} \partial M_t$. Then K is a compact subset of $\Omega \cap B$. Let $\widehat{K}$ be its p-convex hull in $\Omega \cap B$.

Since $\partial M_t \subset K$, Proposition 3.2 implies that each $M_t \subset \widehat{K}$ for $-r \le t < 0$. Since $\widehat{K}$ is closed in $\Omega \cap B$, this proves that x must be in the $\mathbb{R}^n$-closure of $\widehat{K}$; that is, $\widehat{K}$ is not compact. Hence, $\Omega \cap B$ is not p-convex. ◻

Proof of Lemma 3.13. Suppose $\partial\Omega$ is not p-convex at x. Then there is a tangent p-plane W to $\partial\Omega$ at x with

$$\mathrm{tr}_W\{II_{\partial\Omega}\} < 0. \tag{3.6}$$

We may assume that W is the plane spanned by eigenvectors of II with the smallest eigenvalues. We can then choose Euclidean coordinates $(t_1, \dots, t_n)$ with respect to an orthonormal basis $e_1, \dots, e_n$ so that:

(i) x corresponds to the origin 0,
(ii) $n = e_n$ is the outward pointing normal to Ω at x,
(iii) $e_1, \dots, e_{n-1}$ are the eigenvectors of II at x with eigenvalues $\kappa_1 \le \kappa_2 \le \cdots \le \kappa_{n-1}$,
(iv) $W = \mathrm{span}\{e_1, \dots, e_p\}$.

In a neighborhood of 0, our domain can be written as

$$\Omega = \{t_n < f(t_1, \dots, t_{n-1})\}.$$

In particular, $\rho(t) \equiv t_n - f(t_1, \dots, t_{n-1})$ is a local defining function for $\partial\Omega$ near $0 \in \partial\Omega$. By Remark 3.11, since $(\nabla\rho)(0) = e_n$ is a unit vector,

$$D_0^2\rho = -D_0^2 f = II_0. \tag{3.7}$$

Hence f has Taylor expansion

$$f(t) = -\frac{1}{2}(\kappa_1 t_1^2 + \cdots + \kappa_{n-1} t_{n-1}^2) + O(|t|^3). \tag{3.8}$$

By setting $c \equiv -(1/p)(\kappa_1 + \cdots + \kappa_p)$, we obtain a diagonal matrix with trace zero, that is, $\mathrm{diag}(\kappa_1 + c, \dots, \kappa_p + c)$. The hypothesis (3.6) is equivalent to $c > 0$.

We now restrict attention to the linear subspace $P \equiv \mathrm{span}\{e_1, \dots, e_p, e_n\} = W \oplus \mathbb{R}e_n$, and consider graphs $\{t_n = g(t_1, \dots, t_p)\}$ which are minimal hypersurfaces in P (and therefore in $\mathbb{R}^n$). We apply the following basic lemma, whose proof is left as an exercise.

Lemma 3.14. *Given* $A \in \mathrm{Sym}^2(\mathbb{R}^p)$ *with* $\mathrm{tr}\, A = 0$*, there exists a real analytic function* g *defined near the origin with* $g(0) = 0$*,* $(\nabla g)(0) = 0$*, and* $D_0^2 g = A$ *such that* g *satisfies the minimal surface equation.*

We can apply this lemma with $A = -\mathrm{diag}(\kappa_1 + c, \dots, \kappa_p + c)$ obtaining a minimal surface $M = \{(t, g(t)) \in P = \mathbb{R}^{p+1} : |t| < \eta\} \subset \mathbb{R}^n$. The hypothesis $c > 0$ implies that $g(t) < f(t_1, \dots, t_p, 0, \dots, 0)$ if $0 < |t| < \eta$ small. This implies that $M \setminus \{0\} \subset \Omega$, completing the proof of Lemma 3.13 and Theorem 3.9 as well. □

We now commence with the proof of Theorem 3.10. Let $\delta(x)$ denote the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. By the *ε-collar of* $\partial\Omega$ we indicate the set $\{x \in \Omega : 0 < \delta(x) < \varepsilon\}$. Theorem 3.10 is immediate from the next result.

Proposition 3.15.

(1) *If* $\partial\Omega$ *is* p*-convex on a neighborhood of* $x_0 \in \partial\Omega$*, then* $-\log\delta(x)$ *is* p*-plurisubharmonic on the intersection of a neighborhood of* x_0 *in* $\mathbb{R}^n$ *with an* ε*-collar of* $\partial\Omega$*.*

(2) *If* $-\log\delta(x)$ *is* p*-plurisubharmonic on an* ε*-collar of* $\partial\Omega$*, then* Ω *is* p*-convex.*

Summary 3.16. From this proposition and Theorems 3.9 and 3.10, we conclude that

$$
\begin{aligned}
(3.9) \qquad \Omega \text{ is locally } p\text{-convex} &\iff \partial\Omega \text{ is } p\text{-convex} \\
&\iff -\log\delta(x) \text{ is } p\text{-plurisubharmonic} \\
&\iff \Omega \text{ is } p\text{-convex.}
\end{aligned}
$$

Proof of (1). Let II denote the second fundamental form of the hypersurfaces $\{\delta = \varepsilon\}$ for $\varepsilon \geq 0$, and let $n = \nabla\delta$ denote the inward-pointing normal. An arbitrary p-plane V at a point can be put in a canonical form with basis

$$(\cos\theta)n + (\sin\theta)e_1, e_2, \dots, e_p,$$

where $n, e_1, \dots, e_p$ are orthonormal. Set $W \equiv \mathrm{span}\{e_1, \dots, e_p\}$, the *tangential part* of V.

Lemma 3.17.

$$\mathrm{tr}_V\, \mathrm{Hess}(-\log\delta) = \frac{1}{\delta}\sin^2\theta\, \mathrm{tr}_W(II) + \frac{1}{\delta^2}\cos^2\theta.$$

Proof. See Remark after Proposition 5.13 in [HL_2]. □

Note. This formula holds on any Riemannian manifold.

If II has eigenvalues $\kappa_1, \ldots, \kappa_{n-1}$ at a point $x \in \partial\Omega$, then let $\kappa_1(\delta), \ldots, \kappa_{n-1}(\delta)$ denote the eigenvalues of II at the point a distance δ from x along the normal line. A proof of the following lemma can be found in [GT, Section 14.6].

Lemma 3.18. *For small* $\delta \geq 0$, *one has*

$$\kappa_j(\delta) = \frac{\kappa_j}{1 - \delta\kappa_j}, \quad j = 1, \ldots, n-1.$$

Corollary 3.19. *Each* $\lambda_j(\delta)$ *is strictly increasing if* $\kappa_j \neq 0$ *and* $\equiv 0$ *if* $\kappa_j = 0$.

We now combine Lemma 3.17 with Corollary 3.19 to conclude that $-\log\delta$ is p-plurisubharmonic. □

Remark 3.20. Note that each $\partial\Omega_\varepsilon$, where $\Omega_\varepsilon \equiv \{\delta > \varepsilon\}$, is strictly p-convex, and that $-\log\delta$ is strictly p-plurisubharmonic if and only if $\partial\Omega$ has no p-flat points, that is, points where the nullity of $II_{\partial\Omega}$ is $\geq p$.

Proof of (2). By Theorem 3.4, it suffices to prove the existence of a continuous exhaustion function $u : \Omega \to \mathbb{R}^+$ which is smooth and p-plurisubharmonic outside a compact set in Ω. Such a function is given by setting $u(x) = \max\{-\log\delta(x), -\log(\varepsilon/2)\}$. □

Remark 3.21. It would be interesting to determine if Theorem 3.9 remains true for all real numbers p between 1 and n. Most of the other results of this section do extend to all such p by [HL_4].

4. Minimal Varieties and Hulls

There are several notions of the *p-convex hull* of a set, all of which are intimately related to minimal currents. We begin by recalling the following.

Strict convexity. Let X be a Riemannian manifold which is connected and noncompact.

Definition 4.1. We say that a function $u \in \mathrm{PSH}_p^\infty(X)$ is *strictly p-plurisubharmonic at a point* $x \in X$ if $\mathrm{Hess}_x\, u \in \mathrm{Int}\,\mathcal{P}_p(T_xX)$, that is, if one of the following equivalent conditions holds:

(1) $\mathrm{tr}_W \mathrm{Hess}_x\, u > 0$ for all $W \in G(p, T_xX)$,
(2) $\lambda_1(\mathrm{Hess}_x\, u) + \cdots + \lambda_p(\mathrm{Hess}_x\, u) > 0$,
(3) $D_{\mathrm{Hess}_x u} > 0$,

where $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ denote the ordered eigenvalues of A.

Definition 4.2. The manifold X is called *strictly p-convex* if it admits a proper exhaustion function $u : X \to \mathbb{R}$ which is strictly p-plurisubharmonic at every point, and is called *strictly p-convex at infinity* if it admits a proper exhaustion function $u : X \to \mathbb{R}$ which is strictly p-plurisubharmonic outside a compact subset.

Definition 4.3. The *p-core of* X is defined to be the subset

$$\mathrm{Core}_p(X) \equiv \{x \in X : u \text{ is not strict at } x \text{ for all } u \in \mathrm{PSH}_p^\infty(X)\}.$$

Remark 4.4. This concept is useful in conjunction with Definition 4.2.

(1) X admits a smooth strictly p-plurisubharmonic function if and only if

$$\mathrm{Core}(X) = \emptyset.$$

(2) X is strictly p-convex; that is, X admits a smooth strictly p-plurisubharmonic proper exhaustion function if and only if

$$\mathrm{Core}(X) = \emptyset \text{ and } X \text{ is } p\text{-convex.}$$

(3) X is strictly p-convex at infinity if and only if

$$\mathrm{Core}(X) \text{ is compact and } X \text{ is } p\text{-convex.}$$

Part (1) is a special case of Theorem 4.2 in [HL_7]; Part (2) is a special case of 4.8 in [HL_7]; and Part (3) is a special case of Theorem 4.11 in [HL_7].

We note that when X admits a strictly p-plurisubharmonic proper exhaustion function, standard Morse Theory implies that X has the homotopy-type of a complex of dimension $\leq p-1$ (cf. [S], [Wu]).

Proposition 4.5. *Every compact p-dimensional minimal submanifold M without boundary in X is contained in* $\mathrm{Core}_p(X)$. *If instead the boundary $\partial M \neq \emptyset$, and if M is connected, then $M \subset \widehat{\partial M}$.*

Proof. For the first assertion, apply Theorem 2.13 and the maximum principle to conclude the restriction of any smooth p-plurisubharmonic function to M is constant. The second assertion is Proposition 3.2. ◻

This provides an analogue of the support Lemma 3.2 in [HL_3].

Corollary 4.6. *Suppose $M \subset X$ is a compact p-dimensional minimal submanifold with possible boundary. Then $M \subset \widehat{\partial M} \cup \mathrm{Core}(X)$.*

Minimal varieties and their associated hulls. Now we introduce the *minimal current hull* of a compact set K in a Riemannian manifold X, and relate it to the p-convex hull $\widehat{K}$. This second hull will be defined using the group $\mathcal{R}_p(X)$ of p-dimensional rectifiable currents with compact support in X (cf. [F], [Si], [M], etc.). These creatures enjoy many nice properties. They can be usefully considered as compact oriented p-dimensional manifolds with singularities and integer multiplicities, and readers unfamiliar with the general theory can think of them simply as submanifolds.

Of importance here is the following general structure theorem. Associated to each $T \in \mathcal{R}_p(X)$ is a Radon measure $\|T\|$ on X and a $\|T\|$-measurable field of unit p-vectors $\vec{T}$ such that for any smooth p-form ω on X,

$$T(\omega) = \int_X \omega(\vec{T})\, \mathrm{d}\|T\|. \tag{4.1}$$

(Recall [deR] that the p-currents are the topological dual space to the space of smooth p-forms.) In particular, every $T \in \mathcal{R}_p(X)$ has a finite mass

$$\mathbf{M}(T) = \int_X \mathrm{d}\|T\|.$$

Example 4.7. When T corresponds to integration over a compact oriented submanifold with boundary, of finite volume $M \subset X$, one has $\|T\| = \mathcal{H}_p|_M$ ($\mathcal{H}_p$ is the Hausdorff measure). Here, $\vec{T}_x$ corresponds to the oriented tangent plane T_xM, and $\mathbf{M}(T) = \mathcal{H}_p(M)$ is the Riemannian volume of M.

Definition 4.8. A current $T \in \mathcal{R}_p(X)$ is called *minimal* or *stationary* if, for all smooth vector fields V on X which vanish on a neighborhood of the support of ∂T, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}((\varphi_t)_*T))\big|_{t=0} = 0, \tag{4.2}$$

where φ_t denotes the flow generated by V on a neighborhood of the support of T.

Each smooth vector field on X defines a smooth bundle map $\mathcal{A}^V : TX \to TX$ given on a tangent vector W by

$$\mathcal{A}^V(W) \stackrel{\text{def}}{=} \nabla_W V. \tag{4.3}$$

This determines the derivation $D_{\mathcal{A}^V} : \Lambda^p TX \to \Lambda^p TX$ as in Section 2. Proof of the following can be found in [LS] or [L].

Theorem 4.9 (The first variational formula). *Fix $T \in \mathcal{R}_p(X)$, and let V, φ_t be as above. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}((\varphi_t)_*T))\big|_{t=0} = \int_X \langle D_{\mathcal{A}^V}\vec{T}, \vec{T}\rangle\, \mathrm{d}\|T\| = \int_X \mathrm{tr}_{\vec{T}}(\mathcal{A}^V)\, \mathrm{d}\|T\|. \tag{4.4}$$

Suppose now that $V = \nabla u$ for a smooth function u on X. Then

$$\mathcal{A}^{\nabla u} = \mathrm{Hess}\, u, \tag{4.5}$$

considered as an endomorphism of TX. To see this, note that

$$\begin{aligned} \langle \mathcal{A}^{\nabla u}(W), U\rangle &= \langle \nabla_W(\nabla u), U\rangle = W\langle \nabla u, U\rangle - \langle \nabla u, \nabla_W U\rangle \\ &= (WU - \nabla_W U)u = (\mathrm{Hess}\, u)(W, U). \end{aligned}$$

Hence, we have the following result.

Theorem 4.10. *If* $V = \nabla u$, *then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}((\varphi_t)_* T))\,|_{t=0} = \int_X \mathrm{tr}_{\vec{T}}(\mathrm{Hess}\, u)\,\mathrm{d}\|T\|. \tag{4.6}$$

Corollary 4.11. *Suppose* $T \in \mathcal{R}_p(X)$ *is a minimal current, and let* u *be any smooth* p*-plurisubharmonic function that vanishes on a neighborhood of* $\mathrm{supp}(\partial T)$. *Then*

$$\mathrm{tr}_{\vec{T}}(\mathrm{Hess}\, u) \equiv 0 \quad \text{on } \mathrm{supp}(T).$$

If $T = [M]$, *where* M *is a compact connected minimal submanifold of dimension* p, *and if* u *is a smooth* p*-plurisubharmonic function such that* $\nabla u|_{\partial M} = 0$, *then*

$$u\,|_M = \text{constant}.$$

Proof. The first statement follows directly from (4.2), (4.6), and the fact that $\mathrm{tr}_W\,\mathrm{Hess}\, u \geq 0$ on all tangent p planes W.

If $T = [M]$ for a minimal submanifold M, then $\mathrm{tr}_{T_xM}(\mathrm{Hess}_x\, u) = \Delta_M(u|_M)$, where Δ_M is the Laplace-Beltrami operator of M in the induced metric (see Proposition 2.10 in [HL$_2$]). By the first variational formula in the smooth case (e.g., Theorem 1.1 in [L]), we conclude that $u|_M$ is harmonic on M with constant boundary values (when $\partial M \neq \emptyset$), and the conclusion follows from the maximum principle. □

Theorem 4.12. *Let* $K \subset X$ *be a compact subset, and suppose* $T \in \mathcal{R}_p(X)$ *is a minimal current such that* $\mathrm{supp}(\partial T) \subset K$. *Then*

$$\mathrm{supp}(T) \subset \widehat{K} \cup \mathrm{Core}(X).$$

Proof. Suppose $x \notin \widehat{K}$. Then, by the p-plurisubharmonic analogue of Lemma 4.2 in [HL$_2$], there exists a smooth non-negative p-plurisubharmonic function u which is zero on a neighborhood of K and satisfies $u(x) > 0$; furthermore, if $x \notin \mathrm{Core}(X)$, then u can be chosen to be *strict* at x. Therefore, $\mathrm{tr}_{\vec{T}}(\mathrm{Hess}\, u) > 0$ in some neighborhood U of x. Since $\mathrm{tr}_{\vec{T}}(\mathrm{Hess}\, u) \geq 0$ everywhere, it follows from (4.6), (4.2), and minimality that $\|T\|(U) = 0$. Hence, $x \notin \mathrm{supp}(T)$. □

This result can be rephrased in terms of a second hull defined as follows.

Definition 4.13. Given a compact subset $K \subset X$, we define the *minimal* p*-current hull* to be the set $\widehat{K}_{\min} = \bigcup \mathrm{supp}(T)$ where the union is taken over all minimal $T \in \mathcal{R}_p(X)$ with $\mathrm{supp}(\partial T) \subset K$.

Note that $\widehat{K}_{\min}$ contains all compact minimal oriented p-dimensional submanifolds with boundary in K.

Theorem 4.14. *Theorem 4.12 can be restated as follows:*

$$\widehat{K}_{\min} \subset \widehat{K} \cup \mathrm{Core}(X).$$

By Remark 4.4 (1), X supports a global strictly p-plurisubharmonic function if and only if $\mathrm{Core}(X) = \emptyset$. Therefore, $\widehat{K}_{\min} \subset \widehat{K}$ in this case. For example, $|x|^2$ is such a global function on $\mathbb{R}^n$.

Question 4.15. *Suppose that $\Gamma \subset \mathbb{R}^n$ is a compact $(p-1)$-dimensional submanifold which bounds exactly one minimal p-current in $\mathbb{R}^n$, and that this current is an oriented submanifold M. How close does $\widehat{\Gamma}$ come to approximating M?*

General (not necessarily rectifiable) minimal currents. Much of what is said above carries over to general compactly supported currents of finite mass. These are exactly the currents which can be represented as in (4.1), with the provision that the $\|T\|$-measurable field $\vec{T}$ of unit p-vectors is no longer required to be simple $\|T\|$-almost everywhere. Definition 4.8 makes sense for such currents, and the first variational formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}((\varphi_t)_* T))\big|_{t=0} = \int_X \langle D_{\mathcal{A}^V}\vec{T}, \vec{T}\rangle \, \mathrm{d}\|T\|$$

holds. If $V = \nabla u$ where $u \in \mathrm{PSH}_p^\infty(X)$, then by Definition 2.1 (3) we know that $D_{\mathcal{A}^V} \geq 0$. Furthermore, at any point where u is strict, we have $D_{\mathcal{A}^V} > 0$. The arguments for Corollary 4.11 and Theorem 4.12 give the following.

Proposition 4.16. *Let T be a minimal p-dimensional current of finite mass and compact support on X, and let u be any smooth p-plurisubharmonic function which vanishes on a neighborhood of* $\mathrm{supp}(\partial T)$*. Then*

$$\langle D_{\mathrm{Hess}\,u}\vec{T}, \vec{T}\rangle = 0 \quad \|T\|\text{-almost everywhere.}$$

Furthermore,

$$\mathrm{supp}(T) \subset \widehat{\partial T} \cup \mathrm{Core}(X).$$

Thus the minimal current hull $\widehat{K}_{\min}$ can be expanded to contain the supports of all minimal currents with boundary supported in K, and Theorem 4.14 remains true.

Examples 4.17. Minimal nonrectifiable currents abound in geometry. Any positive (p,p)-current on a Kähler manifold X is minimal. This observation extends to positive φ-currents on any calibrated manifold (X,φ) (see [HL$_1$]). Any foliation current whose leaves are minimal p-submanifolds is a minimal current.

There are two basic cases of smooth minimal currents which are interesting. Let T be a smooth d-closed current of dimension $n-1$ (degree 1). Then T is simply a closed 1-form and can be written locally as $T = \mathrm{d}f$ for a smooth function f. In a neighborhood of any point where $\mathrm{d}f \neq 0$, the minimality condition is equivalent to the 1-Laplace equation:

$$d\left(*\frac{\mathrm{d}f}{\|\mathrm{d}f\|}\right) = 0,$$

which says that $*\mathrm{d}f/\|\mathrm{d}f\|$ calibrates the level hypersurfaces of f. In particular, the level sets of f are minimal varieties.

Let T be a smooth d-closed current of dimension 1. Then T can be expressed on a compactly supported 1-form α as $T(\alpha) = \int_X \alpha(V)\,\mathrm{dvol}_X$, where V is a smooth vector field. Minimality is the condition that

$$\nabla_V \left(\frac{V}{\|V\|} \right) = 0,$$

which means exactly that the (reparameterized) flow lines of V are geodesics in X, and that the d-closed condition for T is equivalent to $\mathrm{div}(V) = 0$.

5. THE EXTREME RAYS IN THE CONVEX CONE $\mathcal{P}_p(V)$

Recall the classical fact that the extreme rays in $\mathcal{P}_1(V) \equiv \{A : A \geq 0\}$ are exactly those generated by the orthogonal projections P_e onto the lines spanned by unit vectors $e \in \mathbb{R}^n$. The purpose of this section is to describe the extreme rays in $\mathcal{P}_p(V)$ for other p. Note that $\mathcal{P}_n(V)$ can be excluded from the discussion since it is a closed half-space, and hence has no extreme rays.

First, we state our result in ways that are more suitable for the many applications (see [HL8] and [HL9]).

Theorem 5.1a *(1 < p < n). The convex cone $\mathcal{P}_p(\mathbb{R}^n) \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is the smallest convex cone subequation F with the property that the Riesz kernel K_p is F-subharmonic.*

The second version requires a definition.

Definition 5.1. The *Riesz characteristic* p_F of a subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is defined to be

$$p_F \equiv \sup\{p : I - pP_e \in F \text{ for all } |e| = 1\}.$$

Theorem 5.1b *(1 < p < n). Suppose that $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a convex cone subequation. Then*

$$\mathcal{P}_p \subset F \iff p \leq p_F.$$

Finally, we state the result in terms of extreme rays.

Theorem 5.1c *(1 < p < n). The extreme rays in $\mathcal{P}_p(V)$ are of two types. They are generated by*

(1) *either $I - p(e \circ e) = P_{e^\perp} - (p-1)P_e$,*
(2) *or P_e,*

where e is a unit vector in V. If $n - 1 \leq p < n$, only case (1) occurs.

Proof of Theorem 5.1c. Under the action of O_n on $\mathrm{Sym}^2(\mathbb{R}^n)$, the set $\mathbb{D} \equiv \mathbb{R}^n$ of diagonal matrices forms an n-dimensional cross-section. For any O_n-invariant set $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$, the intersection $\mathbb{F} \equiv F \cap \mathbb{D}$ has orbit $O(\mathbb{F}) = F$. For a convex cone $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$, let $\mathcal{E}\mathrm{xt}(F)$ denote the union of the extreme rays in F.

Lemma 5.2. *If* $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ *is an* O_n*-invariant convex cone and* $\mathbb{F} \equiv F \cap \mathbb{D}$*, then*

$$\mathcal{E}\mathrm{xt}(F) \subseteq O(\mathcal{E}\mathrm{xt}(\mathbb{F})).$$

Proof. Suppose $A \notin O(\mathcal{E}\mathrm{xt}(\mathbb{F}))$. Then $A = gDg^t$ with $g \in \mathrm{O}_n$ implies $D \notin \mathcal{E}\mathrm{xt}(\mathbb{F})$. Thus $D = \alpha D_0 + \beta D_1$ with $\alpha > 0$, $\beta > 0$, $D_0, D_1 \in \mathbb{F}$; but D_0 and D_1 determine different rays. Therefore, $A = \alpha g D_0 g^t + \beta g D_1 g^t = \alpha A_0 + \beta A_1$, $A_0, A_1 \in O(\mathbb{F}) = F$, but A_0 and A_1 determine different rays, proving that $A \notin \mathcal{E}\mathrm{xt}(F)$. ◻

In particular, $\mathcal{E}\mathrm{xt}(\mathcal{P}_p) \subset O(\mathcal{E}\mathrm{xt}(\mathbb{P}_p))$, so that it remains to compute the extreme rays in $\mathbb{P}_p \equiv \mathcal{P}_p \cap \mathbb{D}$. First note that, by definition (see Remark 2.9), we have

$$\mathbb{P}_p = \mathcal{P}_p = \{A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) : \lambda_1^{\uparrow} + \cdots + \lambda_{\bar{p}}^{\uparrow} + (p - \bar{p})\lambda_{\bar{p}+1}^{\uparrow} \geq 0\}, \tag{5.1}$$

where $\lambda_1^{\uparrow} \leq \lambda_2^{\uparrow} \leq \ldots \lambda_n^{\uparrow}$ denotes the rearrangement of the eigenvalues λ_i into ascending order.

Lemma 5.3. *If* $A \in \mathbb{P}_p$ *is extreme, then* A *has at most one strictly negative eigenvalue.*

Proof. Suppose $\lambda_2^{\uparrow} = \lambda_2^{\uparrow}(A) < 0$. To simplify notation, we assume the eigenvalues λ_i are in ascending order, and drop the arrows. Set $\alpha = \lambda_1 + \lambda_2 < 0$, and write $\lambda \equiv (\lambda_1, \ldots, \lambda_n)$. Then $\lambda = sv + (1-s)w$ where $s = \lambda_1/\alpha > 0$, $1-s = \lambda_2/\alpha > 0$, $v = (\alpha, 0, \lambda_3, \ldots, \lambda_n)$, $w = (0, \alpha, \lambda_3, \ldots, \lambda_n)$, and $v, w \in \mathbb{P}_p$. Hence, A is not extreme. ◻

We are now reduced to two cases.

1. *One negative eigenvalue:* By rescaling and permuting, we may assume $\lambda_1 = -1$ and $0 \leq \lambda_2 \leq \cdots \leq \lambda_n$ where $A = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.
   Set $B = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_n)$. Then

$$\lambda_2 + \cdots + \lambda_{\bar{p}} + (p - \bar{p})\lambda_{\bar{p}+1} \geq 1. \tag{5.2}$$

   A similar argument to the one given in the proof of Lemma 5.3 applies to show that if B is extreme in the set of matrices satisfying (5.2), then $\lambda_2 = \cdots = \lambda_n = \mu$, and equality holds in (5.2). Therefore, $(\bar{p} - 1)\mu + (p - \bar{p})\mu = 1$, that is, $\mu = 1/(p-1)$. This proves the following. If $A \in \mathbb{P}_p$ is extreme and has one strictly negative eigenvalue, then after rescaling A and permuting coordinates, $A = \mathrm{diag}(-(p-1), 1, \ldots, 1)$.

2. *All eigenvalues positive:* Consider the hyperplane $\lambda_1 + \cdots + \lambda_{\bar{p}} + (p - \bar{p})\lambda_{\bar{p}+1} = 1$ intersected with the positive quadrant in $\mathbb{R}^{\bar{p}+1}$ (or $\mathbb{R}^{\bar{p}}$ if $p = \bar{p}$). The cone on this set is the positive quadrant. Therefore, the only extreme rays of $\mathbb{P}_p$ that could possibly appear from this set are the axis rays.

This proves that the only possible extreme rays in $\mathcal{P}_p(V)$ are generated by P_e and $I - pP_e$ with $|e| = 1$. By the orthogonal invariance of $\mathcal{P}_p(V)$, the ray generated by $I - pP_e$ (for one unit vector e) is extreme if and only if it is extreme for all unit vectors. Consequently, if $I - pP_e$ is not extreme for one e, then the only possible extreme rays are generated by the rank-one projections P_e. Now $p < n$ implies $\mathcal{P}_p(V) \cap \{A : \operatorname{tr} A = 1\}$ is compact, so that the extreme rays must generate $\mathcal{P}_p(V)$. This forces $\mathcal{P}_p(V) \subset \mathcal{P}(V)$, which contradicts $1 < p$. Summarizing, we have that each $I - pP_e$ generates an extreme ray in $\mathcal{P}_p(V)$.

It remains to show that the axis rays are extreme in $\mathcal{P}_p(V)$ if and only if $1 < p < n - 1$. This is left to the reader. ◻

To see that version (a) of Theorem 5.1 is equivalent to version (c), compute that the second derivative $D_x^2 K_p$ is, up to a positive scalar, equal to $I - pP_x$. The equivalence to version (b) is straightforward.

Acknowledgement. H. Blaine Lawson, Jr. has been partially supported by the NSF.

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Key words and phrases: p-convexity, p-plurisubhamonicity, Dirichlet problem, Levi problem.

2010 Mathematics Subject Classification: 32F10, 32U05, 35J70, 58J32.

Received: November 16, 2011; revised: March 19, 2013.