White's Compactness Theorem for Integral Currents

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Contents

Introduction

The problem of Plateau was to show the existence of a minimal surface given a boundary. In lower dimensions, this problem can be studied with classical tools of differential geometry. However, these tools are insufficient in extending the problem to arbitrary dimensions and co-dimensions.

Singularity formation is inevitable in higher dimensions. This has been the motivation for a new type of geometry. In their landmark paper [FF60], Herbert Federer and Wendell Fleming established the area of Geometric Measure Theory. This theory extends differential geometry to account for singularities - to have n-surfaces where we can talk about it being almost everywhere differentiable in a measure theoretic sense.

The Compactness Theorem is a fundamental theorem in Geometric Measure Theory. It is an elegant solution to the extended problem of Plateau. However, the original proof [FF60] involved the use of the *Structure Theorem*. This dictates that every set $A \subseteq \mathbb{R}^{n+k}$ (which can be written as a union of finite measure sets) can be decomposed into $A = R \cup P$, where R is *rectifiable* (the surfaces of Geometric Measure Theory) and P *purely unrectifiable* (contains only zero measure rectifiable sets). While this result maybe an interest in itself, it is a difficult fact to swallow and makes the proof of compactness difficult to digest.

There have been other proofs of this theorem. Bruce Solomon provided a proof [Sol84] that circumvents this hard fact. However, it uses multivalued functions which are difficult in themselves. The more elegant solution came from Brian White in 1987, where he provides a direct, elegant argument - relying on density facts, deformations, boundary rectifiability, and other more tangible features of Geometric Measure Theory.

Although White's proof is simpler and more elegant than its previous counterparts, the argument in [Whi89] is still terse and inaccessible to a reader who does not have a deep understanding of Geometric Measure Theory. The purpose of this thesis has been to provide, to the extent possible, a self-contained exposition of these facts. Geometric Measure Theory has deep ideas, and in many places it is extremely technical. This thesis aims provides the sufficient background to shed light on the argument.

Other than results proved by White in [Whi89], only a few proofs provided here are taken from books. Most arguments are constructed to justify the important facts of White's proof which maybe lacking in detail. These arguments use non-trivial facts from point-set topology, measure theory, functional analysis and differential geometry. The sources of any theorems used have always been referenced. For that reason, it is recommended that the references are kept in close proximity. In particular, [Sim83], [Mat95], and [Mun96] are important. For convenience, a table of notation has also been included.

Chapter 1

Preliminaries

In this chapter, we develop some preliminary constructs of Geometric Measure Theory. Many results in this section are included for completeness and consequently, only relevant proofs are included.

1.1 Hausdorff Measure

The Hausdorff measure is the central tool of analysis in Geometric Measure Theory. The central motivation behind the Hausdorff measure is to make sense of the measure of an n dimensional subset of an $n+k$ dimensional space.

Definition 1.1.1 (Hausdorff Measure) *Let $n \in \mathbb{R}^+$. Let $A \subseteq \mathbb{R}^{n+k}$. Then we define the* Hausdorff δ-approximation measure *by:*

$$\mathscr{H}^n_\delta(A) = \inf\left\{\sum_{i=1}^{\infty} \omega_n \left(\frac{\operatorname{diam}(C_i)}{2}\right)^n : A \subseteq \bigcup_{i=1}^{\infty} C_i, \operatorname{diam}(C_i) < \delta\right\}$$

where $w_n = \operatorname{Vol}(B_\delta)$ (volume of an n-ball of radius δ) if $n \in \mathbb{N}$.

Then, the Hausdorff measure *is defined by:*

$$\mathscr{H}^n(A) = \lim_{\delta \to 0} \mathscr{H}^n_\delta(A)$$

The following properties of the Hausdorff measure illustrate some important facts:

Theorem 1.1.2 (Properties of Hausdorff measure)

1. *$\mathscr{H}^n$ is Borel Regular*
2. *$\mathscr{H}^n(A) = \mathscr{L}^n(A)$ for $A \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$*
3. *For each $m < n$, $\mathscr{H}^m(A) < \infty \implies \mathscr{H}^n(A) = 0$*
4. *For each $m < n$, $\mathscr{H}^n(A) < \infty \implies \mathscr{H}^m(A) = \infty$*

The last two properties tell us that when the Hausdorff measure is positive, then it is non-zero for a unique $n \in \mathbb{R}^+$. This is often used as a motivation to develop a notion of Hausdorff dimension, which agrees with our definition of dimension in the case where $n \in \mathbb{N}$.

1.2 Lipschitz Functions

Differentiable functions are the functions of differential geometry. Lipschitz functions are the key to the geometry we develop in GMT. We begin with the following definition.

Definition 1.2.1 (Lipschitz Function) *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz if for all $x, y \in R^n$,*

$$\|f(x) - f(y)\| \leq M\|x - y\|$$

The least such constant M is denoted $\mathrm{Lip}\,(f)$.

Next, we quote the famous Rademacher's theorem. The proofs of these results are found in [Sim83, 5.2, 5.3].

Theorem 1.2.2 (Rademacher's Theorem) *Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^m$ be Lipschitz. Then f is differentiable $\mathscr{L}^n-\mathrm{a.e.}\ x \in \mathbb{R}^n$.*

We have the following important consequence:

Lemma 1.2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then,* $\mathrm{ess\ sup}\ \|\nabla f(x)\| \leq \sqrt{n+k}(\mathrm{Lip}\,(f))$.

Proof Since f is Lipschitz, for $h > 0$, we have:

$$\frac{f(x + he_i) - f(x)}{h} \leq \frac{|f(x + he_i) - f(x)|}{h} \leq \frac{(\mathrm{Lip}\,(f))\,|x + he_i - x|}{h} \leq (\mathrm{Lip}\,(f))$$

So, by the definition of partial derivative [Spi65, p25], we have that

$$\frac{\partial f}{\partial x^i} \leq \mathrm{Lip}\,(f)$$

whenever the limit exists. It follows that:

$$\|\nabla f(x)\| = \sqrt{\sum_{i=1}^{n+k}\left(\frac{\partial f}{\partial x^i}(x)\right)^2} \leq \sqrt{\sum_{i=1}^{n+k}(\mathrm{Lip}\,(f))^2} = \sqrt{n+k}(\mathrm{Lip}\,(f))$$

This implies that $\mathrm{ess\ sup}\ \|\nabla f(x)\| \leq \sqrt{n+k}(\mathrm{Lip}\,(f))$. □

The next is an important C^1 approximation result for Lipschitz functions:

Theorem 1.2.4 (C^1 approximation theorem) *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz function. Then for every $\varepsilon > 0$, there exists a $g \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ such that:*

$$\mathscr{L}^n\left(\left\{x \in \mathbb{R}^{n+k} : f(x) \neq g(x)\right\} \cup \left\{x \in \mathbb{R}^{n+k} : \mathrm{D}f(x) \neq \mathrm{D}g(x)\right\}\right) < \varepsilon$$

1.3 Rectifiability

We commence our discussion with the following definition.

Definition 1.3.1 (Countably n-Rectifiable) *A set M is called countably n-Rectifiable if it can be written as:*

$$M = M_0 \cup \left(\bigcup_{i=1}^{\infty} M_i\right)$$

where $\mathscr{H}^n(M_0) = 0$ and for each $i > 0$, $M_i = f_i(\Omega_i)$ with each f_i Lipschitz and each $\Omega_i \subseteq \mathbb{R}^n$.

For convenience, we will usually call such sets n-rectifiable or simply rectifiable (when the dimension context is clear).

Rectifiable sets are the surfaces of Geometric Measure Theory. Intuitively, such surfaces should be $\mathscr{H}^n-$a.e. smooth (in light of the Rademacher's Theorem, Theorem 1.2.2).

The set M_0 is the singular set - it allows the surface to behave sufficiently "badly" up to a set of measure zero. Since the properties of $\mathscr{H}^n$ guarantees that for all sets $B \subseteq R^n$ with $\dim(B) < n$, $\mathscr{H}^n(B) = 0$, the surface can contain any lower dimensional sets with null contribution. Since each $M_i = f_i(\Omega_i)$ is $\mathscr{H}^n-$a.e. smooth, subadditivity of measures ensures that the total set of singular points is a $\mathscr{H}^n$ null set.

The following theorem gives a rigorous formulation of our discussion.

Theorem 1.3.2 (Submanifold Embeddings of Rectifiable Sets) *A set $M \subseteq \mathbb{R}^{n+k}$ is n-rectifiable if and only if:*

$$M \subseteq N_0 \cup \left(\bigcup_{i=1}^{\infty} N_i \right)$$

where $\mathscr{H}^n(N_0) = 0$ and for $i > 0$, N_i is an n-dimensional C^1 embedded submanifold of $\mathbb{R}^{n+k}$.

Corollary 1.3.3 (Rectifiable Decomposition Property) *Let M be a rectifiable set. Then, we can write $M = M_0 \cup (\bigcup_{i=1}^{\infty} M_i)$, where each $M_i \subseteq N_i$ is $\mathscr{H}^n$-measurable, pairwise disjoint, and $\mathscr{H}^n(M_0) = 0$.*

Proof Let $M \subseteq N_0 \cup (\bigcup_{i=1}^{\infty} N_i)$, with N_i as guaranteed by Theorem 1.3.2. Now, define $M_0 = M \setminus \bigcup_{i=1}^{\infty} N_i$. Trivially, $\mathscr{H}^n(M_0) = 0$. Now for $i > 1$, we define $M_i = \left(N_i \setminus \bigcup_{j=1}^{i-1} M_i\right) \cap M$. Measurability of M_i is an easy exercise in induction. □

This decomposition is indeed convenient and its usefulness will become apparent later. We define it formally for convenience.

Definition 1.3.4 (Disjoint Decomposition) *Let $M \subseteq \mathbb{R}^{n+k}$ be a rectifiable set. Then the decomposition promised by Theorem 1.3.3 is called the Disjoint Decomposition of M.*

Before we continue with a discussion of the properties of this geometry, we recall the following important measure theoretic definition.

Definition 1.3.5 (Locally Summable) *Let X be a metric space, and μ a measure on X. Let $f : X \to \mathbb{R}$ be a μ-measurable function. If for every $W \Subset U$,*

$$\int_W |f| \, \mathrm{d}\mu < \infty$$

then we say that f is locally summable.

An important property of submanifolds is that they have a tangent space at every point. The following definition gives a generalisation of a tangent space in the rectifiable setting [FX02, 3.3.3, 3.3.4]. This definition is indeed equivalent in the submanifold setting.

Definition 1.3.6 (Approximate Tangent Space) *Let $M \subseteq \mathbb{R}^{n+k}$ be a $\mathscr{H}^n$ measurable subset, and let $\theta : M \to \mathbb{R}^+$, locally $\mathscr{H}^n$-summable. Given an n-dimensional subspace $P \subseteq \mathbb{R}^{n+k}$, we say that P is the approximate tangent plane with respect to multiplicity $\theta(x)$ at x if*

$$\lim_{\lambda \to 0} \int_{\eta_{x,\lambda} M} f(y) d\mathscr{H}^n(y) = \theta(x) \int_P f(y) d\mathscr{H}^n(y)$$

holds for all $f \in C_c^0(\mathbb{R}^{n+k})$, with $\eta_{x,\lambda}(y) = \lambda^{-1}(y - x)$. We write $P = T_x M$.

We make a remark that θ is unique only up to a set of measure zero. That is, given a function θ' which satisfies the above definition, then the tangent spaces T_xM and T_xM' agree $\mathscr{H}^n-\text{a.e. } x \in M$ [Sim83, 11.5].

The following important results allows us to characterise rectifiable sets in terms of its tangent properties. This result is another confirmation that rectifiable sets are indeed a generalisation of submanifolds.

Theorem 1.3.7 (Existence of Tangent Planes) *Let $M \subseteq \mathbb{R}^{n+k}$ be $\mathscr{H}^n$ measurable. Then M is n-rectifiable if and only if there exists a locally $\mathscr{H}^n$-summable $\theta : M \to \mathbb{R}^+$, such that the tangent plane T_xM exists with respect to θ for $\mathscr{H}^n-\text{a.e. } x \in M$.*

Corollary 1.3.8 (Uniqueness of Tangent Planes) *Let $M \subseteq \mathbb{R}^{n+k}$ be n-rectifiable, and let $M = M_0 \cup (\bigcup_{i=1}^{\infty} M_i)$ be the disjoint decomposition with $M_i \subseteq N_i$, with N_i a C^1 submanifold. Then, for $\mathscr{H}^n-\text{a.e. } x \in M$, $T_xM = T_xN_i$ whenever $x \in M_i$.*

It is worth noting here that if we indeed choose another decomposition N'_j, then whenever $x \in N'_j \cap N_i \neq \emptyset$, we must have $T_xN_i = T_xN'_j$.

1.4 Gradients, Area, Co-area

Recall that for an n-submanifold N in $\mathbb{R}^{n+k}$, with orthonormal tangent basis $v_1, \ldots, v_n$, we can define the N-gradient of a differentiable function $f : N \to \mathbb{R}$ by:

$$\nabla^N f(y) = \sum_{i=1}^{n} (D_{v_i} f) v_i$$

Then, we can make the following definition.

Definition 1.4.1 (M-gradient) *Let $M \subseteq \mathbb{R}^{n+k}$ n-rectifiable and let $M = M_0 \cup (\bigcup_{i=1}^{\infty} M_i)$ be the disjoint decomposition, with $M_i \subseteq N_i$, where N_i is C^1 submanifold for $i > 0$. Let $f : U \to \mathbb{R}$, where $M \subseteq U$ open in $\mathbb{R}^{n+k}$. Then, we define:*

$$\nabla^M f(x) = \nabla^{N_i} f(x)$$

for $\mathscr{H}^n-\text{a.e. } x \in M$ whenever $x \in M_i$.

Such a gradient exists by Rademacher's Theorem, since we are guaranteed that a Lipschitz $f|_{N_i}$ is differentiable $\mathscr{H}^n-\text{a.e. } x \in N_i$. Also, we emphasise that $\nabla^M f$ is independent of the particular decomposition up to a set of measure zero.

We can now proceed to define the differential of Lipschitz functions:

Definition 1.4.2 (M-Differential) *Let $M \subseteq \mathbb{R}^{n+k}$ be an n-rectifiable set, and let $f : U \to \mathbb{R}^m$, where $M \subseteq U$ open in $\mathbb{R}^{n+k}$, be a function with each component f^i locally Lipschitz. Then, we define $\mathrm{d}_x^M f : T_xM \to \mathbb{R}^m$ by:*

$$\mathrm{d}_x^M f(v) = \sum_{i=1}^{m} \langle \nabla^M f_j(x), v \rangle e_i$$

where e_i is the standard basis for $\mathbb{R}^m$, and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ the usual inner product in $\mathbb{R}^m$.

This machinery allows us to start talking about Jacobians.

Definition 1.4.3 *Suppose M and f are as given by Definition 1.4.2, with M an n-rectifiable set. Then, we define the Jacobian and Co-Jacobian respectively for $\mathscr{H}^n-\text{a.e. } x \in M$ by:*

$$\mathrm{J}_M f(x) = \sqrt{\det \mathrm{D}_M^* f(x) \circ \mathrm{D}_M f(x)}\ , m \geq n$$
$$\mathrm{J}_M^* f(x) = \sqrt{\det \mathrm{D}_M f(x) \circ \mathrm{D}_M^* f(x)}\ , m < n$$

where $\mathrm{D}_M^* f(x)$ denotes the adjoint of $\mathrm{D}_M f(x)$.

By virtue of the preceding results and definitions, we know the Jacobian and Cojacobian exists $\mathscr{H}^n-\text{a.e.}$ The characterisation of these quantities allow us to talk about the Area and Co-area formula in the rectifiable setting.

Theorem 1.4.4 (Area and Coarea) *Let $M \subseteq \mathbb{R}^{n+k}$ be n-rectifiable, and suppose $f : U \to \mathbb{R}^m$ is Lipschitz with $M \subseteq U$ open in $\mathbb{R}^{n+k}$. Further, let $A \subseteq M$ be $\mathscr{H}^n$-measurable and suppose $g : M \to \mathbb{R}^+$ $\mathscr{H}^n$-measurable. Then*

1. *If $m \geq n$ then*

$$\int_M g \mathrm{J}_M f \, \mathrm{d}\mathscr{H}^n = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(x)} g \mathrm{d}\mathscr{H}^0 \right) \mathrm{d}\mathscr{H}^m(x) \qquad \text{(Area)}$$

2. *If $m < n$ then*

$$\int_A g \mathrm{J}_M^* f \, \mathrm{d}\mathscr{H}^n = \int_{\mathbb{R}^m} \left(\int_{f^{-1}(x) \cap M} g \mathrm{d}\mathscr{H}^{n-m} \right) \mathrm{d}\mathscr{L}^m(x) \qquad \text{(Co-Area)}$$

Chapter 2

Currents, Varifolds, Densities and Slices

In this chapter, we present some theory and proofs which will be of later use.

2.1 Forms and p-Vectors

We will begin by introducing *Einstein Summation Convention*:

$$a^i e_i = \sum_i a^i e_i$$

That is exactly - whenever there is a raised and lowered index, there is always an implied summation unless otherwise stated. This will prove to be a useful simplification later when we deal with large indices. In general, the significance is the symbol itself, and not its index. That is, we will liberally raise and lower indices on the symbol to assume Einstein Summation.

Let V be a vector space. We denote the p-vectors of V by $\wedge_p V$, and the p-forms of V by $\wedge^p V$.

We define an inner product on $\wedge^p \mathbb{R}^n$.

Definition 2.1.1 (Product in $\wedge^p \mathbb{R}^n$) *Let $\omega, \eta \in \wedge^p \mathbb{R}^n$. Then we write $\omega = \omega_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ and $\eta = \eta_{j_1,\ldots,j_p} dx^{j_1} \wedge \ldots \wedge dx^{j_p}$. We define:*

$$\langle \omega, \eta \rangle_{\wedge^p} = \omega_{i_1,\ldots,i_p} \eta^{i_1,\ldots,i_p}$$

We now prove that this is indeed an inner product on $\wedge^p \mathbb{R}^n$.

Lemma 2.1.2 *The product $\langle \cdot, \cdot \rangle_{\wedge^p}$ defined in Definition 2.1.1 is indeed an inner product on $\wedge^p \mathbb{R}^n$.*

Proof Let ω, η be written as in Definition 2.1.1. For simplicity, let $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\wedge^p}$.

Trivially we have $\langle \eta, \nu \rangle = \langle \nu, \eta \rangle$.

Let $\nu = \nu_{k_1,\ldots,k_p} dx^{k_1} \wedge \ldots \wedge dx^{k_p} \in \wedge^p$. Fix $a, b \in \mathbb{R}$.

$$\begin{aligned}
\langle a\omega + b\eta, \nu \rangle &= \left(a\omega_{i_1,\ldots,i_p} + b\eta_{i_1,\ldots,i_p} \right) \nu^{i_1,\ldots,i_p} \\
&= a\omega_{i_1,\ldots,i_p} \nu^{i_1,\ldots,i_p} + b\eta_{i_1,\ldots,i_p} \nu^{i_1,\ldots,i_p} \\
&= a\langle \omega, \nu \rangle + b\langle \eta, \nu \rangle
\end{aligned}$$

Also, trivially, $\langle \omega, \omega \rangle \geq 0$ and $\langle \omega, \omega \rangle = 0$ if and only if $\omega = 0$.

By [PG96, 4.1], $\langle \cdot, \cdot \rangle_{\wedge^p}$ is an inner product. □

We point out the following topological consequence.

Corollary 2.1.3 *The inner product $\langle \cdot, \cdot \rangle_{\wedge^p}$ induces the usual metric on $\wedge^p \mathbb{R}^n$.*

Proof Trivially,

$$\langle \omega, \omega \rangle = ||\omega||^2$$

where $||\omega||$ is the usual norm on $\wedge^p \mathbb{R}^n$. □

This following result is fundamental in the theory we develop later. It gives an important representation of the inner product in the space of forms.

Lemma 2.1.4 *Let $\omega, \eta \in \wedge^p \mathbb{R}^n$. Then there exists an $\tilde{\eta} \in \wedge_p \mathbb{R}^n$ such that*

$$\langle \omega, \eta \rangle_{\wedge^p} = \langle \omega, \tilde{\eta} \rangle$$

where $\langle \omega, \tilde{\eta} \rangle$ is the usual pairing of a form with a vector. And conversely.

Proof Trivially, we put $\tilde{\eta} = \eta^{i_1,\ldots,i_p} e_{i_1} \wedge \ldots \wedge e_{i_p}$, where $\eta = \eta_{i_1,\ldots,i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}$. □

In light of this result, we can consider the usual paring of a form with a vector "to behave" as the inner product in the space of forms. This notion will be assumed throughout this document.

2.2 Currents

Convergence is best talked in the language of functional analysis. Currents are a way of representing rectifiable sets as linear functionals. In this section we will expose some aspects of the theory of currents.

Definition 2.2.1 (n-Current) *Let U open in $\mathbb{R}^{n+k}$. Define:*

$$\mathscr{D}^n U = \left\{ \omega : U \to \wedge^n \mathbb{R}^{n+k}, \omega \text{ smooth }, \operatorname{spt} \omega \text{ compact in } U \right\}$$
$$\mathscr{D}_n U = \mathscr{D}^n U^*$$

We call $\mathscr{D}_n U$ the set of n-currents of U.

Naturally, $\mathscr{D}^n U = C_c^\infty(U, \wedge^n \mathbb{R}^{n+k})$ is equipped with the standard norm $||\omega|| = \max_{x \in U} ||\omega(x)||$.

We now define an appropriate semi-norm on the space of n-Currents.

Definition 2.2.2 (Mass Norm) *Let U open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$. Let $W \Subset U$. The mass of T in W is defined by:*

$$\mathrm{M}_W(T) = \sup \{T(\omega) : \omega \in \mathscr{D}^n U, \operatorname{spt} \omega \subseteq W, ||\omega|| \leq 1\}$$

and $\mathrm{M}_U(T) = \mathrm{M}(T)$.

Definition 2.2.3 (Locally Finite Mass) *Let U open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$. If for every $W \Subset U$, $\mathrm{M}_W(T) < \infty$, then we say that T has locally finite mass.*

We introduce the following important notion.

Definition 2.2.4 (Measure Functional) *Let U open in $\mathbb{R}^{n+k}$, and let μ be a Radon measure on U. Then, given some n-vectorfield $\nu : \mathbb{R}^{n+k} \to \wedge_n \mathbb{R}^{n+k}$, we define $\mu \wedge \nu$ by:*

$$(\mu \wedge \nu)(\omega) = \int_U \langle \omega(x), \nu(x) \rangle \, \mathrm{d}\mu(x)$$

where $\omega \in \mathscr{D}^n U$.

Theorem 2.2.5 *Let μ be a Radon measure on U open in $\mathbb{R}^{n+k}$, and let $\nu : \mathbb{R}^{n+k} \to \wedge_n \mathbb{R}^{n+k}$ be a locally μ-summable n-vectorfield. Then $(\mu \wedge \nu) \in \mathscr{D}_n U$ has locally finite mass.*

Proof Fix $W \Subset U$. Let $\omega \in \mathscr{D}^n U$ with $\|\omega\| \leq 1$ and $\operatorname{spt} \omega \subseteq W$. We write $\omega(x) = \omega_{i_1,\ldots,i_n}(x) dx^{i_1} \wedge \ldots \wedge dx^{i_n}$ and $\nu(x) = \nu^{j_1,\ldots,j_n}(x) e_{j_1} \wedge \ldots \wedge e_{j_n}$. Also, we have $\langle \omega(x), \nu(x) \rangle = \omega_{i_1,\ldots,i_n}(x) \nu^{i_1,\ldots,i_n}(x)$. In the light of Lemma 2.1.4, we can apply the Cauchy-Schwartz Inequality [PG96, Prop. 1], and it follows that:

$$\begin{aligned}
\int_W \langle \omega(x), \nu(x) \rangle \, \mathrm{d}\mu(x) &\leq \int_W \|\omega(x)\| \|\nu(x)\| \, \mathrm{d}\mu(x) \\
&\leq \int_W \|\nu(x)\| \, \mathrm{d}\mu(x) \\
&= \int_W \sqrt{\sum_{i_1 < \ldots < i_n} (\nu^{i_1,\ldots,i_n}(x))^2} \, \mathrm{d}\mu(x) \\
&< \infty \\
&\quad \text{(Since } \nu \text{ is locally } \mu\text{-summable)}
\end{aligned}$$

Since this holds for every such ω, $\mathrm{M}_W(T) < \infty$. □

We quote the following important theorem. Its proof can be found in [Sim83, 4.1].

Theorem 2.2.6 (Simon's Reisz Representation Theorem) *Let X be locally compact and separable, and let H be a Hilbert space. Let $L : C_c(X, H) \to \mathbb{R}$ be a linear functional such that for every K compact in X,*

$$\sup \left\{ L(f) : f \in C_c^0(X, H), \|f\| \leq 1, \operatorname{spt} f \subseteq K \right\} < \infty$$

Then there exists a Radon measure μ on X and μ-measurable $\nu : X \to H$ with $\|\nu(x)\| = 1$ μ–a.e. $x \in X$ and

$$L(f) = \int_X \langle f(x), \nu(x) \rangle \, \mathrm{d}\mu(x)$$

for all $f \in C_c^0(X, H)$.

Now we prove this important Corollary to the previous theorem.

Corollary 2.2.7 *Let μ be the promised Radon measure, and further suppose that X is Hausdorff. Then for V μ-measurable,*

$$\mu(V) = \sup \left\{ L(f) : f \in C_c^0(X, H), \|f\| \leq 1, \operatorname{spt} f \subseteq V \right\}$$

Proof Fix $f \in C_c^0(X, H)$ with $\|f\| \leq 1$ with $\operatorname{spt} f \subseteq V$, V μ-measurable. We apply the Cauchy-Schwartz Inequality [PG96, 4.1] and find $|\langle f(x), \nu(x)\rangle| \leq \|f\|\|\nu\| \leq 1$ for μ–a.e. $x \in X$.

It follows that,

$$
\begin{aligned}
L(f) &= \int_X \langle f(x), \nu(x)\rangle \, \mathrm{d}\mu(x) \\
&= \int_V \langle f(x), \nu(x)\rangle \, \mathrm{d}\mu(x) \\
&\qquad \textit{(since } \operatorname{spt} f \subseteq V\textit{)} \\
&\leq \int_V |\langle f(x), \nu(x)\rangle| \, \mathrm{d}\mu(x) \\
&\leq \int_V \mathrm{d}\mu(x) \\
&= \mu(V)
\end{aligned}
$$

It follows that $\sup\left\{L(f) : f \in C_c^0(X, H), \|f\| \leq 1, \operatorname{spt} f \subseteq V\right\} \leq \mu(V)$.

Let $K \subseteq V$ be compact in X. Since μ is Radon, K is μ-measurable, and further, $\mu(K) < \infty$. Fix $\varepsilon > 0$. By Lusin's Theorem [Fed96, 2.3.5], there exists compact $C_\varepsilon \subseteq K$ compact in X, with $\mu(K \setminus C_\varepsilon) < \varepsilon$ and $\nu|_{C_\varepsilon}$ continuous. Now, by definition $\operatorname{spt} \nu|_{C_\varepsilon}$ is closed in C_ε and by the Hausdorff hypothesis on X, we have that C_ε is Hausdorff, and it follows that $\operatorname{spt} \nu|_{C_\varepsilon}$ compact in C_ε [Mun96, 26.3]. By definition of subspace topology on C_ε, this implies that $\operatorname{spt} \nu|_{C_\varepsilon}$ is compact in X.

So, we can compute:

$$
L(\nu|_{C_\varepsilon}) = \int_{C_\varepsilon} \langle \nu|_{C_\varepsilon}(x), \nu|_{C_\varepsilon}(x)\rangle \, \mathrm{d}\mu(x) = \int_{C_\varepsilon} \mathrm{d}\mu(x) = \mu(C_\varepsilon)
$$

Since C_ε is μ-measurable, $\mu(K) = \mu(C_\varepsilon) + \mu(K \setminus C_\varepsilon)$, and so it follows that $\mu(K) \leq L(\nu|_{C_\varepsilon}) + \varepsilon$. This implies $\mu(K) = \sup\{L(f) : f \in C_c(X, H), \|f\| \leq 1, \operatorname{spt} f \subseteq K\}$. By [Sim83, 1.4], $\mu(V) = \sup\{\mu(K) : K \subseteq V, K \text{ compact in } X\}$, and the result follows immediately. □

This following corollary illustrates a way to represent currents as an integral against a Radon measure. This result is of central importance and we shall use this representation frequently.

Theorem 2.2.8 *Let U open in $\mathbb{R}^{n+k}$. Let $T \in \mathscr{D}_n U$ with locally finite mass. Then $T = \mu_T \wedge \xi$, where $\xi : \mathbb{R}^{n+k} \to \wedge_n \mathbb{R}^{n+k}$, μ_T a Radon measure on U, and $\mu_T(W) = \mathbf{M}_W(T)$.*

Proof We apply Simon's Reisz Representation Theorem (Theorem 2.2.6), with $X = U \subseteq \mathbb{R}^{n+k}$, $H = \wedge^n \mathbb{R}^{n+k}$. Trivially, we have $\mathscr{D}^n U \subseteq C_c^0(U, \wedge^n \mathbb{R}^{n+k})$. Then, we have a radon measure μ_T on U, and in the light of Lemma 2.1.4, an *n*-vectorfield $\xi : U \to \wedge_n \mathbb{R}^{n+k}$ such that for $\omega \in \mathscr{D}^n U$,

$$
T(\omega) = \int_U \langle \omega(x), \xi(x)\rangle \, \mathrm{d}\mu_T(x)
$$

Now, for any $W \subseteq U$ μ_T-measurable, Corollary 2.2.7 gives us that

$$
\mu_T(W) = \sup\{T(\omega) : \omega \in \mathscr{D}^n U, \|\omega\| \leq 1, \operatorname{spt} \omega \subseteq W\} = \mathbf{M}_W(T)
$$

□

We shall always adhere to the convention that μ_T will represent the Radon measure associated with a locally finite mass current T.

Now we consider representing rectifiable sets as currents. This will allow us to interchange between rectifiable sets and currents under the appropriate conditions.

Definition 2.2.9 (Integral Representation) *Let $M \subseteq \mathbb{R}^{n+k}$ be an n-rectifiable set with $M \subseteq U$, with U open in $\mathbb{R}^{n+k}$. Let ξ be an orientation n-vectorfield for M. Then, we define $[\![M]\!] \in \mathscr{D}_n U$ by:*

$$[\![M]\!](\omega) = \int_M \langle \omega(x), \xi(x) \rangle \, \mathrm{d}\mathscr{H}^n(x)$$

for all $\omega \in \mathscr{D}^n U$.

Motivated by Stokes' Theorem [Spi65, p122], we define the boundary of a current:

Definition 2.2.10 (Boundary of a Current) *Let U open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$. Then, we define $\partial T \in \mathscr{D}_{n-1} U$*

$$\partial T(\omega) = T(\mathrm{d}\omega)$$

for all $\omega \in \mathscr{D}^{n-1} U$.

2.3 Varifolds

Since we allow a set M_0 of measure zero in our definition of a rectifiable set, we allow the possibility that many distinct rectifiable sets may agree up to a set of measure zero. This motivates the notion of a varifold.

Definition 2.3.1 (Varifold) *Let $M \subseteq \mathbb{R}^{n+k}$ be an n-rectifiable set, and $\theta : \mathbb{R}^{n+k} \to \mathbb{R}^+$ be locally $\mathscr{H}^n$-summable with $\theta(x) = 0$ whenever $x \notin M$. Let:*

$$\mathrm{V}(M, \theta) = \left\{ (\tilde{M}, \tilde{\theta}) : \mathscr{H}^n(M \vartriangle \tilde{M}) = 0, \theta = \tilde{\theta} \; \mathscr{H}^n\text{–a.e.} \right\}$$

where $\vartriangle$ is the symmetric set difference. Then $\mathrm{V}(M, \theta)$ is called the Rectifiable n-Varifold associated with M. If $\theta : \mathbb{R}^{n+k} \to \mathbb{Z}^+$, then it is called an Integer Rectifiable n-Varifold.

We associate a special measure and notion of mass to varifolds. In light of our previous concepts of mass and the abstract Radon measure associated with a current, a link will be made in our discussion to follow.

Definition 2.3.2 (Measure of a Varifold) *Let $\mathrm{V}(M, \theta)$ be an n-varifold. Then, we define the measure on V as $\mu_V = \mathscr{H}^n \llcorner \theta$.*

Definition 2.3.3 (Mass of a Varifold) *Let $\mathrm{V}(M, \theta)$ be a varifold. Then, we define mass to be $\mathrm{M}_W(V) = \mu_V(W)$.*

This following measure theoretic result will be of use later.

Lemma 2.3.4 *Let $\theta : \mathbb{R}^{n+k} \to \mathbb{R}^+$ be a locally $\mathscr{H}^n$-summable function. Then, $\mathscr{H}^n \llcorner \theta$ is a Radon measure.*

Proof Let $K \subseteq \mathbb{R}^{n+k}$ be compact. Let $\mathscr{C} = \left\{ U_p \text{ open in } \mathbb{R}^{n+k} : \int_{U_p} \theta \, \mathrm{d}\mathscr{H}^n < \infty, p \in U_p \cap K \right\}$. Such a collection exists by our locally $\mathscr{H}^n$-summable hypothesis. Now, trivially, $\mathscr{C}$ is an open covering of K. By the compactness of K, we find a finite subcover $\mathscr{F} \subseteq \mathscr{C}$. Let $\mathscr{F} = \{F_1, \ldots, F_m\}$. Then,

$$\mathscr{H}^n \llcorner \theta(K) = \int_K \theta \, \mathrm{d}\mathscr{H}^n \le \int_{\bigcup_{i=1}^m F_i} \theta \, \mathrm{d}\mathscr{H}^n \le \sum_{i=1}^m \int_{F_i} \theta \, \mathrm{d}\mathscr{H}^n < \infty$$

□

Definition 2.3.5 (Tangent Space of μ) *Let μ be a Radon measure on $\mathbb{R}^{n+k}$ and $\theta : \mathbb{R}^{n+k} \to \mathbb{R}^+$. For $\lambda > 0$, let $\mu_{n,\lambda}(A) = \lambda^{-1}\mu(x + \lambda A)$. Suppose that for $\mu-$a.e. $x \in \mathbb{R}^{n+k}$, we have an n-dimensional subspace $P_x \subseteq \mathbb{R}^{n+k}$ such that:*

$$\lim_{\lambda \to 0} \int_M f(y)\, \mathrm{d}\mu(y) = \theta(x) \int_{P_x} f(y)\, \mathrm{d}\mathscr{H}^n(y)$$

for every $f \in C_c^0(\mathbb{R}^{n+k})$. Then we say that P_x is the tangent space of μ at x with multiplicity $\theta(x)$.

Definition 2.3.6 (Varifold Tangent Space) *Let $\mathrm{V}(M, \theta)$ be a Varifold. If the tangent space P_x of μ_V exists with multiplicity θ, we define the tangent space $T_x V = P_x$.*

The following result gives a justification to the way in which we defined mass and the measure associated to a varifold.

Theorem 2.3.7 *Let $\mathrm{V}(M, \theta)$ be a Varifold. Then, if μ is any other Radon measure for $\mathrm{V}(M, \theta)$, then $\mu = \mu_V$ if and only if μ has an approximate tangent space P_x with multiplicity θ for $\mu-$a.e. $x \in \mathbb{R}^{n+k}$.*

Proof Suppose that μ has a tangent plane with multiplicity θ. Then by [Sim83, 11.8], $\mu = \mu_V$, since by definition, $\theta(x) = 0$ for $x \notin M$. □

Now we consider the relationship between varifolds and currents. This important notion justifies us dealing almost exclusively with currents.

Definition 2.3.8 (Integer Multiplicity Rectifiable Current) *Let $T \in \mathscr{D}_n U$, for U open in $\mathbb{R}^{n+k}$. Suppose there exists a countably n-rectifiable $\mathscr{H}^n$-measurable set $M \subseteq \mathbb{R}^{n+k}$, a $\theta : \mathbb{R}^{n+k} \to \mathbb{Z}^+$ with $\theta(x) = 0$ for $x \notin M$, a simple n-vectorfield $\xi : M \to \wedge_n \mathbb{R}^{n+k}$, which can be written as $\xi(x) = \tilde{\xi}(x)\tau_1 \wedge \ldots \wedge \tau_n$, with $\tau_1, \ldots, \tau_n$ an orthonormal basis for $T_x M$, and for all $\omega \in \mathscr{D}^n U$, we can write:*

$$T(\omega) = \int_M \langle \omega(x), \xi(x) \rangle \theta(x)\, \mathrm{d}\mathscr{H}^n$$

Then, we say that T is an integer multiplicity rectifiable current (or simply rectifiable current), and we write $T = T(M, \theta, \xi)$. We call θ the multiplicity, and ξ the orientation for T.

And now, we present the following important result, although trivial in proof.

Theorem 2.3.9 *If $T(M, \theta, \xi) \in \mathscr{D}_n U$ is an integer multiplicity rectifiable current, then there is an associated integer rectifiable varifold $V(M, \theta)$.*

Proof Simply, we have that M is countable n-rectifiable. Then, by definition, we can construct the integer rectifiable varifold $V = V(M, \theta)$. □

2.4 Slicing

Here, we consider the way in which we could naturally talk about $(n-1)$-dimensional slices of an n-rectifiable set. We begin with a few preliminary results.

We begin by quoting the following important fact. See [Sim83, 28.1].

Theorem 2.4.1 *Let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be Lipschitz, and let M be countably n-rectifiable. Then for $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$,*

$$M_t = f^{-1}(t) \cap M$$

is countably $(n-1)$-rectifiable.

Corollary 2.4.2 *Let $M_+ = \left\{x \in M : \left|\nabla^M f(x)\right| > 0\right\}$. Then $f^{-1}(t) \cap M_+$ is $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$ $(n-1)$-rectifiable.*

Proof M_+ is trivially n-rectifiable, and the result follows. □

Since such sets $f^{-1}(t) \cap M_+$ are countably $(n-1)$-rectifiable, they have tangent properties. The following result makes a useful connection to the tangent plane of M.

Theorem 2.4.3 *Let M be n-rectifiable, $f : \mathbb{R}^{n+k} \to \mathbb{R}$ Lipschitz, and let $M_t = f^{-1}(t) \cap M_+$. Then for $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$ and $\mathscr{H}^{n-1}$–a.e. $x \in M_t$, T_xM, T_xM_t exist with T_xM_t an $(n-1)$ dimensional subspace of T_xM, and*

$$T_xM = \left\{y + \lambda \nabla^M f(x) : y \in T_xM_t, \lambda \in \mathbb{R}\right\}$$

Proof Since by Theorem 2.4.1, M_t is $(n-1)$-rectifiable, we write $M = M_0 \cup (\bigcup_{i=1}^{\infty} M_i)$ and $M_t = M_0^t \cup \left(\bigcup_{i=1}^{\infty} M_i^t\right)$ with each $\{M_i\}$ and $\left\{M_i^t\right\}$ pairwise disjoint.

Now, for each $i > 0$, we can find C^1 submanifolds N_i n-dim and N_i^t $(n-1)$-dim respectively, with each $M_i \subseteq N_i$ and $M_i^t \subseteq N_i^t$. By [Sim83, 11.6] we have that that $T_xM_i = T_xN_i, \mathscr{H}^n$–a.e. $x \in M$ and $T_xM_i^t = T_xN_i^t, \mathscr{H}^{n-1}$–a.e. $x \in M_t$. Now, for $x \in M_i^t \cap M_j \neq \emptyset$ where $T_xM_i^t$ exists, we consider $\tau \in T_xM_i^t$. So, there is a curve $\gamma : I \to N_i^t$ with $\gamma(0) = x, \dot{\gamma}(0) = \tau$. Now, we can restrict this to $\gamma : I' \to N_i^t \cap N_j$. Trivially then, $\tau \in T_xM_j$. This establishes that T_xM_t is an $(n-1)$-dimensional subspace of T_xM.

Now, consider the original curve. Then, it follows that:

$$\frac{d}{dt}|_{t=0}(f \circ \gamma)(t) = \frac{\partial f}{\partial x^i}|_x\dot{\gamma}^i(0) = \frac{\partial f}{\partial x^i}|_x\tau^i = \langle \nabla f(x), \tau\rangle$$

By construction, $f(x) = t$ for all $x \in M_t$. It then follows that $\langle \nabla f(x), \tau\rangle = 0$.

Now, we can write $\nabla^M f(x) = \nabla f(x) - \sum_{i=1}^{k}\langle \nabla f(x), \nu^i\rangle\nu^i$, where $\nu^i \perp T_xM$. Then, it follows that:

$$\langle \nabla^M f(x), \tau\rangle = \langle \nabla f(x) - \sum_{i=1}^{k}\langle \nabla f(x), \nu^i\rangle\nu^i, \tau\rangle = \langle \nabla f(x), \tau\rangle - \sum_{i=1}^{k}\langle \nu^i, \tau\rangle\nu^i$$

and since we have established that $T_xM_t \subseteq T_xM$, the result follows immediately. □

We now introduce the following important notion.

Definition 2.4.4 (Restriction of a p-vector) *Let N be an n-manifold and let $v \in \wedge_p N$, $w \in T_xN$. Then, we define $v \llcorner w \in \wedge_{p-1}N$ by $\langle v \llcorner w, \alpha\rangle = \langle v, w \wedge \alpha\rangle$ for all $\alpha \in \wedge_{p-1}N$ with $\langle \cdot, \cdot\rangle$ as the usual inner product in $\wedge_p N$.*

Lemma 2.4.5 *Let N be an n-manifold, and let $\xi : N \to \wedge_p N$ be a simple, unit length p-vectorfield. Then for $\omega \in T_xN$, $\xi \llcorner \tilde{\omega}$ is a simple, unit length $(p-1)$-vectorfield, where $\tilde{\omega} = \frac{\omega}{\|\omega\|}$.*

Proof Trivially, by definition, $\xi \llcorner \omega$ is simple. To prove that it has unit length, consider:

$$\langle \xi \llcorner \tilde{\omega}, \xi \llcorner \tilde{\omega}\rangle = \langle \xi, \tilde{\omega} \wedge \xi \llcorner \tilde{\omega}\rangle = \langle \xi, \xi\rangle = 1$$

□

We now introduce the notion of an $(n-1)$-dimensional “slice” of an n-dimensional current.

Definition 2.4.6 (Current Associated with Slice) *Let U open in $\mathbb{R}^{n+k}$, and let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be Lipschitz. Further, let $T(M, \theta, \xi) \in \mathscr{D}_n U$ be an integer multiplicity rectifiable current. Then, we define:*

$$M_t = f^{-1}(t) \cap M_+$$
$$\theta_t(x) = \begin{cases} 0 & \text{if } \nabla^M f(x) = 0 \text{ or } x \notin M_t \\ \theta(x) & \text{otherwise} \end{cases}$$
$$\xi_t(x) = \xi \llcorner \frac{\nabla^M f(x)}{\|\nabla^M f(x)\|}$$

We define $\langle T, f, t\rangle \in \mathscr{D}_{n-1} U$ by:

$$\langle T, f, t\rangle = T(M_t, \theta_t, \xi_t)$$

In the light of Lemma 2.4.5, ξ_t does indeed orient $\langle T, f, t\rangle$.

We now make a connection to the Co-Area formula.

Lemma 2.4.7 *For $M \subseteq \mathbb{R}^{n+k}$ n-rectifiable, $f : \mathbb{R}^{n+k} \to \mathbb{R}$ Lipschitz and M_t as defined previously, the following equation holds:*

$$\int_{-\infty}^{\infty} \left(\int_{M_t} g \, \mathrm{d}\mathscr{H}^{n-1} \right) \mathrm{d}t = \int_M \left| \nabla^M f \right| g \, \mathrm{d}\mathscr{H}^n$$

for any $g \geq 0$ and $\mathscr{H}^n$-measurable.

Proof We note that $\mathrm{J}_M^* f = \left|\nabla^M f\right|$, since $f : \mathbb{R}^{n+k} \to \mathbb{R}$. So, we apply the Co-Area formula (Theorem 1.4.4), and the result follows. □

Lemma 2.4.8 *Let $M \subseteq \mathbb{R}^{n+k}$ be n-rectifiable, $f : \mathbb{R}^{n+k} \to \mathbb{R}$ Lipschitz. Define $A_t = \left\{x \in \mathbb{R}^{n+k} : f(x) < t\right\}$ for $t \in \mathbb{R}$. Then,*

$$\int_{M \cap A_t} \left|\nabla^M f\right| g \, \mathrm{d}\mathscr{H}^n = \int_{-\infty}^{t} \int_{M_s} g \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s$$

for $g \geq 0$ and $\mathscr{H}^n$-measurable.

Proof We associate $g\chi_{A_t}$ with g in Lemma 2.4.7. Consider

$$\int_{-\infty}^{\infty} \int_{M_s} g\chi_{A_t} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s = \int_{-\infty}^{t} \int_{M_s} g\chi_{A_t} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s + \int_{t}^{\infty} \int_{M_s} g\chi_{A_t} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s$$

Now, note that for $x \geq t$, $\chi_{A_t}(x) = 0$. This implies that

$$\int_{t}^{\infty} \int_{M_s} g\chi_{A_t} \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}s = 0$$

and the result follows. □

Lemma 2.4.9 *Let μ be a measure on X and let $f : X \to \mathbb{R}$ be a μ-measurable function. Then,*

$$\int_X f \, \mathrm{d}\mu \leq (\text{ess sup } f)\mu(X)$$

Proof If $\mu(X) = \infty$ or $\operatorname{ess\,sup} f = \infty$, then the result is trivial. So, we assume $\mu(X) < \infty$ and $\operatorname{ess\,sup} f < \infty$. For every α with $f \leq \alpha$ μ–a.e. , we define $H_\alpha = \{x \in X : f(x) \leq \alpha\}$. Let $H'_\alpha = X \setminus H_\alpha$. By construction, $\mu(H'_\alpha) = 0$. Now, $\mu(X) \leq \mu(H_\alpha) + \mu(H'_\alpha) = \mu(H_\alpha)$, and so it follows that $\mu(X) = \mu(H_\alpha)$. Then,

$$\int_X f \,\mathrm{d}\mu \leq \int_{H_\alpha} f \,\mathrm{d}\mu + \int_{H'_\alpha} f \,\mathrm{d}\mu \leq \int_{H_\alpha} \alpha \,\mathrm{d}\mu = \alpha\mu(H_\alpha) = \alpha\mu(X)$$

Now, taking an $\inf$ over all α, we attain the desired result. □

We tally these results together to obtain the following important result.

Theorem 2.4.10 *Let U open in $\mathbb{R}^{n+k}$, $T(M, \theta, \xi) \in \mathscr{D}_n U$, and W open in U. Then,*

$$\int_{-\infty}^{\infty} \mathrm{M}_W(\langle T, f, t\rangle) \,\mathrm{d}t = \int_{M\cap W} \left|\nabla^M f\right| \theta \,\mathrm{d}\mathscr{H}^n \leq (\operatorname{ess\,sup}_{M\cap W} \left|\nabla^M f\right|)\mathrm{M}_W(T)$$

Proof We put $\theta_+(x) = \theta(x)$ whenever $\left|\nabla^M f(x)\right| > 0$, and $\theta_+(x) = 0$ otherwise. By invoking Lemma 2.4.7, and identifying $\theta_+\chi_W$ with g in the lemma, we find that

$$\int_{-\infty}^{\infty} \left(\int_{M_t\cap W} \theta_+ \,\mathrm{d}\mathscr{H}^{n-1}\right) \mathrm{d}t = \int_{M\cap W} \left|\nabla^M f\right| \theta \,\mathrm{d}\mathscr{H}^n$$

The result follows by definition of $\mathrm{M}_W(\langle T, f, t\rangle)$ (since $\langle T, f, t\rangle$ is integer rectifiable for $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$), and by invoking Lemma 2.4.9. □

The following result illustrates an important algebraic expression for slices. The proof can be found in [Sim83, 28.5].

Theorem 2.4.11 (Slicing Formula) *Let U be open in $\mathbb{R}^{n+k}$, and let $T(M, \theta, \xi) \in \mathscr{D}_n U$. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be Lipschitz, and suppose $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$. Then:*

1. $\langle T, f, t\rangle = \partial (T \llcorner R) - (\partial T)\llcorner R$, *where* $R = \{x : f(x) < t\}$
2. $\langle \partial T, f, t\rangle = -\partial\langle T, f, t\rangle$

Motivated by this, we can now define slicing for a general current. We define the upper and lower slices:

Definition 2.4.12 (Slice of a Current) *Let U open in $\mathbb{R}^{n+k}$ and suppose $T \in \mathscr{D}_n U$ with $\mathrm{M}_W(T)+\mathrm{M}_W(\partial T) < \infty$, for all $W \Subset U$. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be Lipschitz. For $t \in \mathbb{R}$, let $S_l = \left\{x \in \mathbb{R}^{n+k} : f(x) < t\right\}$, and $S_u = \left\{x \in \mathbb{R}^{n+k} : f(x) > t\right\}$. Then, we define the upper and lower slices respectively by:*

$$\langle T, f, t\rangle_- = \partial(T \llcorner S_l) - (\partial T)\llcorner S_l$$
$$\langle T, f, t\rangle_+ = -\partial(T \llcorner S_u) + (\partial T)\llcorner S_u$$

And we write $\langle T, f, t\rangle$ when we have $\langle T, f, t\rangle_- = \langle T, f, t\rangle_+$.

It is important to note that we have $\langle T, f, t\rangle_- = \langle T, f, t\rangle_+$ for all but countably many values of t. So, indeed, we can write $\langle T, f, t\rangle$, for $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$. See [Sim83, p161].

2.5 Densities

We begin our discussion of measure theoretic densities by considering some facts about coverings.

Definition 2.5.1 (Fine Covering) *Let X be a metric space. Let $\mathscr{C}$ be a covering of X with closed balls. Then, if $\inf\{\operatorname{diam}(B) : B \in \mathscr{C}\} = 0$, then $\mathscr{C}$ is called a fine covering of X.*

The following two are important covering facts. Their proofs can be found in [Sim83, 3.3, 3.4].

Lemma 2.5.2 *Let $\mathscr{B}$ a family of closed balls in a metric space X. Suppose $R = \sup\{\operatorname{diam}(B) : B \in \mathscr{B}\} < \infty$. Then there exists a pairwise disjoint subcollection $\mathscr{C} \subseteq \mathscr{B}$ such that*

$$\bigcup \mathscr{B} \subseteq \bigcup 5\mathscr{C}$$

such that if $B \in \mathscr{B}$, there exists an $C \in \mathscr{C}$ such that $B \cap C \neq \emptyset$ and $B \subseteq 5F$.

Corollary 2.5.3 *Suppose $\mathscr{B}$ covers $A \subseteq X$. For every such subcover $\mathscr{C}$ as given in Lemma 2.5.2, given a finite subcollection $\{F_1, \ldots, F_n\} \subseteq \mathscr{C}$, we have:*

$$A \setminus \bigcup_{i=1}^{n} F_i \subseteq \bigcup 5(\mathscr{C} \setminus \{F_1, \ldots, F_n\})$$

We now present the following important result.

Corollary 2.5.4 *Let μ be a Borel measure on X such that $\mu(X) < \infty$. Suppose that for each $B \in \mathscr{B}$, $\mu(X \cap B) > 0$. Then the disjoint subcollection $\mathscr{C}$ is countable.*

Proof Suppose $\mathscr{C}$ is uncountable. For each $\varepsilon > 0$, we define:

$$\mathscr{C}_\varepsilon = \{B \in \mathscr{C} : \mu(X \cap B) > \varepsilon\}$$

Now, we can find a $\delta > 0$ such that $\mathscr{C}_\delta$ is uncountable. Such a collection must exist, because otherwise, we can consider $\bigcup \mathscr{C}_{\frac{1}{n}} = \mathscr{C}$, a countable union of countable sets which is again countable.

Now, let $\{C_1, C_2, \ldots\} \subseteq \mathscr{C}_\delta$ be a countably infinite subset. Since μ is Borel and $\mathscr{C}_\delta$ pairwise disjoint, we have:

$$\mu\left(X \cap \bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} \mu(X \cap C_i) = \infty$$

since each $\mu(X \cap C_i) > \delta > 0$. But, we have that:

$$\mu\left(X \cap \bigcup_{i=1}^{\infty} C_i\right) \leq \mu\left(X \cap \bigcup \mathscr{C}\right) \leq \mu(X) < \infty$$

which is a contradiction. □

Now, we consider densities and some important facts about densities.

Definition 2.5.5 (Upper/Lower Density) *Let μ be a Borel Regular measure on $\mathbb{R}^{n+k}$. Let $A \subseteq \mathbb{R}^{n+k}$. For $x \in \mathbb{R}^{n+k}$, we define*

$$\Theta^{*n}(\mu, A, x) = \limsup_{\sigma \to 0} \frac{\mu(A \cap B_\sigma(x))}{\omega_n \sigma^n}$$
$$\Theta^{n}_{*}(\mu, A, x) = \liminf_{\sigma \to 0} \frac{\mu(A \cap B_\sigma(x))}{\omega_n \sigma^n}$$

where ω_n is the volume of an n-ball in X. We call $\Theta^{*n}(\mu, A, x)$ the upper n-density of x in A, and $\Theta^{n}_{*}(\mu, A, x)$ the lower n-density. Where these quantities agree, we simply call it the density $\Theta^n(\mu, A, x)$.

We now illustrate some important density results.

Theorem 2.5.6 *Let μ a Borel Regular measure on $\mathbb{R}^{n+k}$. Fix $t \geq 0$. Then, if $A_1 \subseteq A_2 \subseteq \mathbb{R}^{n+k}$, and $\Theta^{*n}(\mu, A_2, x) \geq t$ for all $x \in A_1$, then $t\mathscr{H}^n(A_1) \leq \mu(A_2)$.*

Proof Our proof is essentially the same as [Sim83, 3.2].

Now, if $\mu(A_2) = \infty$ or $t = 0$, there's nothing to do. So, suppose $\mu(A_2) < \infty$, and $t > 0$. Now, fix $\infty > \varepsilon > 0$, and we construct:

$$\mathscr{B}_\varepsilon = \left\{ B_\delta(x) \text{ closed in } \mathbb{R}^{n+k} : x \in A_1, 0 < \delta < \frac{\varepsilon}{2}, \mu(A_2 \cap B_\delta(x)) > t\omega_n\delta^n \right\}$$

Now, trivially, $\mathscr{B}_\varepsilon \neq \emptyset$, since $\Theta^{*n}(\mu, A, x) > t$. By construction, $\mathscr{B}_\varepsilon$ is a fine covering and $R = \sup\{\text{diam}(B) : B \in \mathscr{B}\} < \varepsilon < \infty$.

We invoke Corollary 2.5.3 and Lemma 2.5.4 (since $\mu(\mathbb{R}^{n+k} \cap B) > 0$, for $B \in \mathscr{B}$) to find a countable pairwise disjoint subset $\mathscr{C} = \{C_1, C_2, \ldots\} \subseteq \mathscr{B}$ such that:

$$A_1 \subseteq \bigcup_{i=1}^{n} C_i \cup \bigcup_{i=n+1}^{\infty} 5C_i$$

Now, by the definition of the Hausdorff measure,

$$\mathscr{H}^n_{5\varepsilon}(A_1) \leq \sum_{i=1}^{n} \omega_n \left(\frac{\text{diam}(C_i)}{2} \right)^n + 5^n \sum_{i=n+1}^{\infty} \omega_n \left(\frac{\text{diam}(C_i)}{2} \right)^n \to \sum_{i=1}^{\infty} \omega_n \left(\frac{\text{diam}(C_i)}{2} \right)^n$$

under the limit as $n \to \infty$. By the construction of $\mathscr{B}_\varepsilon$, we have

$$\mathscr{H}^n_{5\varepsilon}(A_1) \leq \sum_{i=1}^{\infty} \omega_n \left(\frac{\text{diam}(C_i)}{2} \right)^n \leq \sum_{i=1}^{\infty} t^{-1}\mu(A_2 \cap C_i) \leq t^{-1}\mu\left(\bigcup_{i=1}^{\infty} A_2 \cap C_i \right) \leq t^{-1}\mu(A_2)$$

We let the limit $\varepsilon \to 0$, and the result follows. □

Theorem 2.5.7 *Let $A \subseteq \mathbb{R}^{n+k}$ with $\mathscr{H}^n(A) < \infty$. Then $\Theta^{*n}(\mathscr{H}^n, A, x) \leq 1$ for $\mathscr{H}^n-$a.e. $x \in A$.*

Proof This proof is similar to [Sim83, 3.6].

Fix $\varepsilon, t > 0$. Define:

$$A_t = \{x \in A : \Theta^{*n}(\mathscr{H}^n, A, x) \geq t\}$$

By [Sim83, 1.3], we find an U_ε open in X with $A_t \subseteq U_\varepsilon$ with

$$\mathscr{H}^n(A \cap U_\varepsilon) \leq \mathscr{H}^n(A_t) + \varepsilon$$

By Theorem 2.5.6, with $A_1 = A_t, A_2 = A \cap U_\varepsilon$, we have that

$$t\mathscr{H}^n(A_t) \leq \mathscr{H}^n(A \cap U_\varepsilon) \leq \mathscr{H}^n(A_t) + \varepsilon$$

Let $\{t_i\}$ be a decreasing sequence with $t_i \to 1$. Trivially, $A_1 = \bigcup_{i=1}^{\infty} A_{t_i}$. So, we have the estimate:

$$\begin{aligned} &\mathscr{H}^n(A_{t_i}) \leq t_i\mathscr{H}^n(A_{t_i}) \leq \mathscr{H}^n(A \cap U_\varepsilon) \leq \mathscr{H}^n(A_1) + \varepsilon \\ \implies &\mathscr{H}^n(A_1) \leq \mathscr{H}^n(A \cap U_\varepsilon) \leq \mathscr{H}^n(A_1) + \varepsilon \end{aligned}$$

Now, as $\varepsilon \to 0$, we have $\mathscr{H}^n(A \cap U_\varepsilon) \to 0$, and it follows that $\mathscr{H}^n(A_1) = 0$. □

2.6 Lebesgue Points

We begin with the following definition.

Definition 2.6.1 (Lebesgue Point) *Let X be a metric space, and μ a Radon measure on X. Let $f : X \to \mathbb{R}$ be μ-measurable. If for some $x \in X$,*

$$\lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \|f(y) - f(x)\| \, \mathrm{d}\mu(y) = 0$$

then x is called a Lebesgue point of f.

Our aim in this section is to introduce enough theory to prove an important fact about Lebesgue points of f in $\mathbb{R}^{n+k}$.

Definition 2.6.2 (Symmetric Vitali Property) *Let X be a metric space and μ a Radon measure on X. Let $\mathscr{B}$ be a collection of closed balls in X, and let $\mathscr{C} = \{x \in X : B_\varepsilon(x) \in \mathscr{B}\}$ denote the centres of $\mathscr{B}$. If whenever $\mu(C) < \infty$ we can find a countable subcollection $\mathscr{B}'$ covering $\mu-\text{a.e. } x \in C$, then we say that X has the Symmetric Vitali Property with respect to μ.*

Definition 2.6.3 (Absolutely Continous) *Let μ, ν be measures on a set X. If whenever $\mu(F) = 0$ we have $\nu(F) = 0$, then we say that ν is absolutely continuous with respect to μ. We write $\nu \ll \mu$.*

Now we prove a series of necessary results.

Lemma 2.6.4 *Let X be a metric space, and let μ be a Borel Regular measure on X. Let $f : X \to \mathbb{R}^+$ be a μ-measurable function. Define:*

$$\nu(B) = \int_B f \, \mathrm{d}\mu, \ B \text{ Borel}$$
$$\nu(A) = \inf\{\nu(B) : A \subseteq B, B \text{ Borel}\}$$

Then ν is:

1. *Borel Regular measure and absolutely continuous w.r.t μ*
2. *Radon if f is μ-summable.*

Proof Trivially, $\nu(\emptyset) = 0$, and if $A \subseteq B$, $\nu(A) \leq \nu(B)$ since $f \geq 0$. Equally as trivially, we have $\int_{\bigcup_{i=1}^\infty A_i} f \, \mathrm{d}\mu \leq \sum_{i=1}^\infty \int_{A_i} f \, \mathrm{d}\mu$ which illustrates that ν is subadditive. These facts establish that ν is indeed a measure.

We now show that ν is Borel. Let $A, B \subseteq X$ be such that $d(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\} > 0$, where, ρ is the metric on X. Since μ is Borel Regular, we find $A', B' \subseteq X$ Borel such that $\mu(A) = \mu(A')$ and $\mu(B) = \mu(B')$.

Now, we can assume that $A' \cap B' = \emptyset$. This follows from the fact that $d(A, B) = d(\overline{A}, \overline{B}) > 0$ which implies $\overline{A} \cap \overline{B} = \emptyset$, and since $\overline{A}, \overline{B}$ are Borel, so are $A' \cap \overline{A}$ and $B' \cap \overline{B}$.

Now, since μ is Borel, we have that A', B' are measurable and it follows that since $A' \cap B' = \emptyset$ and f is μ-measurable,

$$\int_{A' \cup B'} f \, \mathrm{d}\mu = \int_{A'} f \, \mathrm{d}\mu + \int_{B'} f \, \mathrm{d}\mu$$

Now, $\nu(A \cup B) = \inf\left\{\int_{A' \cup B'} f \, \mathrm{d}\mu\right\}$, since if $A \subseteq C \subseteq A' \cup B'$ with C Borel, then $A \subseteq A' \cap C$ and $B \subseteq B' \cap C$, so under infimum over such A', B', we have $\nu(A \cup B) = \nu(A) + \nu(B)$. By application of Caratheodoré's criterion [Fed96, 2.3.2], we establish that ν is Borel.

Next, we prove that ν is Borel Regular. Let $A \subseteq X$. Define,

$$F_A = \left\{ \int_B f \, \mathrm{d}\mu : A \subseteq B, B \text{ Borel} \right\}$$

If $\nu(A) = \inf F_A = \infty$, then we can simply take X as our Borel set and we're done. So, assume that $\nu(A) = \inf F_A < \infty$. Now $F_A \subseteq \mathbb{R}$ bounded, and since $\inf F_A$ is a limit point, by the fact that $\mathbb{R}$ is first countable we invoke the Sequence Lemma [Mun96, 21.2] and find a sequence of Borel B_i such that

$$\int_{B_i} f \, \mathrm{d}\mu \to \nu(A)$$

By construction, $\int_{B_i} f \, \mathrm{d}\mu \geq \nu(A)$ and since each B_i is Borel, and the space of Borel sets is a σ-algebra, $B = \bigcap_{i=1}^{\infty} B_i$ is Borel. It follows then that:

$$\begin{aligned}
&\int_{B_i} f \, \mathrm{d}\mu \geq \int_B f \, \mathrm{d}\mu \\
&\implies \lim_{i\to\infty} \int_{B_i} f \, \mathrm{d}\mu \geq \int_B f \, \mathrm{d}\mu \\
&\implies \nu(A) \geq \int_B f \, \mathrm{d}\mu
\end{aligned}$$

From the fact that $A \subseteq B$, it follows that $\nu(A) = \nu(B)$.

Now, suppose $F \subseteq X$ Borel with $\mu(F) = 0$. Then trivially,

$$\nu(F) = \int_F f \, \mathrm{d}\mu = \sup \left\{ \sum_{i=1}^{n} a_i \mu(\phi^{-1}(a_i)) : \phi < f, \phi \text{ simple} \right\} = \sup\{0\} = 0$$

which establishes the absolute continuity conclusion.

Lastly, if f is μ-summable, ν attains a finite measure on all sets, and it follows that ν is Radon. □

Now we have the following result for positive locally summable functions.

Lemma 2.6.5 *Let X be a second countable metric space, and μ a measure on X having Symmetric Vitali Property with respect to μ. Let $f : X \to \mathbb{R}^+$ be locally μ-summable. Then,*

$$\lim_{r\to 0} \frac{1}{\mu(B_r(0))} \int_{B_r(0)} f(y) \, \mathrm{d}\mu(y) = f(x)$$

Proof Since X is second countable, let $\mathscr{B}$ be a countable basis for X. By the locally summable hypothesis, for each $x \in X$, there exists a basis $B_x \in \mathscr{B}$ such that $f|_{B_x}$ is μ-summable. Let,

$$\mathscr{C} = \{B_x \in \mathscr{B} : f|_{B_x} \text{ is } \mu\text{-summable}, x \in X\}$$

Trivially, $\mathscr{C}$ is a countable open covering of X.

Now define $\nu(A) = \inf\left\{\int_B f \, \mathrm{d}\mu : A \subseteq B, B \text{ Borel}\right\}$. Then, for every $B_i \in \mathscr{C}$, since $f|_{B_i}$ is μ-summable, we invoke Lemma 2.6.4, to find $\nu \llcorner B_i$ Radon measure and $\nu \llcorner B_i \ll \mu$. By Radon-Nikodym Theorem [Sim83, 4.7], we find:

$$\begin{aligned}
\mathrm{D}_\mu \nu \llcorner B_i(x) &= \lim_{r\to 0} \frac{\nu(B_r(x)}{\mu(B_r(x)} \\
\nu \llcorner B_i(A) &= \int_A \mathrm{D}_\mu \nu \llcorner B_i(x) \, \mathrm{d}\mu(x), A \text{ Borel}
\end{aligned}$$

for μ−a.e. $x \in B_i$. Since

$$\nu \llcorner B_i(A) = \int_{A\cap B_i} f(x) \, \mathrm{d}\mu(x) = \int_A \lim_{r\to 0} \frac{1}{\mu B_r(x)} \left(\int_{B_r(x)} f(y) \, \mathrm{d}\mu(y) \right) \mathrm{d}\mu(x)$$

for every Borel A, it follows that

$$f(x) = \lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f(y)\, \mathrm{d}\mu(y)$$

Now, for each B_i, let C_i be the largest set for which this equality does not hold. Since we are guaranteed that equality holds for μ−a.e. $x \in B_i$, it follows that $\mu(C_i) = 0$. By the fact that $\mathscr{C}$ is an open covering of X, it follows that equality fails on $\bigcup_{i=1}^{\infty} C_i$ and subadditivity of μ ensures that this is a set of measure zero. Then, we have that the required result holds for μ−a.e. $x \in X$. □

We now prove this important fact about Lebesgue points of locally summable functions in $\mathbb{R}^{n+k}$.

Theorem 2.6.6 (Lebesgue Points) *Let μ be Radon in $\mathbb{R}^{n+k}$, and let $f : \mathbb{R}^{n+k} \to \mathbb{R}^m$ be locally μ-summable. Then μ−a.e. $x \in \mathbb{R}^{n+k}$ is a Lebesgue point of f.*

Proof Since μ is Radon by hypothesis, by [Sim83, 4.6], we have that $\mathbb{R}^{n+k}$ has Symmetric Vitali Relation w.r.t. μ. Define $g_z : \mathbb{R}^{n+k} \to \mathbb{R}$ by:

$$g_z(x) = \| f(x) - z\|$$

for $z \in \mathbb{R}$. Now, g_z is trivially locally μ-summable. We apply Lemma 2.6.5 to g_z to find that

$$g_z(x) = \lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} g_z(y)\, \mathrm{d}\mu(y)$$
$$\iff \| f(x) - z\| = \lim_{r\to 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} \| f(y) - z\|\, \mathrm{d}\mu(y)$$

for μ−a.e. $x \in \mathbb{R}^{n+k}$. But $z \in \mathbb{R}$ was arbitrary, and so we put $z = f(x)$ and the result follows. □

Chapter 3

Some Important Lemmas

3.1 The Slicing Lemma

We now continue to prove an important result regarding convergence in sequences of slices. Firstly, we present two lemmas.

Lemma 3.1.1 *Let U open in $\mathbb{R}^{n+k}$ and let $T \in \mathscr{D}_n U$. Then $\mathrm{M}_W(T) \geq 0$ for $W \Subset U$.*

Proof Assume that $\mathrm{M}_W(T) < 0$. So, for every $\omega \in \wedge^n \mathbb{R}^{n+k}$ with $\|\omega\| \leq 1$ and $\mathrm{spt}\,\omega \subseteq W$, $T(\omega) < 0$. We note that $\|-\omega\| = \|\omega\|$, and trivially, $\mathrm{spt}\,(-\omega) \subseteq W$. By linearity of T, we have $T(-\omega) = -T(\omega) > 0$. But clearly, this is a contradiction. □

Lemma 3.1.2 *Let U open in $\mathbb{R}^{n+k}$ and $T \in \mathscr{D}_n U$. Further, assume $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$. Then for $f : \mathbb{R}^{n+k} \to \mathbb{R}$ Lipschitz,*

$$\mathrm{M}_W(\langle T, f, r \rangle) \leq \sqrt{n+k}(\mathrm{Lip}\,(f))\mathrm{M}_W(T)$$

Proof By Lemma 3.1.1 and by the lower semicontinuity of mass [Sim83, 26.13], for some $\varepsilon > 0$,

$$\mathrm{M}_W(\langle T, f, r \rangle) \leq \int_{r-\varepsilon}^{r+\varepsilon} \mathrm{M}_W(\langle T, f, t \rangle)\,\mathrm{d}t$$

Now, for $\omega \in \mathscr{D}^n U$, and $\|\omega\| \leq 1$ with $\mathrm{spt}\,\omega \subseteq W$, with $\langle \omega(x), \xi(x) \rangle \geq 0$, we have that:

$$\begin{aligned} T \llcorner \{x : r - \varepsilon < f(x) < r + \varepsilon\}(\omega) &= \int_{T \cap \{x : r-\varepsilon < f(x) < r+\varepsilon\}} \langle \omega(x), \xi(x) \rangle\,\mathrm{d}\mu_T \\ &\leq \int_T \langle \omega(x), \xi(x) \rangle\,\mathrm{d}\mu_T \\ &= T(\omega) \end{aligned}$$

So, it follows then that $\mathrm{M}_W(T \llcorner \{x : r - \varepsilon < f(x) < r + \varepsilon\}) \leq \mathrm{M}_W(T)$.

Now, since f is Lipschitz, we apply Lemma 1.2.3, to find $\mathrm{ess\,sup}_W \|\nabla f(x)\| \leq \sqrt{n+k}(\mathrm{Lip}\,(f))$. The result follows by combining our previous estimates, in the light of [Sim83, 28.10]. □

Now we present the important Slicing Lemma:

Theorem 3.1.3 (Slicing Lemma) *Let U open in $\mathbb{R}^{n+k}$, and $f : U \to \mathbb{R}$ be Lipschitz. Let $\{T_i\} \subseteq \mathscr{D}_n U$ be a sequence of currents such that for every $W \Subset U$,*

$$\sup\{\mathrm{M}_W(T_i) + \mathrm{M}_W(\partial T_i) : i \in \mathbb{N}\} < \infty \qquad \textit{(and)}$$
$$T_i \rightharpoonup T$$

Then, for $\mathscr{L}^1$−a.e. $r \in \mathbb{R}$, there is a subsequence i' such that:

$$\langle T_{i'}, f, r\rangle \rightharpoonup \langle T, f, r\rangle \qquad \textit{(and)}$$
$$\sup\{\mathrm{M}_W(\langle T_{i'}, f, r\rangle) + \mathrm{M}_W(\partial\langle T_{i'}, f, r\rangle) : i \in \mathbb{N}\} < \infty$$

Further, if for some $W_0 \Subset U$ $\lim_{i\to\infty}(\mathrm{M}_{W_0}(T_i) + \mathrm{M}_{W_0}\partial T_i)) = 0$, then we can chose i' such that

$$\lim_{i'\to\infty}(\mathrm{M}_{W_0}(\langle T_{i'}, f, r\rangle) + \mathrm{M}_{W_0}\partial\langle T_{i'}, f, r\rangle) = 0$$

Proof We firstly note that the convergence $\rightharpoonup$ is in the sense of pointwise convergence of measures $\mu(f) = \int_X f\,\mathrm{d}\mu$.

Since $T_i, \partial T_i$ are currents, we invoke Theorem 2.2.8, to find measures $\mu_{T_i}, \mu_{\partial T_i}$, and ξ, ξ' respectively n and $(n-1)$-vectorfields such that we can represent T_i and ∂T_i as an integral. Now, by [Sim83, 4.4], we can find a subsequence i' such that $\mu_{T_{i'}} \rightharpoonup \mu_T$ and $\mu_{\partial T_{i'}} \rightharpoonup \mu_{\partial T}$. Now, fix $\omega \in \mathscr{D}^{n-1}U$, and set $f = \langle \mathrm{d}\omega, \xi\rangle\chi_R$ and $g = \langle \omega, \xi'\rangle\chi_R$, where $R = \{x : f(x) < r\}$. Note from our construction that:

$$\mu_{T_{i'}}(f) = \int_{M_{i'}} \langle \mathrm{d}\omega(x), \xi(x)\rangle\,\mathrm{d}\mu_{T_{i'}} = \partial(T_{i'}\llcorner R)(\omega)$$
$$\mu_{\partial T_{i'}}(g) = \int_{\tilde{M}_{i'}} \langle \omega(x), \xi'(x)\rangle\,\mathrm{d}\mu_{\partial T_{i'}} = (\partial T_{i'})\llcorner R(\omega)$$

It follows then that:

$$\begin{aligned}
\lim_{i'\to\infty}\langle T_{i'}, f, r\rangle(\omega) &= \lim_{i'\to\infty}(\partial(T_{i'}\llcorner R) - (\partial T_{i'})\llcorner R)(\omega) \\
&= \lim_{i'\to\infty}\left(\mu_{T_{i'}}(f) - \mu_{\partial T_{i'}}(g)\right) \\
&= \mu_T(f) - \mu_{\partial T}(g) \\
&= \partial(T\llcorner R)(\omega) - (\partial T)\llcorner R(\omega) \\
&= \langle T, f, r\rangle(\omega)
\end{aligned}$$

and $\langle T_{i'}, f, r\rangle \rightharpoonup \langle T, f, r\rangle$ is established.

Fix $W \Subset U$. Then by Lemma 3.1.2,

$$\mathrm{M}_W(\langle T_{i'}, f, r\rangle) + \mathrm{M}_W(\langle \partial T_{i'}, f, r\rangle) \le \sqrt{n+k}(\mathrm{Lip}\,(f))\,(\mathrm{M}_W(T_{i'}) + \mathrm{M}_W(\partial T_{i'}, f, r)) < \infty$$

and trivially follows that $\sup\{\mathrm{M}_W(\langle T_{i'}, f, r\rangle + \mathrm{M}_W\partial\langle T_{i'}, f, r\rangle : i' \subseteq i\} < \infty$.

Now suppose for some $W_0 \Subset U$, we have $\lim_{i\to\infty}(\mathrm{M}_{W_0}(T_i) + \mathrm{M}_{W_0}\partial T_i)) = 0$. Then, we can choose a monotonically decreasing subsequence i'' such that:

$$\mathrm{M}_{W_0}(T_{i''+1}) + \mathrm{M}_{W_0}(\partial T_{i''+1}) \le \mathrm{M}_{W_0}(T_{i''}) + \mathrm{M}_{W_0}(\partial T_{i''})$$

Then we apply our earlier construction to this sequence i'', and the conclusion follows. □

3.2 The Lower Density Lemma

We begin with an important result relating the mass of a current to its cone. A general treatment of cones can be found [Sim83, 26.26]. Now we introduce the notion of a cone over a point.

Definition 3.2.1 (Cone over a point) *Let U open in $\mathbb{R}^{n+k}$ with $T \in \mathscr{D}_n U$ and $\mathbf{M}_W(T) + \mathbf{M}_W(\partial T) < \infty$. Let $q \in U$, then the cone centred at q is given by:*

$$q \times T = 0 \times f_\sharp T$$

where $f(x) = x - q$.

Now we present the following important results.

Lemma 3.2.2 *Let $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function. Then, $\|\mathrm{d}(\mathrm{id}_x)\| = 1$, for all $x \in \mathbb{R}^n$.*

Proof We write $\mathrm{d}(\mathrm{id}_x)$ explicitly. Note, we have $\mathrm{D}(\mathrm{id})(x) = I$ for all $x \in \mathbb{R}^n$. It follows then that: $\mathrm{d}(\mathrm{id}_x) = \sum_{i=1}^n \mathrm{d}x^i$. It follows that:

$$\|\mathrm{d}(\mathrm{id}_x)\| = \sup\{\langle \mathrm{d}(\mathrm{id}_x), v\rangle : \|v\| \leq 1, v \in \wedge_n \mathbb{R}^n\} = 1$$

□

Lemma 3.2.3 *Let U open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$. Further suppose that $\mathbf{M}_W(T) + \mathbf{M}_W(\partial T) < \infty$ for all $W \Subset U$. Let $B_r(q)$ be a ball of radius r > 0. Then,*

$$\mathbf{M}(q \times T \llcorner B_r(q)) \leq r\mathbf{M}(T \llcorner B_r(q))$$

Proof Let $f(x) = x - q$. We firstly prove that $\mathbf{M}(f_\sharp T) \leq \mathbf{M}(T)$. Fix $W \Subset f(U)$. We note that $\mathrm{D}f = I$, and $\operatorname{ess\,sup}_{f^{-1}(W)} \|\mathrm{D}f\| = 1$. The conclusion follows by applying [Sim83, 26.25], since $\operatorname{spt} f$ is trivially proper [Sim83, p137].

Now, define $h : [0,1] \times U \to \mathbb{R}^{n+k}$ by $h(t,x) = tx$. We have $k(x) = h(0,x) = 0$ and $g(x) = h(1,x) = x = \mathrm{id}_x$, and by definition:

$$q \times T \llcorner B_r(q) = h_\sharp([\![0,1]\!] \times f_\sharp T \llcorner B_r(q)) = h_\sharp([\![0,1]\!] \times (f_\sharp T) \llcorner B_r(0))$$

Now, we note that $\|\mathrm{d}d_x\| = 0$ and $\|\mathrm{d}g_x\| = 1$ for all $x \in U$ by Lemma 3.2.2. Also, $\sup\left\{\|(k-g)(x)\| : x \in \operatorname{spt}(f_\sharp T) \llcorner B_r(0)\right\} \leq r$. Then:

$$\begin{aligned}
\mathbf{M}(q \times T \llcorner B_r(q)) &= \mathbf{M}(0 \times (f_\sharp T) \llcorner B_r(0)) \\
&\leq \sup\left\{\|(k-g)(x)\| : x \in \operatorname{spt}(f_\sharp T) \llcorner B_r(0)\right\} \sup\left\{\|\mathrm{d}k_x\| + \|\mathrm{d}g_x\| : x \in \operatorname{spt}(f_\sharp T) \llcorner B_r(0)\right\} \mathbf{M}((f_\sharp T) \llcorner B_r(0)) \\
&\qquad \textit{(By [Sim83, 26.23])} \\
&= r\mathbf{M}((f_\sharp T) \llcorner B_r(0)) \\
&\leq r\mathbf{M}(T \llcorner B_r(q))
\end{aligned}$$

□

The following topological result is useful in the discussion to follow.

Lemma 3.2.4 *Let X be a second countable, Hausdorff, locally compact space. Then,*

$$X = \bigcup_{i=1}^{\infty} V_i$$

where each $V_i \Subset X$.

Proof Since X is Hausdorff and locally compact, we invoke [Mun96, 29.2] and for each $x \in X$, we can find V_x open in X and $\overline{V}_x$ compact in X. Let $\mathscr{C} = \left\{V_x \text{ open in } X : \overline{V}_x \text{ compact in } X, x \in X\right\}$. Trivially, $\mathscr{C}$ is an open covering for X.

Now, by the second countable hypothesis on X, we apply [Mun96, 30.3] to find that X is Lindelöf. So, there exists a countable subcover $\mathscr{C}' = \{V_1, V_2, \ldots\} \subseteq \mathscr{C}$. By construction of $\mathscr{C}$, the result follows. □

Lemma 3.2.5 *Let U open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$. Suppose $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$ for all $W \Subset U$. Then for μ_T-a.e. $x \in U$,*

$$\lim_{r\to 0} \frac{\lambda(x,r)}{\mu_T(B_r(x))} = 1$$

where $\lambda(x,r) = \inf\{\mathrm{M}(S) : \partial S = \partial(T \llcorner B_r(x)), S \in \mathscr{D}_n U\}$.

Proof Our proof is a detailed exposition of [Whi89, p210]. For contradiction, assume that for μ_T-a.e. $x \in U$,

$$\lim_{r\to 0} \frac{\lambda(x,r)}{\mu_T(B_r(x))} \neq 1$$

We firstly note that for any $r > 0$, $\mu_T(B_r(x)) = \mathrm{M}_{B_r(x)}(T) = \mathrm{M}(T \llcorner B_r(x))$. Then, since $\mathrm{M}(T \llcorner B_r(x)) \in \{\mathrm{M}(S) : \partial S = \partial(T \llcorner B_r(x)), S \in \mathscr{D}_n U\}$, we have $\lambda(x,r) \leq \mathrm{M}(T \llcorner B_r(x)) = \mu_T(B_r(x))$.

So, this implies that there exists $X \subseteq U$, with $\mu_T(X) > 0$ such that for all $x \in X$:

$$\lim_{r\to 0} \frac{\lambda(x,r)}{\mu_T(B_r(x))} < 1$$

Take an $\varepsilon \in (0, 1 - \lim_{r\to 0} \frac{\lambda(x,r)}{\mu_T(B_r(x))})$., and it follows that there exists an $R > 0$ such that whenever $r < R$ implies:

$$\frac{\lambda(x,r)}{\mu_T(B_r(x))} < (1-\varepsilon) \iff \lambda(x,r) < (1-\varepsilon)\mathrm{M}(T \llcorner B_r(x))$$

Now, we show that X can be chosen $X \subseteq W$ for some $W \Subset U$. Trivially, $\mathbb{R}^{n+k}$ is second countable, locally compact, and Hausdorff. Since U open in $\mathbb{R}^{n+k}$, it is also locally compact, second countable, and Hausdorff in the subspace topology. So, by Lemma 3.2.4, we can write:

$$U = \bigcup_{i=1}^{\infty} V_i$$

with each $V_i \Subset U$. So,

$$\mu_T(X) = \mu_T(U \cap X) \leq \sum_{i=1}^{\infty} \mu_T(V_i \cap X)$$

and since $\mu_T(X) > 0$, there must exist a V_i with $\mu_T(V_i \cap X) > 0$. So, we can identify X with $V_i \cap X \subseteq V_i \Subset U$.

Now, let

$$\mathscr{B} = \left\{\overline{B}_r(x) : r < R, x \in X\right\}$$

Then, by the Besicovitch Covering Lemma [Sim83, 4.6], there exists a countable subcollection $\mathscr{B}' \subseteq \mathscr{B}$, such that $\bigcup \mathscr{B}'$ covers μ_T-a.e. $x \in X$.

We show that for each $\overline{B}_i \in \mathscr{B}'$ (note: i is an index and not a radius), there exists an $S_i \in \mathscr{D}_n U$ such that $\partial S_i = \partial(T \llcorner B_i)$ with $\mathrm{M}(S_i) < (1-\varepsilon)\mathrm{M}(T \llcorner B_i)$. Now, by construction, for all $\overline{B_i} \in \mathscr{B}'$, $\frac{\mathrm{diam}(B_i)}{2} < R$, and so by previous argument,

$$\frac{\inf\{\mathrm{M}(S) : \partial S = \partial(T \llcorner B_i)\}}{\mathrm{M}(T \llcorner B_i)} < (1-\varepsilon)$$

By definition, for all $\delta > 0$, we have $\mathrm{M}S$ with $\mathrm{M}(S) < \inf\{\mathrm{M}(S) : \partial S = \partial(T \llcorner B_i)\} + \delta$. So, we choose $\delta \in (0, (1 - \varepsilon)\mathrm{M}(T \llcorner B_i) - \lambda(x,r))$ and we have a S_i such that $\mathrm{M}(S_i) < (1-\varepsilon)\mathrm{M}(T \llcorner B_i)$.

Now, let $T_R = T - \sum_{i=1}^{\infty} T\llcorner B_i(q_i) + \sum_{i=1}^{\infty} S_i$. Then, for all $\omega \in \mathscr{D}^n U$,

$$
\begin{aligned}
(T - T_R)(\omega) &= \sum_{i=1}^{\infty} (T\llcorner B_i(q_i) - S_i)(\omega) \\
&= \sum_{i=1}^{\infty} (\partial(q_i \times\!\!\!\times (T\llcorner B_i(q_i) - S_i)))(\omega) \\
&\qquad \textit{(By [Sim83, 26.26], since } \partial(T\llcorner B_i - S_i) = 0\textit{)} \\
&= \sum_{i=1}^{\infty} (q_i \times\!\!\!\times (T\llcorner B_i(q_i) - S_i))(\mathrm{d}\omega) \\
&\le \sum_{i=1}^{\infty} \mathrm{M}(q_i \times\!\!\!\times (T\llcorner B_i(q_i) - S_i)) \sup\{\|\mathrm{d}\omega\| : \|\omega\| \le 1\} \\
&\le \sum_{i=1}^{\infty} \frac{\operatorname{diam}(B_i(q_i))}{2} \mathrm{M}(T\llcorner B_i(q_i) - S_i) \sup\{\|\mathrm{d}\omega\| : \|\omega\| \le 1\} \\
&\qquad \textit{(By Lemma 3.2.3)} \\
&\le R \sum_{i=1}^{\infty} \mathrm{M}(T\llcorner B_i(q_i) - S_i) \sup\{\|\mathrm{d}\omega\| : \|\omega\| \le 1\} \\
&\le 2R \sum_{i=1}^{\infty} \mathrm{M}(T\llcorner B_i(q_i)) \sup\{\|\mathrm{d}\omega\| : \|\omega\| \le 1\} \\
&\qquad \textit{(Since } \mathrm{M}(S) < (1-\varepsilon)\mathrm{M}(T\llcorner B_i)\textit{)} \\
&\le 2R\mathrm{M}_W(T) \sup\{\|\mathrm{d}\omega\| : \|\omega\| \le 1\} \\
&\qquad \textit{(Since each } B_i, B_j \textit{ disjoint)}
\end{aligned}
$$

So, as $R \to 0$, $T_R \rightharpoonup T$.

By [Sim83, 26.13], we have that mass is lower semicontinuous with respect to weak convergence, and $\mathrm{M}_W(T) \le \liminf_{R\to 0} \mathrm{M}_W(T_R)$. But by construction,

$$
\begin{aligned}
\mathrm{M}_W(T_R) &\le \mathrm{M}_W(T - \sum_{i=1}^{\infty} T\llcorner B_i(q_i)) + \sum_{i=1}^{\infty} \mathrm{M}_W(S_i) \\
&\le \mathrm{M}_W(T) - \sum_{i=1}^{\infty} \mathrm{M}_W(T\llcorner B_i) + (1-\varepsilon) \sum_{i=1}^{\infty} \mathrm{M}_W(T\llcorner B_i) \\
&\le \mathrm{M}_W(T) - \varepsilon \mathrm{M}_W(T) \\
&= \mathrm{M}_W(T) - \varepsilon \mu_T(X)
\end{aligned}
$$

But this implies that $\mathrm{M}_W(T_R) + \varepsilon\mu_T(X) \le \mathrm{M}_W(T)$ with $\varepsilon > 0$ and $\mu_T(X) > 0$, contradicting the lower semicontinuity of mass. □

We now establish a few more auxiliary results.

Lemma 3.2.6 *Let U open in $\mathbb{R}^{n+k}$, and let $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$ for all $W \Subset U$. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}$ be Lipschitz, and let $R(t) = \left\{x \in \mathbb{R}^{n+k} : f(x) < t\right\}$. Then:*

1. $\mathrm{M}_W(T\llcorner R(t))$ *is differentiable* $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$
2. $\mathrm{M}_W(\langle T, f, t\rangle_-) \le (\operatorname{ess\,sup}_W |\mathrm{D}f|) \frac{d}{dt} \mathrm{M}_W T\llcorner R(t)$

Proof *1.* Fix $W \Subset U$. Trivially, if $t < s$, then $\mathrm{M}_W(T\llcorner R(t)) \le \mathrm{M}_W(T\llcorner R(s))$, so $\mathrm{M}_W(T\llcorner R(t))$ is monotone. Define $f : \mathbb{R} \to \mathbb{R}^+$ by $f(t) = \mathrm{M}_W(T\llcorner R(t) - 0$. So, f is the difference of two monotone functions, and by [Roy88, §5 (p103)], whenever $a < b$, f is of bounded variation on $[a, b]$.

Now, fix $\varepsilon > 0$. Then for each $q_i \in \mathbb{Q}$, f is bounded variation on $[q_i - \varepsilon, q_i + \varepsilon]$. Now, by [Roy88, 6 (p104)], f is differentiable $\mathscr{L}^1$–a.e. $t \in [q_i - \varepsilon, q_i + \varepsilon]$. Since $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$, we have $\bigcup [q_i - \varepsilon, q_i + \varepsilon] = \mathbb{R}$. Now, let $F_i \subseteq [q_i - \varepsilon, q_i + \varepsilon]$ be the the null set where f fails to be differentiable. It follows that f fails to be differentiable on $\bigcup_{i=1}^{\infty} F_i \subseteq \mathbb{R}$. The conclusion follows by the countable subadditivity of $\mathscr{L}^1$.

2. We note that by the translation $t \mapsto t - h$, the definition of differentiability becomes:

$$\frac{d}{dt} f = \lim_{h \to 0} \frac{f(t) - f(t-h)}{h}$$

Now, since $\mathrm{M}_W(T) < \infty$, we have $\mu_T(W) = \mathrm{M}_W(T)$. It follows that $\mathrm{M}_W(T \llcorner R(t)) = \mu_T(W \cap R(t))$, and it follows that since f is μ_T-measurable,

$$\begin{aligned} \mu_T(W \cap R(t)) &= \mu_T(W \cap R(t) \cap R(t-h)) + \mu_T(W \cap R(t) \setminus R(t-h)) \\ &= \mu_T(W \cap R(t-h)) + \mu_T(W \cap \{x : t - h < f(x) < t\}) \end{aligned}$$

which implies that $\mathrm{M}_W(T \llcorner \{x : t-h < f(x) < t\}) = \mathrm{M}_W(T \llcorner R(t) - \mathrm{M}_W(T \llcorner R(t-h))$.

Since we've established that $\mathrm{M}_W(T \llcorner R(t))$ is differentiable $\mathscr{L}^1$–a.e. $t \in \mathbb{R}$, for such a point t,

$$\begin{aligned} \liminf_{h \to 0} \frac{\mathrm{M}_W(T \llcorner \{x : t-h < f(x) < t\})}{h} &= \lim_{h \to 0} \frac{\mathrm{M}_W(T \llcorner \{x : t-h < f(x) < t\})}{h} \\ &= \lim_{h \to 0} \frac{\mathrm{M}_W(T \llcorner R(t) - \mathrm{M}_W(T \llcorner R(t-h))}{h} \\ &= \frac{d}{dt} \mathrm{M}_W(T \llcorner R(t)) \end{aligned}$$

The result follows by [Sim83, 28.9].

□

Lemma 3.2.7 *Let $x \in \mathbb{R}^n$, and $g_x : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g_x(y) = \|x - y\|$. Then $|\nabla g_x(y)| = 1$ for $\mathscr{L}^n$–a.e. $y \in \mathbb{R}^{n+k}$.*

Proof Trivially, g_x is Lipschitz, so $\nabla g_x(y)$ exists $\mathscr{L}^n$–a.e. $y \in \mathbb{R}^{n+k}$ by Rademacher's Theorem, Theorem 1.2.2. Write $x = (x^1, \ldots, x^n)$, and $y = (y^1, \ldots, y^n)$. Then, for $g_x(y) \neq 0$,

$$\begin{aligned} \frac{\partial g}{\partial y^i}(y) &= \frac{\partial}{\partial y^i}\Big|_y \left[\sum_{j=1}^{n} (x^j - y^j)^2 \right]^{\frac{1}{2}} \\ &= \frac{1}{2} \left[\sum_{j=1}^{n} (x^j - y^j)^2 \right]^{-\frac{1}{2}} \frac{\partial}{\partial y^i}\Big|_y (x^i - y^i)^2 \\ &= -\frac{1}{g_x(y)} (x^i - y^i) \end{aligned}$$

It then follows that:

$$\begin{aligned} \nabla g_x(y) &= \frac{\partial g}{\partial y^i}(y) e_i = -\frac{1}{g_x(y)} (x^i - y^i) e_i \\ \implies |\nabla g_x(y)| &= \frac{1}{g_x(y)} g_x(y) = 1 \end{aligned}$$

□

We now present the important Theorem of this section. Our proof is a detailed description of [Whi89, p211].

Theorem 3.2.8 (Lower Density Lemma) *Let U open in $\mathbb{R}^{n+k}$, and $\mathrm{M}_W(T) < \infty$ for all $W \Subset U$. Further, suppose that $\partial T = 0$. Then if $\partial(T \llcorner B_r(x))$ is rectifiable for every $x \in \mathbb{R}^{n+k}$ and $\mathscr{L}^1-\text{a.e. } r \in \mathbb{R}^+$, then there exists $\delta > 0$ such that:*

$$\Theta^n_*(\mu_T, x) > \delta$$

for $\mu_T-\text{a.e. } x \in U$.

Proof Recall from Lemma 3.2.5, $\lambda(x,r) = \inf\{\mathrm{M}S : \partial S = \partial T \llcorner B_r(x)\}$. Now, let x be a point where:

$$\lim_{r\to 0} \frac{\lambda(x,r)}{\mathrm{M}(T \llcorner B_r(x)}) = 1$$

Firstly, we claim that there exists an $R > 0$ such that for all $r < R$, $\mathrm{M}(T \llcorner B_r(x)) < 2\lambda(x,r)$. Assume the converse. That is, suppose that for all $r > 0$, $\mathrm{M}(T \llcorner B_r(x)) \geq 2\lambda(x,r)$. Then,

$$\begin{aligned} 1 &\geq \frac{2\lambda(x,r)}{\mathrm{M}(T \llcorner B_r(x))} \\ &\implies 1 \geq 2 \lim_{r\to 0} \frac{\lambda(x,r)}{\mathrm{M}(T \llcorner B_r(x))} \\ &\implies 1 \geq 2 \end{aligned}$$

which is a contradiction.

Trivially, $B_r(x) = \{y : \|x-y\| < r\}$. Let $g : \mathbb{R}^{n+k} \to \mathbb{R}^+$ be given by $g(y) = \|x-y\|$. Now, since $\partial T = 0$, we have $\langle T, g, t\rangle_- = \partial(T \llcorner B_r(x))$. By Lemma 3.2.7, we have $\operatorname{ess\,sup} |\nabla g| = 1$ for $\mathscr{L}^n-\text{a.e. } x \in \mathbb{R}^n$, and by Lemma 3.2.6, we have:

$$\mathrm{M}(T \llcorner B_r(x)) \leq f'(r)$$

where $f(r) = \mathrm{M}(T \llcorner B_r(x))$.

Now, since $\partial(T \llcorner B_r(x))$ is rectifiable $\mathscr{L}^1-\text{a.e. } r \in \mathbb{R}^+$, we apply the Isoperimetric Inequality [Sim83, 30.1] to find a constant $c = c(n,k)$ such that

$$\mathrm{M}(R)^{\frac{n-1}{n}} \leq c\mathrm{M}(\partial(T \llcorner B_r(x)))$$

for $R \in \mathscr{D}_n U$ with $\partial R = \partial(T \llcorner B_r(x))$. It follows that:

$$\begin{aligned} \mathrm{M}(R) &\leq [c\mathrm{M}(\partial(T \llcorner B_r(x)))]^{\frac{n}{n-1}} \\ &\implies \lambda(x,r) \leq [c\mathrm{M}(\partial(T \llcorner B_r(x)))]^{\frac{n}{n-1}} \\ &\implies \lambda(x,r)^{\frac{n-1}{n}} \leq c\mathrm{M}(\partial(T \llcorner B_r(x))) \end{aligned}$$

for $\mathscr{L}^1-\text{a.e. } r \in \mathbb{R}^+$.

Further note (by the chain rule):

$$\frac{d}{dr} f(r)^{\frac{1}{n}} = \frac{1}{n} f(r)^{\frac{1}{n}-1} \frac{d}{dr} f(r)$$

Now, we compute for $r < R$:

$$\begin{aligned} f(r) < 2\lambda(x,r) &\iff \frac{1}{2}f(r) < \lambda(x,r) \\ &\iff \left[\frac{1}{2}f(r)\right]^{1-\frac{1}{n}} \leq \lambda(x,r)^{1-\frac{1}{n}} \\ &\implies \left[\frac{1}{2}f(r)\right]^{1-\frac{1}{n}} \leq c f'(r) \\ &\implies \frac{1}{2c} \leq f(r)^{\frac{1}{n}-1} f'(r) \\ &\implies \frac{1}{2nc} \leq \frac{1}{n} f(r)^{\frac{1}{n}-1} f'(r) \\ &\implies \frac{1}{2nc} \leq \frac{d}{dr}\left(f(r)\right)^{\frac{1}{n}} \\ &\implies \frac{r}{2nc} \leq f(r)^{\frac{1}{n}} \\ &\implies r^n \left(\frac{1}{2nc}\right)^n \leq f(r) \end{aligned}$$

Now, $f(r) = \mathrm{M}(T \llcorner B_r(x)) = \mu_T(B_r(x))$. Now, set:

$$\delta = \frac{\left(\frac{r}{2nc}\right)^n}{\omega_n}$$

and by definition of lower density, the result follows. □

3.3 Constant Vectorfield Lemma

We begin with the following measure theoretic results.

Lemma 3.3.1 *Let X be a metric space, and let μ be a Radon measure on X. Let $f : X \to [-1,1]$ be a μ-measurable function. Then, $\mu \llcorner f = \mu \llcorner f^+ - \mu \llcorner f^-$ with at least one of $\mu \llcorner f^+$ or $\mu \llcorner f^-$ finite, and both measures Radon.*

Proof By [dB00, 8.1], $\mu \llcorner f$ is a signed measure, and by [dB00, 4 (p137)], we fine a unique $\mu \llcorner f^-$, $\mu \llcorner f^+$ measures mutually singular, which establishes that at least one is finite. In fact, $\mu \llcorner f^- = \mu \llcorner (f^-)$ and $\mu \llcorner f^+ = \mu \llcorner (f^+)$.

Trivially, we have $f^+(x), f^-(x) \in [0,1]$, and since μ is Borel Regular, Lemma 2.6.4 gives us that each $\mu \llcorner f^+$ and $\mu \llcorner f^-$ is Borel Regular. Let K be compact in X. Then,

$$\mu \llcorner f^+(K) = \int_K f^+ \, \mathrm{d}\mu \leq \int_K \mathrm{d}\mu = \mu(K) < \infty$$

which establishes that $\mu \llcorner f^+$ is indeed Radon. Similarly for $\mu \llcorner f^-$. □

We now introduce some notation from Mollification theory.

Definition 3.3.2 (Mollification of a Radon Measure) *Let X be a metric space with a Radon measure μ. Let f be a μ-measurable function. Then, we define:*

$$(f * \mu)(x) = \int_X f(x-y) \, \mathrm{d}\mu(y)$$

as the mollification of μ by f.

Definition 3.3.3 (Standard Symmetric Mollifier) *For $\varepsilon > 0$, define $\eta^\varepsilon : \mathbb{R}^n \to \mathbb{R}$,*

$$\eta^\varepsilon(x) = \begin{cases} c(\varepsilon) \exp \frac{1}{\varepsilon^2 - |x|^2} & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon \end{cases}$$

We call η^ε the standard symmetric mollifier.

Now, we present some important facts about mollification of currents. The general theory is discussed at length in [EG92, §4], and [Mat95, §1.25].

Theorem 3.3.4 (Current Mollification) *Let U be open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$ with locally finite mass. Then, there exists a $\theta_\varepsilon : \mathbb{R}^{n+k} \to \wedge_n \mathbb{R}^{n+k}$, with $\theta_\varepsilon^{i_1,\dots,i_n} \in C_C^\infty(\mathbb{R}^{n+k})$ with*

$$T_\varepsilon(\omega) = \int_U \langle \omega x, \theta_\varepsilon(x) \rangle \, \mathrm{d}\mathscr{L}^{n+k}(x) \to \int_U \langle \omega x, \xi x \rangle \, \mathrm{d}\mu_T(x) = T(\omega)$$

where:

$$\theta_\varepsilon^{i_1,\dots,i_n} = (\eta^\varepsilon * \mu_T \llcorner \xi^{i_1,\dots,i_n})$$

with η^ε the standard symmetric mollifier, and where $\xi(x) = \xi^{i_1,\dots,i_n} e_{i_1,\dots i_n}$.

Proof Fix $\omega \in \mathscr{D}_n U$. We write: $\omega(x) = \omega_{j_1,\dots,j_n} \mathrm{d}x^{j_1,\dots,j_n}$. So, we have, $\langle \omega(x), \xi(x) \rangle = \omega_{i_1,\dots,i_n}(x) \xi^{i_1,\dots,i_n}(x)$, and it follows that:

$$T(\omega) = \int_U \omega_{i_1,\dots,i_n}(x) \xi^{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T(x) = \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T \llcorner \xi^{i_1,\dots,i_n}(x)$$

We note that $\|\xi\| = 1$ for μ_T−a.e. $x \in \mathbb{R}^{n+k}$, which implies $\xi^{i_1,\dots,i_n} \in [-1, 1]$. Now, for the signed measure $\mu_T \llcorner \xi^{i_1,\dots,i_n}$, by Lemma 3.3.1, we have:

$$\mu_T \llcorner \xi^{i_1,\dots,i_n} = \mu_T^+ \llcorner \xi^{i_1,\dots,i_n} - \mu_T^- \llcorner \xi^{i_1,\dots,i_n}$$

with $\mu_T^+ \llcorner \xi^{i_1,\dots,i_n}$, and $\mu_T^- \llcorner \xi^{i_1,\dots,i_n}$ guaranteed to be Radon measures.

Now, for each multi-index $i_1, \dots, i_n$, by [Mat95, 1.26], we have:

$$\lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T^+ \llcorner \xi_{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) = \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T^+ \llcorner \xi_{i_1,\dots,i_n}(x)$$
$$\lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T^- \llcorner \xi_{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) = \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T^- \llcorner \xi_{i_1,\dots,i_n}(x)$$

Combining these, we obtain:

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T^+ \llcorner \xi_{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) - \lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T^- \llcorner \xi_{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\ = \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T^+ \llcorner \xi_{i_1,\dots,i_n}(x) - \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T^- \llcorner \xi_{i_1,\dots,i_n}(x) \\ \iff \lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T \llcorner \xi_{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) = \int_U \omega_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T \llcorner \xi_{i_1,\dots,i_n}(x) = \int_U \omega_{i_1,\dots,i_n}(x) \xi_{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T(x) \end{aligned}$$

Now, summing over all multi-indices, we conclude:

$$\lim_{\varepsilon \to 0} T_\varepsilon(\omega) = \lim_{\varepsilon \to 0} \int_U \omega_{i_1,\dots,i_n}(x) (\eta^\varepsilon * \mu_T \llcorner \xi^{i_1,\dots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) = \int_U \omega_{i_1,\dots,i_n}(x) \xi^{i_1,\dots,i_n}(x) \, \mathrm{d}\mu_T(x) = T(\omega)$$

Trivially, by construction:

$$T_\varepsilon(\omega) = \int_U \langle \omega(x), \theta_\varepsilon(x) \rangle \, \mathrm{d}\mathscr{L}^{n+k}(x)$$

By [Mat95, 1.26], we are guaranteed that for every $\varepsilon > 0$, $(\eta^\varepsilon * \mu_T^+ \llcorner \xi_{i_1,\ldots,i_n}) \in C_C^\infty(\mathbb{R}^{n+k})$ and $(\eta^\varepsilon * \mu_T^- \llcorner \xi_{i_1,\ldots,i_n}) \in C_C^\infty(\mathbb{R}^{n+k})$ which implies that $\theta_\varepsilon^{i_1,\ldots,i_n} \in C_C^\infty(\mathbb{R}^{n+k})$. □

Corollary 3.3.5 *We can write:*

$$T_\varepsilon(\omega) = T(\eta^\varepsilon * \omega)$$

where $\eta^\varepsilon * \omega = (\eta^\varepsilon * \omega_{i_1,\ldots,i_n})\mathrm{d}x^{i_1,\ldots,i_n}$

Proof

$$\begin{aligned}
T_\varepsilon(\omega) &= \int_U \langle \omega(x), \theta_\varepsilon(x) \rangle \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int_U \omega_{i_1,\ldots,i_n}(x)(\eta^\varepsilon * \mu_T \llcorner \xi^{i_1,\ldots,i_n})(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int_U \omega_{i_1,\ldots,i_n}(x) \int_U \eta^\varepsilon(x-y)\xi^{i_1,\ldots,i_n}(y) \, \mathrm{d}\mu_T(y) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int_U \int_U \omega_{i_1,\ldots,i_n}(x)\eta^\varepsilon(x-y)\xi^{i_1,\ldots,i_n}(y) \, \mathrm{d}\mu_T(y) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int_U \int_U \omega_{i_1,\ldots,i_n}(x)\eta^\varepsilon(x-y)\xi^{i_1,\ldots,i_n}(y) \, \mathrm{d}\mathscr{L}^{n+k}(x) \, \mathrm{d}\mu_T(y) \\
&\quad \textit{(By Fubini Theorem [EG92, 1.4])} \\
&= \int_U \xi^{i_1,\ldots,i_n}(y) \int_U \eta^\varepsilon(y-x)\omega_{i_1,\ldots,i_n}(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) \, \mathrm{d}\mu_T(y) \\
&\quad \textit{(Since } \eta^\varepsilon \textit{ is an even function)} \\
&= \int_U (\eta^\varepsilon * \omega_{i_1,\ldots,i_n})(y)\xi^{i_1,\ldots,i_n} \, \mathrm{d}\mu_T(y) \\
&= \int_U \langle \eta^\varepsilon * \omega(y), \xi(y) \rangle \, \mathrm{d}\mu_T(y) \\
&= T(\eta^\varepsilon * \omega)
\end{aligned}$$

□

We conclude this section by proving the following result, its proof taken from [Whi89, p213].

Theorem 3.3.6 (Constant Vectorfield Lemma) *Let $T \in \mathscr{D}_n\mathbb{R}^{n+k}$ with locally finite mass and with $\partial T = 0$. Write $T = \mu_T \wedge \xi$, and suppose for all $x \in \mathbb{R}^{n+k}$, we have $\xi(x) = \tau \in \wedge_n\mathbb{R}^{n+k}$. Let $V \subseteq \mathbb{R}^{n+k}$ be a subspace of vectors such that T is invariant in the direction of $v \in V$. Then $\tau \in \wedge_n V$.*

Proof As with the proof given by [Whi89], we assume V has a basis consisting of a sub collection of the standard basis for $\mathbb{R}^{n+k}$, by an orthogonal basis transform.

Now, it suffices to show that if $e_i \notin V$ implies τ has no $e_i \wedge e_{j_1,\ldots,j_n}$ coefficients.

Fix $\varepsilon > 0$. Suppose $e_i \notin V$. Let $f \in C_C^\infty(\mathbb{R}^{n+k})$. For $1 \leq l \leq n+k$ with $e_l \wedge e_{j_1,\ldots,\hat{l},\ldots,j_n} \in \wedge_n V$, let τ^l denote the coefficients of τ of basis $e_l \wedge e_{j_1,\ldots,\hat{l},\ldots,j_n}$. Now, by Corollary 3.3.5 and since $\partial T = 0$, we have $0 = \partial T = \partial T_\varepsilon$. Let

$\phi_\varepsilon = \eta^\varepsilon * \mu_T$, it follows that:

$$
\begin{aligned}
0 &= \partial T_\varepsilon(f \mathrm{d}x^{j_1,\ldots,\hat{i},\ldots,j_n}) \\
&= T_\varepsilon\left(\frac{\partial f}{\partial x^l}\mathrm{d}x^l \wedge \mathrm{d}x^{j_1,\ldots,\hat{i},\ldots,j_n}\right) \\
&= \int \frac{\partial f}{\partial x^l}\theta^l_\varepsilon \, \mathrm{d}\mathscr{L}^{n+k} \\
&= \int \frac{\partial f}{\partial x^l}(x)\left(\int \eta^\varepsilon(x-y)\tau^l \, \mathrm{d}\mu_T(y)\right) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int \tau^l \frac{\partial f}{\partial x^l}(x)(\eta^\varepsilon * \mu_T)(x) \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= \int \tau^l \frac{\partial f}{\partial x^l}(x)\phi^\varepsilon \, \mathrm{d}\mathscr{L}^{n+k}(x) \\
&= -\int \tau^l \frac{\partial \phi^\varepsilon}{\partial x^l} f \, \mathrm{d}\mathscr{L}^{n+k}(x)
\end{aligned}
$$

(Integration by Parts, and by [Mat95, 1.26], $\phi^\varepsilon \in C^\infty_C$)

Since f was chosen arbitrarily, this implies:

$$\tau^l \frac{\partial \phi^\varepsilon}{\partial x^l} = 0$$

and so T_ε is invariant in the direction $\tau^l e_l$. Since this holds for all $\varepsilon > 0$, by invoking Theorem 3.3.4, we can conclude that T is invariant in the direction $\tau^l e_l$. It follows then that $\tau^i = 0$. □

Chapter 4

Closure and Compactness

In this chapter, we give an exposition of Brian White's proof of the Closure Theorem and the Compactness Theorem.

4.1 Preliminary Results

For the convenience of the reader, we prove some parts of the theorem as the following general results.

Lemma 4.1.1 *Let U be open in $\mathbb{R}^{n+k}$, $T \in \mathscr{D}_n U$ with $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$ for all $W \Subset U$. Suppose E closed in U and $\mathscr{H}^n(E) = 0$. Then $\mu_T(E) = 0$.*

Proof We write $\xi(x) = \xi^{i_1,\dots,i_n} e_{i_1,\dots,i_n}$. Define:

$$\omega^\varepsilon(x) = (\eta^\varepsilon * \xi_{i_1,\dots,i_n}) \mathrm{d}x^{i_1,\dots,i_n}$$

where η^ε is the standard symmetric mollifier. Trivially, $\omega^\varepsilon \in \mathscr{D}_n U$. Since $\mathscr{H}^n(E) = 0$, we invoke [Sim83, 26.29, 26.30] and:

$$0 = T \llcorner E(\omega^\varepsilon) = \int_E \langle \omega^\varepsilon, \xi \rangle \, \mathrm{d}\mu_T$$

Now, by [EG92, Theorem 1, §4.2], we have:

$$0 = \lim_{\varepsilon \to 0} \int_E \langle \omega^\varepsilon, \xi \rangle \, \mathrm{d}\mu_T = \int_E \|\xi\| \, \mathrm{d}\mu_T = \mu_T(E)$$

□

Corollary 4.1.2 *We have further that $\mu_T \ll \mathscr{H}^n$.*

Proof Let $A \subseteq \mathbb{R}^{n+k}$ be an arbitrary set with $\mathscr{H}^n(A) = 0$. Since $\mathscr{H}^n$ is Borel Regular we find a Borel B such that $A \subseteq B$ and $\mathscr{H}^n(B) = \mathscr{H}^n(A) = 0$. If $C \subseteq B$ closed, we have $\mathscr{H}^n(C) = 0$, and by Lemma 4.1.1, we find $\mu_T(C) = 0$. By Borel Regularity of μ_T, B is μ_T-measurable, and by [Sim83, 1.3], we have

$$\mu_T(B) = \sup \left\{ \mu_T(C) : C \subseteq B, C \text{ closed in } \mathbb{R}^{n+k} \right\} = 0$$

By construction, $A \subseteq B$ which implies $\mu_T(A) = 0$. □

Lemma 4.1.3 *Let $T \in \mathscr{D}_n\mathbb{R}^{n+k}$, $\partial T = 0$ and $\mathrm{M}(T) < \infty$. Suppose further that for every Lipschitz $f : \mathbb{R}^{n+k} \to \mathbb{R}$, $\langle T, f, r\rangle$ is $(n-1)$-integer rectifiable for $\mathscr{L}^1$–a.e. $r \in \mathbb{R}$. Then, there exists a $\delta > 0$ such that letting $M = \left\{x \in \mathbb{R}^{n+k} : \Theta^n_*(\mu_T, x) \geq \delta\right\}$, we have $\mu_T(\mathbb{R}^{n+k} \setminus M) = 0$, $\mu_T \ll \mathscr{H}^n \llcorner M$, and $\mathscr{H}^n(M) < \infty$.*

Proof Put $f_y(x) = \|y - x\|$ trivially Lipschitz, and we have $\langle T, f, r\rangle = \partial(T \llcorner B_r(x))$, since $\partial T = 0$. We can then invoke the Lower Density Lemma (Theorem 3.2.8), to find a $\delta > 0$ such that $\Theta^n_*(\mu_T, x) \geq \delta$ for μ_T–a.e. $x \in \mathbb{R}^{n+k}$. Then, by construction of M, we have that $\mu_T(\mathbb{R}^{n+k} \setminus M) = 0$.

Now, note that $\Theta^{*n}(\mu_T, x) \geq \Theta^n_*(\mu_T, x) \geq \delta > 0$ So, we can apply Theorem 2.5.6 with $A_1 = A_2 = M$, $\mathscr{H}^n(M) \leq \delta^{-1}\mu_T(M) = \delta^{-1}\mathrm{M}(T) < \infty$.

Now, since $\mathrm{M}(T) + \mathrm{M}(\partial T) = \mathrm{M}(T) < \infty$, we invoke Corollary 4.1.2, and since by construction $\mu_T(\mathbb{R}^{n+k} \setminus M) = 0$, we have that $\mu_T \ll \mathscr{H}^n \llcorner M$. □

Lemma 4.1.4 *Let U be open in $\mathbb{R}^{n+k}$, and let $T \in \mathscr{D}_n U$ be locally finite mass. Further, suppose $\mu_T \ll \mathscr{H}^n \llcorner M$, for some $M \subseteq \mathbb{R}^{n+k}$. If $\|\xi\| = 1$, for μ–a.e. $x \in \mathbb{R}^{n+k}$, then*

$$\tau = \xi \mathrm{D}_{\mathscr{H}^n \llcorner M}\mu$$

is locally $\mathscr{H}^n \llcorner M$ summable.

Proof Fix $W \Subset U$. Then,

$$\infty > \mathrm{M}_W(T) = \mu_T(W) = \int_W \|\xi\| \, \mathrm{d}\mu_T = \int_W \|\xi\| \mathrm{D}_{\mathscr{H}^n \llcorner M}\mu_T \, \mathrm{d}\mathscr{H}^n \llcorner M = \int_W \|\tau\| \, \mathrm{d}\mathscr{H}^n \llcorner M$$

□

Lemma 4.1.5 *Let μ be a Radon measure on $\mathbb{R}^{n+k}$, and let $M = \left\{x \in \mathbb{R}^{n+k} : \Theta^n_*(\mu, x) > 0\right\}$, and $\mu(M) > 0$. Further, suppose that $\mathscr{H}^n \llcorner M$ is Radon, and $\mu \ll \mathscr{H}^n \llcorner M$. Let f be such that $\|f\| = 1$ μ–a.e. , and $\tau = f \mathrm{D}_{\mathscr{H}^n \llcorner M}\mu$. Then for $a \in M$ a Lebesgue point of τ,*

$$\|\tau(a)\| \Theta^n_*(\mathscr{H}^n, M, a) = \Theta^n_*(\mu, a) > 0$$

Proof We firstly note that τ is locally μ-summable by Lemma 4.1.4. So, we can apply the Lebesgue points formula Theorem 2.6.6 which trivially implies Lemma 2.6.5 since we are in $\mathbb{R}^{n+k}$. So, we can write:

$$\|\tau(a)\| = \lim_{r \to 0} \frac{1}{\mathscr{H}^n(M \cap B_r(a))} \int_{M \cap B_r(a)} \|\tau(x)\| \, \mathrm{d}\mathscr{H}^n = \liminf_{r \to 0} \frac{\mu(B_r(a))}{\mathscr{H}^n(M \cap B_r(a))}$$

since $\mu \ll \mathscr{H}^n \llcorner M$, and $\|f\| = 1, \mu$–a.e. $x \in \mathbb{R}^{n+k}$.

Firstly, we show that $\tau(a) \neq 0$. For contradiction, assume the converse. Our observation above implies that $\mu(B_r(a)) = 0$, which implies that $\Theta^n_*(\mu, a) = 0$. But then, $a \notin M$ which is a contradiction.

Now, we compute:

$$\begin{aligned}
\|\tau(a)\| \Theta^n_*(\mathscr{H}^n, M, a) &= \left(\liminf_{r \to 0} \frac{\mu(B_r(a))}{\mathscr{H}^n(M \cap B_r(a))}\right)\left(\liminf_{r \to 0} \frac{\mathscr{H}^n(M \cap B_r(a))}{\omega_n r^n}\right) \\
&= \liminf_{r \to 0} \frac{\mu(B_r(a))}{\mathscr{H}^n(M \cap B_r(a))} \frac{\mathscr{H}^n(M \cap B_r(a))}{\omega_n r^n} \\
&\qquad \textit{(Since } \tau(a) \neq 0 \textit{ and } \mathscr{H}^n(M \cap B_r(a)) \neq 0\textit{)} \\
&= \liminf_{r \to 0} \frac{\mu(B_r(a))}{\omega_n r^n} \\
&= \Theta^n_*(\mu, a) \\
&> 0
\end{aligned}$$

The result follows immediately. □

Lemma 4.1.6 *Let $M \subseteq \mathbb{R}^{n+k}$, and let $\eta_{\lambda,a} = \lambda^{-1}(x-a)$ for $\lambda > 0$. Then,*

$$\mathscr{H}^n(\eta_{\lambda,a}M \cap B_r(0)) = \lambda^{-n}\mathscr{H}^n(M \cap B_{\lambda r}(a))$$

Proof Fix $\lambda > 0$. Define $M_\lambda = \{x - a : x \in M\}$. Then, observe that:

$$\lambda^{-1}(M_\lambda \cap B_{\lambda r}(0)) = \eta_{\lambda,a}M \cap B_r(0)$$
$$\lambda^{-1}(M_\lambda \cap B_{\lambda r}(0) + a) = \lambda^{-1}(M \cap B_{\lambda r}(a))$$

By the translation invariance of the Hausdorff measure,

$$\begin{aligned}\mathscr{H}^n(\eta_{\lambda,a}M \cap B_r(0)) &= \mathscr{H}^n(\lambda^{-1}(M \cap B_{\lambda r}(a))) \\ &= \lambda^{-n}\mathscr{H}^n(M \cap B_{\lambda r}(a))\end{aligned}$$

□

Lemma 4.1.7 *Let $T \in \mathscr{D}_n\mathbb{R}^{n+k}$ such that $T = \mathscr{H}^n \llcorner M \wedge \tau(x)$ where $M \subseteq \mathbb{R}^{n+k}$. Further, let $\eta_{\lambda,a}(x) = \lambda^{-1}(x-a)$, and let Λ be a positive sequence converging to zero. Suppose that $a \in M$ is a Lebesgue Point of τ, and $\Theta^{*n}(\mathscr{H}^n, M, a) \leq 1$. Then,*

$$\lim_{\lambda\in\Lambda} \mathrm{M}_{B_R(0)}(\eta_{\lambda,a\sharp}T - (\mathscr{H}^n \llcorner \eta_{\lambda,a}M) \wedge \tau(a)) = 0$$

Proof We firstly make note of an important density estimate. Fix $\delta > 0$ and let:

$$L_\delta = \left\{\frac{\mathscr{H}^n(M \cap B_r(a))}{\omega_n r^n} : 0 < r < \delta\right\}$$

In light of this notation, we note that $\Theta^{*n}(\mathscr{H}^n, M, a) = \lim_{\delta\to 0}\sup L_\delta \leq 1$. It follows then that for $\delta < \gamma$, we have $\sup L_\delta \leq \sup L_\gamma$. So, it follows that there must exist a small $\delta > 0$ such that for all $0 < r < \delta$, $\mathscr{H}^n(M \cap B_r(a)) \leq \omega_n r^n$. So, we have:

$$\frac{\omega_n}{\mathscr{H}^n(M \cap B_r(a))} \geq \frac{1}{r^n} \implies \frac{\omega_n}{\mathscr{H}^n(M \cap B_r(a))}\int_{M\cap B_r(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n \geq \frac{1}{r^n}\int_{M\cap B_r(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n$$

Now, fix $0 < R < \infty$. For $\lambda > 0$ such that $0 < \lambda R < \lambda$, we associate r with λR, and it follows that:

$$\frac{R^n\omega_n}{\mathscr{H}^n(M \cap B_{\lambda R}(a))}\int_{M\cap B_{\lambda R}(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n \geq \frac{1}{\lambda^n}\int_{M\cap B_{\lambda R}(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n$$

Fix $R > 0$. Now, we compute the mass:

$$\begin{aligned}\mathrm{M}_{B_R(0)}(\eta_{\lambda,a\sharp}T - \mathscr{H}^n \llcorner \eta_{\lambda,a}M \wedge \tau(a)) &\leq \mathrm{M}_{B_R(0)}(\eta_{\lambda,a\sharp}T - \eta_{\lambda,a\sharp}\mathscr{H}^n \llcorner M \wedge \tau(a)) \\ &\qquad \textit{(Since } \tau(a) \textit{ is a constant)} \\ &\leq (\operatorname{ess\,sup}\|\mathrm{D}\eta_{\lambda,a}\|)^n \mathrm{M}_{B_{\lambda R}(a)}(T - \mathscr{H}^n \llcorner M \wedge \tau(a)) \\ &\qquad \textit{(By [Sim83, 26.25])} \\ &\leq \lambda^{-n}\mathrm{M}_{B_{\lambda R}(q)}(T - \mathscr{H}^n \llcorner M \wedge \tau(a)) \\ &\leq \lambda^{-n}\int_{M\cap B_{\lambda R}(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n \\ &\qquad \textit{(Since } T = \mathscr{H}^n \llcorner M \wedge \tau(x)\textit{)} \\ &\leq \frac{R^n\omega_n}{\mathscr{H}^n(M \cap B_{\lambda R}(a))}\int_{M\cap B_{\lambda R}(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n\end{aligned}$$

Since $a \in M$ is a Lebesgue point of τ,

$$\lim_{\lambda\in\Lambda} \mathrm{M}_{B_R(0)}(\eta_{\lambda,a\sharp}T - \mathscr{H}^n \llcorner \eta_{\lambda,a}M \wedge \tau(a)) \leq R^n\omega_n \lim_{\lambda\in\Lambda}\frac{1}{\mathscr{H}^n(M \cap B_{\lambda R}(a))}\int_{M\cap B_{\lambda R}(a)} \|\tau(x) - \tau(a)\| \, \mathrm{d}\mathscr{H}^n = 0$$

and the conclusion follows. □

Lemma 4.1.8 *Let $T \in \mathscr{D}_n\mathbb{R}^{n+k}$, and let $\partial T = 0$. If for every Lipschitz $f : \mathbb{R}^{n+k} \to \mathbb{R}$, and $\mathscr{L}^1-$a.e. $r \in \mathbb{R}$, $\langle T, f, r\rangle$ is $(n-1)$-integer rectifiable, then so is $\langle \eta_{\lambda,a\sharp}T, f, r\rangle$, where $\eta_{\lambda,a} = \lambda^{-1}(x-a)$.*

Proof Fix f Lipschitz, and let $r \in \mathbb{R}$ be a point where $\langle T, f, r\rangle$ is $(n-1)$ integer rectifiable. Since $\partial T = 0$, we note that we can write the slice as: $\langle T, f, r\rangle = \partial(T\llcorner R)$ where $R = \left\{x \in \mathbb{R}^{n+k} : f(x) < r\right\}$. Now, by [Sim83, 26.21], we have:

$$\eta_{\lambda,a\sharp}\langle T, f, r\rangle = \eta_{\lambda,a\sharp}\partial(T\llcorner R) = \partial(\eta_{\lambda,a\sharp}T\llcorner R) = \langle \eta_{\lambda,a\sharp}T, f, r\rangle$$

□

Lemma 4.1.9 *Let $T = \mu \wedge \tau \in \mathscr{D}_n\mathbb{R}^{n+k}$ with $\mathbf{M}(T) < \infty$, where $\tau \in \wedge_n\mathbb{R}^{n+k}$, with $\partial T = 0$, and $\langle T, f, r\rangle$ integer $(n-1)$-rectifiable for $\mathscr{L}^1-$a.e. $r \in \mathbb{R}$. Then T is translation invariant in exactly n directions*

Proof Let $V \subseteq \mathbb{R}^{n+k}$ be the maximal vector subspace of $\mathbb{R}^{n+k}$ such that T is translation invariant. Since $\partial T = 0$, we invoke the Constant Vectorfield Lemma (Theorem 3.3.6) to find $\tau \in \wedge_n V$. So T is invariant in at least n directions.

Since our hypothesis satisfies Lemma 4.1.3, we invoke the lemma. So there exists a $\delta > 0$, with $M = \left\{x \in \mathbb{R}^{n+k} : \Theta^n_*(\mu, x) \geq \delta > 0\right\}$ and $\mathscr{H}^n(M) < \infty$. So, by Theorem 1.1.2, $\mathscr{H}^{n+k}(M) = 0$ for $k > 0$, and $\mathscr{H}^{n-k}(M) = \infty$, for $k < n$. Together with these facts, and since $\mu \ll \mathscr{H}^n\llcorner M$, we can represent T as an $\mathscr{H}^n\llcorner M$ integral, it follows that T is translation invariant in at most n directions. □

Lemma 4.1.10 *Let $M \subseteq \mathbb{R}^{n+k}$, and $\eta_{\lambda,a}(x) = \lambda^{-1}(x-a)$. Suppose $\mu = \sum_{i=1}^p \alpha_i \mathscr{H}^n\llcorner P_i$, $1 \leq p \leq \infty$ where each P_i is a parallel n-plane. If for some $a \in M$, $\mathscr{H}^n\llcorner\eta_{\lambda,a}M \rightharpoonup \mu$, where $\lambda \in \Lambda$ a positive sequence converging to zero, and $\Theta^n_*(\mu, x) > \delta > 0$ for $\mu-$a.e. $x \in M$ and $\sum_{i=1}^p a_i \leq \Theta^{*n}(\mathscr{H}^n, M, a) \leq 1$, then $p < \infty$.*

Proof Suppose $p = \infty$. We make a lower density observation. Let $x \in P_j$ such that $\delta < \Theta^n_*(\mu, x)$. Then,

$$0 < \delta < \Theta^n_*(\mu, x) = \liminf_{r\to 0} \frac{\sum_{i=1}^n \alpha_i \mathscr{H}^n\llcorner P_j(B_r(x))}{\omega_n r^n} = \liminf_{r\to 0} \frac{\mathscr{H}^n\llcorner P_j(B_r(0))}{\omega_n r^n} = \alpha_j$$

since $\{P_i\}$ are distinct and parallel, and $P_i \cap B_r(x)$ yields an n-ball since P_i is an n-plane. Now note that:

$$\sum_{i=1}^p \alpha_i = p\delta = \infty$$

Now, since $\mathscr{H}^n\llcorner\eta_{\lambda,a}M \rightharpoonup \mu$, for some sequence Λ:

$$\begin{aligned}
\mu(B_r(0)) &= \lim_{\lambda\in\Lambda} \mathscr{H}^n\llcorner\eta_{\lambda,a}M(B_r(0)) \\
&= \limsup_{\lambda\in\Lambda} \lambda^{-n}\mathscr{H}^n(M \cap B_{\lambda r}(a)) \\
&\qquad \text{(By Lemma 4.1.6)} \\
&\leq \omega_n r^n \Theta^{*n}(\mathscr{H}^n, M, a) \\
&= \omega_n r^n
\end{aligned}$$

Since we assume $p = \infty$, we can take r large enough so that for $i > N$ (by rearranging index),

$$\frac{\mathscr{H}^n\llcorner P_i(B_r(0))}{\omega_n r^n} > \frac{1}{i}$$

which implies that

$$\sum_{i=1}^p \alpha_i \leq \frac{\mu(B_r(0))}{\omega_n r^n} \leq 1$$

which contradicts our previous estimate. □

Lemma 4.1.11 *Let $\{T_i\} \subseteq \mathscr{D}_n\mathbb{R}^{n+k}$, each with locally finite mass, $\partial T_i = 0$, and $T_i \rightharpoonup T$, with $\partial T = 0$. Suppose f is Lipschitz with $\operatorname{ess\,sup} \|\mathrm{D}f\| < \infty$. Let $R_{a,b} = \left\{x \in \mathbb{R}^{n+k} : a < f(x) < b\right\}$ where $a < b$. Suppose further that:*

$$\lim_{i\to\infty} \mathrm{M}_W(T_i \llcorner R_{a,b}) = 0$$

for some $W \Subset \mathbb{R}^{n+k}$. Then, there exists a $t \in (a,b)$ and a subsequence i' such that $T_{i'} \llcorner R \rightharpoonup T \llcorner R_t$, and

$$\lim_{i'\to\infty} \mathrm{M}_W(\partial(T_{i'} \llcorner R_t)) = 0$$

where $R_t = \left\{x \in \mathbb{R}^{n+k} : f(x) < t\right\}$

Proof Trivially, $T_i \rightharpoonup T$ implies $T_i \llcorner R_t \rightharpoonup T \llcorner R_t$, by considering forms ω with $\operatorname{spt} \omega \subseteq R_t$, for any $t \in (a,b)$.

By the Slicing Lemma (Theorem 3.1.3), we can choose a subsequence i' such that:

$$\lim_{i'\to\infty} \mathrm{M}_W(\partial(T_{i'} \llcorner R_{a,b})) = 0$$

since $\partial T_i = 0$.

Now, by [Sim83, 28.10], and since $\|\mathrm{D}f\| < \infty$,

$$\lim_{i'\to\infty} \int_{[a,b]} \mathrm{M}_W(T_i \llcorner R_t)\, \mathrm{d}\mathscr{L}^1(t) \leq \lim_{i'\to\infty} \operatorname{ess\,sup} \|\mathrm{D}f\| \mathrm{M}_W(T_i \llcorner R_{a,b}) \implies \lim_{i'\to\infty} \int_{[a,b]} \mathrm{M}_W(T_i \llcorner R_t)\, \mathrm{d}\mathscr{L}^1(t) = 0$$

Now, suppose for contradiction that for all $t \in (a,b)$, $\lim_{i'\to\infty} \mathrm{M}_W(\partial(T_{i'} \llcorner R_t)) \neq 0$. But then, by the lower semicontinuity of mass [Sim83, 26.13], this would imply $\lim_{i'\to\infty} \int_{[a,b]} \mathrm{M}_W(T_i \llcorner R_t)\, \mathrm{d}\mathscr{L}^1(t) \neq 0$, which is a contradiction.

We observe that $T_i \llcorner R_t \rightharpoonup T \llcorner R_t$ implies $T_{i'} \llcorner R_t \rightharpoonup T \llcorner R_t$. □

Lemma 4.1.12 *Let U be open, bounded and convex in $\mathbb{R}^n$. Let $T \in \mathscr{D}_n U$, with $\mathrm{M}_W(T) + \mathrm{M}_W(\partial T) < \infty$ for all $W \Subset U$. Then, for $W \Subset U$, there exists a $\beta \in \mathbb{R}$ and a constant $c > 0$ such that:*

$$\mathrm{M}_W(T - \beta[\![W]\!]) \leq c\mathrm{M}_W(\partial T)$$

Proof For $\omega \in C_c^\infty(U)$, we note that we can write:

$$T(\omega \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) = \int_U \omega\eta\, \mathrm{d}\mathscr{L}^n$$

where $\eta \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$ by mollification of the measure and $\mathrm{M}_W(\partial T) = |\mathrm{D}\eta|(W)$ (see [Sim83, 26.28]).

Now, fix $\theta \in (0,1)$. Then, by [Sim83, 6.4], there exists a $\beta \in \mathbb{R}$ and a $c > 0$ such that

$$\int_U \|\eta - \beta\|\, \mathrm{d}\mathscr{L}^n \leq c\mathrm{M}_W(\partial T)$$

Fix $\omega \in C_c^\infty(U)$, with $\|\omega\| \leq 1$, with $\operatorname{spt} \omega \subseteq W$. We note that:

$$(T - \beta[\![W]\!])(\omega \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n) = \int_U \omega\eta - \beta\omega\, \mathrm{d}\mathscr{L}^n \leq \int_U \|\eta - \beta\|\, \mathrm{d}\mathscr{L}^n \leq c\mathrm{M}_W(\partial T)$$

The result follows by taking a $\sup$ over all such ω. □

Lemma 4.1.13 *Let $M \subseteq \mathbb{R}^{n+k}$, and $\eta_{\lambda,a}(x) = \lambda^{-1}(x-a)$. Suppose $\mu = \sum_{i=1}^p \alpha_i \mathscr{H}^n \llcorner P_i$, $1 \leq p \leq \infty$ where each P_i is a parallel n-plane. If for some $a \in M$, $\mathscr{H}^n \llcorner \eta_{\lambda,a} M \rightharpoonup \mu$, for $\lambda \in \Lambda$, a positive sequence converging to zero, then:*

$$\Theta_*^n(\mathscr{H}^n, M, a) \leq \sum_{i=1}^p \alpha_i$$

Proof Fix $0 < r < \infty$. We observe that since $\mathcal{H}^n \llcorner \eta_{\lambda,a} M \rightharpoonup \mu$,

$$\begin{aligned}
\mu(B_r(0)) &= \lim_{\lambda\in\Lambda} (\mathcal{H}^n \llcorner \eta_{\lambda,a} M)(B_r(0)) \\
&= \lim_{\lambda\in\Lambda} \inf(\mathcal{H}^n(\eta_{\lambda,a} M \cap (B_r(0)) \\
&\geq \omega_n r^n \Theta^n_*(\mathcal{H}^n, M, a)
\end{aligned}$$

(By applying Lemma 4.1.6)

Since each P_i is parallel and distinct and since each $P_i \cap B_r(0)$ is an n-ball, we can choose r sufficiently small such that:

$$\frac{\mathcal{H}^n \llcorner P_i(B_r(0))}{\omega_n r^n} \leq 1$$

By definition of μ, the result follows. □

4.2 The Closure Theorem

Theorem 4.2.1 (The Lesser Closure Theorem) *Let $\{T_i\} \subseteq \mathscr{D}_n \mathbb{R}^{n+k}$ be a sequence of n-integer rectifiable currents with*

$$\sup\{\mathrm{M}(T_i) + \mathrm{M}(\partial T_i)\} < \infty$$

Suppose $T_i \rightharpoonup T$, and $\partial T = 0$. Then T is an n-integer rectifiable current.

Proof We proceed by induction. Note that the base case $n = 0$ is trivial, since every 0 dimensional current is an integer rectifiable current.

Now, assume that the theorem holds for the $(n-1)$ dimensional case. Firstly, we notice that for every Lipschitz $f : \mathbb{R}^{n+k} \to \mathbb{R}$, and for $\mathcal{L}^1$–a.e. $r \in \mathbb{R}$, $\langle T_i, f, r\rangle$ is $(n-1)$-rectifiable by the definition of a slice for integer rectifiable currents (Definition 2.4.6). By the Slicing Lemma (Theorem 3.1.3), we are guaranteed a subsequence i' such that

$$\begin{aligned}
&\sup\{\mathrm{M}(\langle T_{i'}, f, r\rangle) + \mathrm{M}(\partial\langle T_{i'}, f, r\rangle)\} < \infty \\
&\langle T_{i'}, f, r\rangle \rightharpoonup \langle T, f, r\rangle
\end{aligned}$$

for $\mathcal{L}^1$–a.e. $r \in \mathbb{R}$. By the induction hypothesis, we have that $\langle T, f, r\rangle$ is indeed $(n-1)$-integer rectifiable. Since $\partial T = 0$, we note that $\langle T, f, r\rangle = \partial(T \llcorner R_r)$, where $R_r = \left\{x \in \mathbb{R}^{n+k} : f(x) < r\right\}$.

Now, we have that $T \in \mathscr{D}_n \mathbb{R}^{n+k}$, $\mathrm{M}(T) < \infty$ since $\sup\{\mathrm{M}(T_i) + \mathrm{M}(\partial T_i)\} < \infty$, and that for every Lipschitz f, and $\mathcal{L}^1$–a.e. $r \in \mathbb{R}$, $\langle T, f, r\rangle$ is $(n-1)$-integer rectifiable. So we invoke Lemma 4.1.3 to find a $\delta > 0$, and for $M = \left\{x \in \mathbb{R}^{n+k} : \Theta^n_*(\mu_T, x) > \delta\right\}$, $\mu_T(\mathbb{R}^{n+k} \setminus M) = 0$. Further we are guaranteed that $\mu_T \ll \mathcal{H}^n \llcorner M$ and $\mathcal{H}^n(M) < \infty$. Note that this implies that $\mathcal{H}^n \llcorner M$ is a Radon Measure.

Then, by Radon-Nikodym Theorem [Sim83, 4.7], by letting $\theta(x) = \mathrm{D}_{\mathcal{H}^n \llcorner M} \mu(x)$, we can write

$$T(\omega) = \int_{\mathbb{R}^{n+k}} \langle \omega(x), \xi(x)\rangle \theta(x)\, \mathrm{d}\mathcal{H}^n \llcorner M = \int_M \langle \omega(x), \tau(x)\rangle\, \mathrm{d}\mathcal{H}^n$$

where $\tau = \xi\theta$.

Since $\mathcal{H}^n(M) < \infty$, by invoking Theorem 2.5.7 we find that for $\mathcal{H}^n$–a.e. $a \in M$,

$$\Theta^{*n}(\mathcal{H}^n, M, a) \leq 1$$

Now, by Lemma 4.1.4, τ is locally $\mathcal{H}^n \llcorner M$ summable, and we find that $\mathcal{H}^n$–a.e. $a \in M$ is a Lebesgue Point of τ. Explicitly, for $\mathcal{H}^n$–a.e. $a \in M$,

$$\lim_{r\to 0} \frac{1}{\mathcal{H}^n(M \cap B_r(0))} \int_{M\cap B_r(a)} \|\tau(a) - \tau(x)\|\, \mathrm{d}\mathcal{H}^n(x) = 0$$

Trivially, by the subadditivity of the measure, where both these statements fail is a null set. Fix $a \in M$, a point which satisfies both statements.

Now, note that by Lemma 4.1.5, by associating f with ξ, we find:

$$\|\tau(a)\|\Theta^n_*(\mathscr{H}^n, M, a) = \Theta^n_*(\mu_T, a) > 0$$

We now establish some convergence results about our measures. Let $\eta_{\lambda,a}(x) = \lambda^{-1}(x-a)$, and let Λ be a sequence converging to zero. Note that for each $\lambda \in \Lambda$, each measure $\mathscr{H}^n \llcorner \eta_{\lambda,a}M = \lambda^{-n}\mathscr{H}^n \llcorner (x-a)M$ and so it is a Radon Measure. Fix $0 < r < \infty$. Then,

$$\begin{aligned}
\limsup_{\lambda\in\Lambda}(\mathscr{H}^n \llcorner \eta_{\lambda,a}M)(B_r(0)) &= \limsup_{\lambda\in\Lambda}\lambda^{-n}\mathscr{H}^n(M\cap B_{\lambda r}(a)) \\
&\qquad \textit{(By Lemma 4.1.6)} \\
&= \omega_n r^n \limsup_{\lambda\in\Lambda}\frac{\mathscr{H}^n(M\cap B_{\lambda r}(a))}{(\lambda r)^n\omega_n} \\
&\le \omega_n r^n \Theta^{*n}(\mathscr{H}^n, M, a) \\
&< \infty
\end{aligned}$$

by our previous estimate on upper density. So, we can apply [Sim83, 4.4], to find that there exists a subsequence $\Lambda' \subseteq \Lambda$ and a Radon measure μ such that:

$$\mathscr{H}^n \llcorner \eta_{\lambda,a}M \rightharpoonup \mu \qquad (\lambda \in \Lambda')$$

Since this is pointwise convergence, this implies:

$$\mathscr{H}^n \llcorner \eta_{\lambda,a}M \wedge \tau(a) \rightharpoonup \mu \wedge \tau(a) \qquad (\lambda \in \Lambda')$$

Let $T_\lambda = \eta_{\lambda,a\sharp}T$. Since we fix $0 < r < \infty$, by Lemma 4.1.7,

$$\begin{aligned}
T_\lambda &\rightharpoonup \mu \\
\mu_{T_\lambda} &\rightharpoonup \|\tau a\|\mu
\end{aligned}$$

for $\lambda \in \Lambda'$.

Since we have that $\partial T = 0$, and $\langle T, f, r\rangle$ is $(n-1)$-integer rectifiable, by Lemma 4.1.8 gives us that for each $\lambda \in \Lambda'$, $\langle T_\lambda, f, r\rangle$ is $(n-1)$-rectifiable for every Lipschitz f and $\mathscr{L}^1$–a.e. $r \in \mathbb{R}$. Now, we have $\mathbf{M}(T_\lambda) + \mathbf{M}(\partial T_\lambda) < \infty$ since $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$. Further, we have $T_\lambda \rightharpoonup \mu \wedge \tau(a)$. So, by invoking the Slicing Lemma (Theorem 3.1.3), there exists a further subsequence $\Lambda'' \subseteq \Lambda'$ such that:

$$\langle T_\lambda, f, r\rangle \rightharpoonup \langle \mu \wedge \tau(a), f, r\rangle \qquad (\lambda \in \Lambda'')$$

for $\mathscr{L}^1$–a.e. $r \in \mathbb{R}$, and every Lipschitz f. Again, by our inductive hypothesis, $\langle \mu \wedge \tau(a), f, r\rangle$ is $(n-1)$-rectifiable. Furthermore, by the Lower Density Lemma (Theorem 3.2.8), we have a $\delta > 0$ such that

$$\Theta^n_*(\mu, x) > \frac{\delta}{\|\tau(a)\|} > 0$$

for μ–a.e. $x \in M$.

Now, since $\partial T_\lambda = 0$, and $T_\lambda \rightharpoonup \mu \wedge \tau(a)$, it follows that $\partial(\mu \wedge \tau(a)) = 0$. So, we apply Lemma 4.1.9 to find that $\mu \wedge \tau(a)$ is translation invariant in exactly n directions. Letting $V \subseteq \mathbb{R}^{n+k}$ denote the vector subspace of vectors such that $\mu \wedge \tau(a)$ is translation invariant, the Constant Vectorfield Lemma (Theorem 3.3.6) guarantees that $\tau(a) \in \wedge_n V$. Coupling these facts imply that $\tau(a)$ is indeed a simple n-vector and $\dim(V) = n$. So, we have that $\tau(a)$ determines the n-plane V.

Now, let $M_\mu = \left\{x \in M : \Theta^n_*(\mu, x) > \delta\|\tau(a)\|^{-1}\right\}$. By our previous lower density estimate, $\mu(M \setminus M_\mu) = 0$. Define:

$$\mathscr{P} = \left\{P\ n\text{-plane} : \exists x \in M_\mu \text{ with } x \in P, P \text{ parallel to } V\right\}$$

Fix $P \in \mathscr{P}$. Let $x \in P \cap M_\mu$, so we have $\Theta^n_*(\mu, x) > 0$ and it follows that $\mu \llcorner P(B_r(x)) > 0$. Since μ is Radon, $\mu \llcorner P(B_r(x)) < \infty$. But since P is parallel to V and μ translation invariant in direction of $v \in V$, we have $0 <$

$\mu \llcorner P(B_r(x)) = \mu \llcorner (B_r(y)) < \infty$ for $y \in P \cap M_\mu$. So, μ is uniformly distributed Borel Regular for all $x, y \in M_\mu \cap P$, and by [Mat95, 3.4 (p45)], we can write $\mu \llcorner P = \alpha \mathscr{H}^n \llcorner P$ for α a constant. By the translation invariance, it follows that:

$$(\mu \wedge \tau(a))(\omega) = \int_{M_\mu} \langle \omega(x), \tau(a) \rangle \, d\mu(x) = \int_{M_\mu \cap (\bigcup \mathscr{P})} \langle \omega(x), \tau(a) \rangle \, d\mu(x)$$

Now, we argue that $\mathscr{P}$ is finite. Without loss of generality, we assume that $\mathscr{P}$ is countable and show that this implies that $\mathscr{P}$ is finite. We write $\mu = \sum_{i=1}^{p} \alpha_i \mathscr{H}^n \llcorner P_i$, where $\alpha_i > 0$, and $1 \leq p \leq \infty$. Since $\Theta_*^n(\mu, x) > \delta \|\tau(a)\|^{-1} > 0$ for μ−a.e. $x \in M$, we can apply Lemma 4.1.10 to find that $p < \infty$. (Note: our countable assumption does not lose generality for the reason that in our proof of Lemma 4.1.10, a lower density argument is used. An uncountable collection would only contribute a larger value used as a contradiction within the argument).

We now argue that $p = 1$ and $\alpha_1 = 1$. Suppose for contradiction that $p > 1$. For the simplicity of the argument, we assume that P_i is parallel to $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+k}$. (We can otherwise achieve this by rotation and since $\mathscr{H}^n$ is invariant under rotations).

Fix $0 < r < \infty$. Let $W = B_r^n(0) \times \mathbb{R}^k$, where $B_r^n(0)$ denotes the n-ball of radius r. Note that we can consider $\overline{W}$ as a product over $\mathbb{R}^k$ of closed balls n-balls which are compact in $\mathbb{R}^{n+k}$. Then by the Tychonoff Theorem [Mun96, 37.3], $\overline{W}$ is compact.

Fix $\varepsilon > 0$ such that $3\varepsilon < \min\left\{\mathrm{dist}(P_i, P_j) : P_i \neq P_j\right\}$. Fix j. Let $f(x) = \mathrm{dist}(P_j, x)$. Let $S_\varepsilon = \left\{x \in \mathbb{R}^{n+k} : \varepsilon < f(x) < 2\varepsilon\right\}$. Now, note that if $x \in S_\varepsilon$, then $\varepsilon < \mathrm{dist}(x, P_j) < 2\varepsilon$ which implies $P_j \cap S = \emptyset$ and $P_j \cap S \cap \overline{W} = \emptyset$. Furthermore, if $i \neq j$, then $x \in P_i$ implies $\mathrm{dist}(x, P_i) > 3\varepsilon$, and it follows that $P_i \cap S \cap \overline{W} = \emptyset$. So, we have that $\mu(\overline{W} \cap S) = 0$. Since $\mu_{T_\lambda} \rightharpoonup \|\tau(a)\|\mu$,

$$\lim_{\lambda \in \Lambda'} \mathrm{M}_W(T_\lambda \llcorner S_\varepsilon) \leq \|\tau(a)\| \mu(\overline{W} \cap S_\varepsilon) = 0$$

We note that $\mathrm{Lip}(f) = 1$, and by Lemma 1.2.3, $\operatorname{ess\,sup} \|\mathrm{D}f\| \leq \sqrt{n+k} < \infty$, and along with the fact that $\overline{W}$ is compact, we can invoke Lemma 4.1.11. We then have a $t \in (\varepsilon, 2\varepsilon)$, and a subsequence $\Lambda'' \subseteq \Lambda$ such that $T_\lambda \llcorner R_t \rightharpoonup \mu \wedge \tau(a) \llcorner R_t$, and:

$$\lim_{\lambda \in \Lambda''} \mathrm{M}_W(\partial(T_\lambda \llcorner R_t)) = 0$$

where $R_t = \left\{x \in \mathbb{R}^{n+k} : f(x) < t\right\}$.

Now, note that we are guaranteed $t \in (\varepsilon, 2\varepsilon)$. So, $R_t \cap P_i = \emptyset$, provided $P_i \neq P_j$. So, it follows that:

$$\mu \wedge \tau(a) = \sum_{i=1}^{p} \alpha_i \mathscr{H}^n \llcorner P_i \wedge \tau(a) = \alpha_j \|\tau(a)\| [\![P_j]\!]$$

Let $\Pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the orthogonal projection, and let $T'_\lambda = \Pi(T_\lambda \llcorner R_t)$. Since $B_r^n(0)$ is open, convex and bounded in $\mathbb{R}^n$, and also since $T'_\lambda \in \mathscr{D}_n \mathbb{R}^n$, for each λ we invoke Lemma 4.1.12 and find a $\beta_\lambda \in \mathbb{R}$ such that:

$$\mathrm{M}_{B_r^n(0)}(T'_\lambda - \beta_\lambda [\![B_r^n(0)]\!]) \leq c \mathrm{M}_{B_r^n(0)}(\partial T'_\lambda)$$

Note that:

$$\begin{aligned}
&\lim_{\lambda \in \Lambda''} \mathrm{M}_{B_r^n(0)}(T'_\lambda - \beta_\lambda [\![B_r^n(0)]\!]) \leq c \lim_{\lambda \in \Lambda''} \mathrm{M}_{B_r^n(0)}(\partial T'_\lambda) \\
&\implies \lim_{\lambda \in \Lambda''} \mathrm{M}_{B_r^n(0)}(T'_\lambda - \beta_\lambda [\![B_r^n(0)]\!]) = 0 \\
&\qquad \textit{(Since } \mathrm{M}_W(\partial T_\lambda \llcorner R_t) \to 0\textit{)} \\
&\iff \lim_{\lambda \in \Lambda''} (T'_\lambda - \beta_\lambda [\![B_r^n(0)]\!]) = 0 \\
&\iff \Pi(\mu \wedge \tau(a) \llcorner R_t) - \lim_{\lambda \in \Lambda''} \beta_\lambda [\![B_r^n(0)]\!]) = 0 \\
&\iff \alpha_j \|\tau(a)\| [\![B_r^n(0)]\!] - (\lim_{\lambda \in \Lambda''} \beta_\lambda) [\![B_r^n(0)]\!]) = 0 \\
&\qquad \textit{(Since we've assumed that } P_j \textit{ parallel to } \mathbb{R}^n \times \{0\}\textit{)} \\
&\implies \lim_{\lambda \in \Lambda''} \beta_\lambda = \alpha_j \|\tau(a)\|
\end{aligned}$$

Letting $\beta = \lim_{\lambda\in\Lambda''}\beta_\lambda$, we have

$$\lim_{\lambda\in\Lambda''} \mathbf{M}_{B_r^n(0)}(T'_\lambda - \beta[\![B_r^n(0)]\!]) = 0$$

We note that $\mathscr{L}^n$ is a Radon Measure, and since $\mathscr{H}^n(\eta_{\lambda,a}M) = \lambda^{-n}\mathscr{H}^n(M) < \infty$, we have $\mathscr{L}^n(\Pi(\eta_{\lambda,a}M\cap R_t)) < \infty$. So,

$$\begin{aligned}\beta\mathscr{L}^n(B_r^n(0)\setminus\Pi(\eta_{\lambda,a}M\cap R_t)) &= \beta\int_{B_r^n(0)} 1\,\mathrm{d}\mathscr{L}^n - \beta\int_{\Pi(\eta_{\lambda,a}M\cap R_t)} 1\,\mathrm{d}\mathscr{L}^n \\ &\le \sup\left\{(\beta[\![B_r^n(0)]\!] - T'_\lambda)(\omega) : \|\omega\| \le, \omega\in\mathscr{D}^n\mathbb{R}^{n+k}\right\} \\ &= \mathbf{M}_{B_r^n(0)}(T'_\lambda - \beta[\![B_r^n(0)]\!])\end{aligned}$$

Now, under the limit:

$$\lim_{\lambda\in\Lambda''}\beta\mathscr{L}^n(B_r^n(0)\setminus\Pi(\eta_{\lambda,a}M\cap R_t)) = 0$$

It follows then that:

$$\begin{aligned}\mathscr{L}^n(B_r^n(0)) &\le \lim_{\lambda\in\Lambda''}\mathscr{L}^n(B_r^n(0)\cap\Pi(\eta_{\lambda,a}M\cap R_t)) + \lim_{\lambda\in\Lambda''}\mathscr{L}^n(B_r^n(0)\setminus\Pi(\eta_{\lambda,a}M\cap R_t)) \\ &\le \liminf_{\lambda\in\Lambda''}\mathscr{H}^n(W\cap\eta_{\lambda,a}M\cap R_t) \\ &\le \mu(\overline{W}\cap\overline{R}_t) \\ &\qquad \textit{(Since } \mathscr{H}^n\eta_{\lambda,a}M \rightharpoonup \mu\textit{)} \\ &= \alpha_j\mathscr{H}^n(B_r^n(0)) \\ &\qquad \textit{(Since } t\in(\varepsilon,2\varepsilon),\ \mathscr{H}^n(\overline{W}\cap\overline{R}_t\cap P_j) = \mathscr{H}^n(B_r^n(0))\textit{)}\end{aligned}$$

This establishes that:

$$\mathscr{L}^n(B_r^n(0)) \le \alpha_j\mathscr{H}^n(B_r^n(0)) = \alpha_j\mathscr{L}^n(B_r^n(0))$$

which implies that $1\le a_j$ for each $1\le j\le p$.

We invoke Lemma 4.1.13 to establish that $\Theta_*^n(\mathscr{H}^n, M, a) \le \sum_{i=1}^p \alpha_i$. Since $\mu = \sum_{i=1}^p \alpha_i\mathscr{H}^n \llcorner P_i$ we have that $\mu \ll \mathscr{H}^n$ and we can invoke Lemma 4.1.5. By combining the upper density estimate given by Lemma 4.1.10,

$$0 < \Theta_*^n(\mathscr{H}^n, M, a) \le \sum_{i=1}^p \alpha_i \le \Theta^{*n}(\mathscr{H}^n, M, a) \le 1$$

To avoid contradiction with our previous estimate that each $\alpha_j \ge 1$, we must have $p = 1$ and $\alpha_1 = 1$.

We also note that $P = P_1$ passes through the origin since for every $r > 0$,

$$\begin{aligned}\mu(B_{2r}(0)) &\ge \liminf_{\lambda\in\Lambda}(\mathscr{H}^n\llcorner\eta_{\lambda,a}M)(B_r(0)) \\ &\ge \omega_n r^n\Theta_*^n(\mu_T, a) \\ &> 0 \\ &\qquad \textit{(By Lemma 4.1.5)}\end{aligned}$$

Also, P is independent of the subsequence Λ, since P is uniquely determined by the simple n-vector $\tau(a)$.

Now, fix $f\in C_c^0(\mathbb{R}^{n+k})$. Then, observe that:

$$\lim_{\lambda\to 0}\int_{\eta_{\lambda,a}M} f(x)\,\mathrm{d}\mathscr{H}^n(x) = \int_{\mathbb{R}^{n+k}} f(x)\,\mathrm{d}\mu(x) = \int_P f(x)\,\mathrm{d}\mathscr{H}^n(x)$$

since $\mathscr{H}^n\llcorner\eta_{\lambda,a} \rightharpoonup \mu$. By definition Definition 1.3.6, P is the approximate plane of M at a. Since this holds for all Lebesgue points of τ, and by Theorem 2.6.6 $\mathscr{H}^n$−a.e. $a\in M$ are Lebesgue points of τ, M has an approximate tangent plane $\mathscr{H}^n$−a.e. $a\in M$ By Theorem 1.3.7, M is n-rectifiable.

Now, we need to establish that $\theta(a)$ is an integer. Note that:

$$\langle \mu \wedge \tau(a), f, r\rangle = \partial(\mu \wedge \tau(a) \llcorner R_t)(\omega) = \int_M \langle \mathrm{d}\omega, \tau(a)\rangle \, \mathrm{d}\mathscr{H}^n \llcorner P = \int_M \langle \mathrm{d}\omega, \xi(a)\rangle \theta(a) \, \mathrm{d}\mathscr{H}^n \llcorner P$$

is an $(n-1)$-integer rectifiable current by the induction hypothesis. So, $\theta(a) \in \mathbb{Z}$.

This establishes that T is indeed an n-integer rectifiable current, thereby completing the proof. □

Leading up to the general case of the Closure Theorem, we present the following key result.

Theorem 4.2.2 (Weaker Boundary Rectifiability Theorem) *Let $T \in \mathscr{D}_n \mathbb{R}^{n+k}$ an integer rectifiable current. Suppose that $\mathrm{M}(\partial T) < \infty$, Then $\partial T \in \mathscr{D}_{n-1} U$ is an integer rectifiable current.*

Proof We use the Weak Polyhedral Approximation Theorem [Sim83, 30.2] to find a polyhedral (and integer rectifiable by Deformation Theorem [Sim83, 29.3]) sequence $\{\partial P_k\} \subseteq \mathscr{D}_{n-1}\mathbb{R}^{n+k}$ such that $\partial P_k \rightharpoonup \partial T$. Now, trivially, $\partial(\partial P_k) = 0$, and since $\mathrm{M}(\partial P_k) \leq \mathrm{M}(\partial T)$, we can apply Theorem 4.2.1 to conclude that ∂T is an $(n-1)$-integer rectifiable current. □

Theorem 4.2.3 (The Closure Theorem) *Let U be open in $\mathbb{R}^{n+k}$, and let Let $\{T_i\} \subseteq \mathscr{D}_n U$ be a sequence of n-integer rectifiable currents with*

$$\sup \{\mathrm{M}_W(T_i) + \mathrm{M}_W(\partial T_i)\} < \infty$$

Suppose $T_i \rightharpoonup T$. Then T is an n-integer rectifiable current.

Proof We firstly illustrate how to relax the $\partial T = 0$ hypothesis of Theorem 4.2.1. So, suppose still that $U = \mathbb{R}^{n+k}$. We invoke the Weak Boundary Rectifiability Theorem (Theorem 4.2.2) to find that each ∂T_i is $(n-1)$-integer rectifiable. Since $\partial(\partial T) = 0$ and $\partial T_i \rightharpoonup \partial T$, we apply Theorem 4.2.1 to conclude that ∂T is $(n-1)$-integer rectifiable. By application of the Isoperimetric Inequality [Sim83, 30.1], we find an n-integer rectifiable $R \in \mathscr{D}_n \mathbb{R}^{n+k}$ such that $\partial T = \partial R$. This implies $\mathrm{M}(\partial R) = \mathrm{M}(\partial T) < \infty$. So, again by Theorem 4.2.2, ∂R is $(n-1)$-integer rectifiable.

Now, we have that $T_i - R \rightharpoonup T - R$, and by construction $\partial(T - R) = 0$. So, by applying Theorem 4.2.1, we find $(T - R)$ n-integer rectifiable which implies that T is n-integer rectifiable.

Now, we show how to relax the condition $U = \mathbb{R}^{n+k}$, and the finite mass hypothesis. Fix $W \Subset U$. Let $\mathscr{C} = \{B_r(x) : B_r(x) \Subset U\}$. be a cover for $\overline{W}$. Such a collection must exist since $\overline{W} \subseteq U$ and U open in $\mathbb{R}^{n+k}$. So, we have simply reduced the problem to considering balls $B_r(x) \Subset U$.

Fix $x \in U$, and let $B_r(x) \Subset U$.

$$\sup \{\mathrm{M}_{B_r(x)}(T_i) + \mathrm{M}_{B_r(x)}(\partial T_i)\} < \infty \iff \sup \{\mathrm{M}(T_i \llcorner B_r(x)) + \mathrm{M}((\partial T_i) \llcorner B_r(x))\} < \infty$$

Now, by the Slicing Lemma (Theorem 3.1.3), we have a subsequence $i' \subseteq i$ such that:

$$\begin{aligned} &T_{i'} \llcorner B_r(x) \rightharpoonup T \llcorner B_r(x) \\ &\sup \{\langle T, \mathrm{dist}(x, .), r\rangle\} = \sup \{\mathrm{M}(\partial(T_{i'} \llcorner B_r(x)) - (\partial T_{i'}) \llcorner B_r(x))\} < \infty \end{aligned}$$

By combining these, we have:

$$\sup \{\mathrm{M}(T_{i'} \llcorner B_r(x)) + \mathrm{M}(\partial(T_{i'} \llcorner B_r(x))\} < \infty$$

Now, by the first part of our argument, by putting we can put $U = \mathbb{R}^{n+k}$, $T_{i'} \llcorner B_r(x) \rightharpoonup T \llcorner B_r(x)$. is a n-integer rectifiable current. Since we chose x arbitrarily, we conclude T is n-integer rectifiable. □

4.3 Compactness

Here, we present a proof of the compactness theorem. One difference in our discussion is that unlike [Fed96, 4.2.16], we talk about compactness in the usual Weak* sense. A better discussion on the Weak* Topology can be found in [Pry, §6]. An important fact to note is that convergence in Weak* corresponds to pointwise convergence [Pry, 6.2.4].

We introduce the following auxiliary results.

Lemma 4.3.1 *Let $(X, \mathbb{R})$ be a normed linear space over $\mathbb{R}$, and X^* the Dual of X. For $k \in \mathbb{R}$, let $L_k : X^* \to X^*$ be defined by $L_k(x) = kx$. Then L_k is continuous in the Weak* topology.*

Proof By [Pry, 6.2.2], the basis elements of the Weak* topology are given by:

$$N(x, \xi, r) = \{\nu \in X^* : \|\nu(x) - \xi(x)\| < r\}$$

where $x \in X$, and $\xi \in X^*$. We note that $L_k^{-1} = k^{-1}x$, and so it suffices to prove that $L_k(N(x, \xi, r))$ is open. But this is trivially true since $L_k(N(x, \xi, r)) = N(x, \xi, |k|\, r)$. □

Corollary 4.3.2 *For $0 < r < \infty$, the usual (metric) closed ball $\overline{B}_r(0) \subseteq X^*$ is compact in the Weak* topology.*

Proof We know from Banach-Alaoglu Theorem [Pry, 6.3] that the unit ball $\overline{B}_1(0)$ is Weak* compact. Let $r > 0$. By Lemma 4.3.1 we have that $L_r = rx$ is continuous. We apply [Mun96, 26.5] to find that the image of a compact domain under a continuous map is compact and conclude that $L_r(\overline{B}_1(0)) = \overline{B}_r(0)$ is compact. □

We make the following important definition.

Definition 4.3.3 (Integral Currents) *Let U be open in $\mathbb{R}^{n+k}$. Define:*

$$\mathscr{I}_{n,k} = \{T \in \mathscr{D}_n U : T, \partial T \text{ integer rectifiable }, \mathrm{M}(T) \le k, \mathrm{M}(\partial T) \le k\}$$

We call $\mathscr{I}_{n,k}$ the n-integer currents with k normal mass.

Theorem 4.3.4 (Compactness Theorem for Integral Currents) *Let U open in $\mathbb{R}^{n+k}$. Let $0 < k < \infty$. Then $\mathscr{I}_{n,k}$ is Weak* compact in $\mathscr{D}_n U$.*

Proof Firstly, we show that $\mathscr{I}_{n,k}$ is Weak* closed in $\mathscr{D}_n U$. Let $\{T_i\} \subseteq \mathscr{I}_{n,k}$, with $T_i \rightharpoonup T$. Since $\sup\{\mathrm{M}(T_i) + \mathrm{M}(\partial T_i)\} \le 2k < \infty$, we apply of the Closure Theorem (Theorem 4.2.3) to find that T is integer rectifiable. Also $\sup\{\mathrm{M}(T_i)\} \le k$ and $\sup\{\mathrm{M}(\partial T_i)\} \le k$, which implies $T \in \mathscr{I}_{n,k}$. By [Pry, 6.2.4], $\mathscr{I}_{n,k}$ is Weak* closed.

Now, by Corollary 4.3.2, $\overline{B}_k(0) \subseteq \mathscr{D}_n U$ is Weak* compact. Trivially, $\mathscr{I}_{n,k} \subseteq \overline{B}_k(0)$ and since by [Mun96, 26.2], closed subsets of compact sets are compact, we conclude that $\mathscr{I}_{n,k}$ is Weak* compact. □

The compactness result is often discussed in literature in terms of a topology called the flat-metric topology. Our discussion in Weak* topology was for a matter of convenience. Our formulation is in fact equivalent to the discussion in the flat-metric. A more detailed discussion can be found in [Sim83, 31.3].

Finally, note that although we have proceeded our discussion finite mass rather than locally finite mass, we have not lost any generality. By considering compactness in the set $\mathscr{D}_n W$ for $W \Subset U$, one can attain a similar result for locally finite mass currents.

Notation

$\operatorname{ess\,sup}$	Essential Supremum
$C^r(X, Y)$	r-differentiable (continuous if $r = 0$) functions from X to Y
$C^r_c(X, Y)$	Compactly supported r-differentiable functions from X to Y
$C(X, Y)$	Same as $C^0(X, Y)$
$\operatorname{img} \phi$	Image of function ϕ
$\mathrm{D}_v f$	Directional derivative of f in the direction v
μ–a.e.	μ almost everywhere
$\wedge_n V$	n-Vectors of vectorspace V
$\wedge^n V$	n-Forms of vectorspace V
$e_{i_1,\dots,i_n}$	Same as $e_{i_1} \wedge \dots \wedge e_{i_n}$
$\mathrm{d}x^{i_1,\dots,i_n}$	Same as $\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_n}$
$\tau \in \wedge_n V$ simple	$\tau = \tilde{\tau} e_{i_1} \wedge \dots \wedge e_{i_n}$, where $\left\{e_{i_j}\right\}_{j=1}^{n} \subseteq \{e_i\}_{i=1}^{n+k}$, the basis for V
$\langle \cdot, \cdot \rangle$	Pairing of a form with a vector
$\langle \cdot, \cdot \rangle_H$	Inner product of Hilbert Space H
$\operatorname{spt} \omega$	Support of ω
$W \Subset U$	W open in U and $\overline{W}$ compact in U
$\mu \llcorner R$	Measure μ restricted to R
$\operatorname{Vol}(M)$	Volume of M
$T_i \rightharpoonup T$	$T_i(\omega) \to T(\omega)$, pointwise convergence
$B_r(x)$	Metric ball centred at x with radius r
$B^n_r(x)$	Metric n-ball centred at x with radius r
$\mathrm{D}_\mu \nu$	Radon-Nikodym derivative of ν w.r.t μ
$f_\sharp T$	Push forward of current T by f
$\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^n)$	Locally bounded variation real-valued functions on $\mathbb{R}^n$
$\mathbb{N}$	Natural Numbers not including 0
$\mathbb{Z}^+$	Positive Integers including 0
$\mathbb{R}^+$	Real Numbers including 0
$\mathscr{H}^n$	Hausdorff n measure

$\mathscr{L}^n$	Lebesgue n measure
$\mathrm{Lip}\,(f)$	Lipschitz constant of a Lipschitz function f
$\nabla^M f$	M-Gradient of f
$\mathrm{d}_x^M f$	M-differential of f at x
$\mathrm{J}_M f$	Jacobian of f over M
$\mathrm{J}_M^* f$	Co-Jacobian of f over M
$\mathscr{D}^n U$	Compactly supported smooth n forms over open set U
$\mathscr{D}_n U$	Dual to $\mathscr{D}^n U$
$\mathrm{M}_W(T)$	Mass of current T in W
$\mu \wedge \xi$	Current defined by Radon measure μ and n-vectorfield ξ
$[\![M]\!]$	Current defined by Rectifiable set M
$\mathrm{V}\,(M, \theta)$	Varifold defined by M with multiplicity θ
$T(M, \xi, \theta)$	Integer rectifiable current with orientation ξ, rectifiable set M and multiplicity θ
$\langle T, f, r\rangle$	Slice of current T by Lipschitz f at $f = r$
$\Theta_*^n \mu, A, x$	Lower density of μ at x in A
$\Theta^{*n} \mu, A, x$	Upper density of μ at x in A
$\nu \ll \mu$	ν absolutely continuous w.r.t μ
$\eta^\varepsilon * \omega$	Mollification of ω by η^ε

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