# ANALYTIC DISKS AND THE PROJECTIVE HULL

Blaine Lawson* and John Wermer**

**ABSTRACT**

Let $X$ be a complex manifold and $\gamma$ a simple closed curve in $X$. We address the question: What conditions on $\gamma$ insure the existence of a 1-dimension complex variety $\Sigma$ with boundary $\gamma$ in $X$. When $X = \mathbf{C}^n$, an answer to this question involves the polynomial hull of gamma. When $X = \mathbf{P}^n$, complex projective space, the projective hull $\widehat{\gamma}$ of $\gamma$, a generalization of polynomial hull, comes into play. One always has $\Sigma \subset \widehat{\gamma}$, and for analytic $\gamma$ they conjecturally coincide.

In this paper we establish an approximate analogue of this idea which holds without analyticity. We characterize points in $\widehat{\gamma}$ as those which lie on a sequence of analytic disks whose boundaries converge down to $\gamma$. This is in the spirit of work of Poletsky and of Larusson-Sigurdsson, whose results are essential here.

The results are applied to construct a remarkable example of a closed curve $\gamma \subset \mathbf{P}^2$, which is real analytic at all but one point, and for which the closure of $\widehat{\gamma}$ is $W \cup L$ where $L$ is a projective line and $W$ is an analytic (non-algebraic) subvariety of $\mathbf{P}^2 - L$. Furthermore, $\widehat{\gamma}$ itself is the union of $W$ with two points on $L$.

**Introduction.**

Let $\gamma$ be a simple closed real curve in a complex manifold $X$. Consider the problem of finding conditions which guarantee that $\gamma$ forms the boundary of a complex analytic subvariety in $X$. When $X$ is $\mathbf{C}^n$ (or, more generally, Stein), there is a solution [W] which involves the polynomial hull of $\gamma$. When $X$ is $\mathbf{P}^n$ (or, more generally, projective), there is a notion of the projective hull of $\gamma$, denoted $\widehat{\gamma}$, which is related to the polynomial hull and has the following property. If

$$f : \Sigma \ \rightarrow \ X$$

is a map of a compact Riemann surface with boundary, which is holomorphic on $\mathrm{Int}\Sigma$ and continuous up to the boundary with

$$f(\partial\Sigma) = \gamma,$$

then

$$f(\Sigma) \ \subseteq \ \widehat{\gamma}.$$

*Partially supported by the N.S.F.

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(See [HL$_2$].) It is conjectured that if $X = \mathbf{P}^n$ and $\gamma$ is real analytic, then either $\widehat{\gamma} = \gamma$ or $\widehat{\gamma}$ is a 1-dimensional analytic subvariety with boundary $\gamma$ or $\widehat{\gamma}$ is an algebraic curve containing $\gamma$. Real analyticity is fundamental for this conjecture. It fails for any $C^\infty$-curve $\gamma$ which is not pluripolar [HL$_2$, Cor. 4.4]. However, the conjecture has been established for real analytic curves which are "stable" by using arguments of E. Bishop (see [HLW]).

In this paper we show that for any closed curve $\gamma \subset \mathbf{P}^n$, all points $z_0$ in the projective hull $\widehat{\gamma}$ are essentially characterized by the fundamental property discussed above. We show that if $z_0 \in \gamma$, then for any sequence $\epsilon_j \searrow 0$, there exists a sequence of holomorphic maps of the unit disk

$$f_j : \Delta \ \rightarrow \ \mathbf{P}^n$$

with

$$f_j(0) \ = \ z_0$$

and

$$\text{dist}\,(f_j(\partial\Delta), \gamma) \ \leq \ \epsilon_j.$$

In fact, from any such sequence we extract limiting holomorphic maps $f : \Delta \rightarrow \mathbf{P}^n$, possibly constant $\equiv z_0$, and show in Theorem 2 that

$$f(\Delta) \subset \widehat{\gamma}.$$

If we choose an affine coordinate chart $\mathbf{C}^n \subset \mathbf{P}^n$ containing $\gamma$, then the sequence of maps $f_j$ can be chosen so that the poles $\zeta_1^j, ..., \zeta_{N_j}^j$ of $f_j$, viewed as a $\mathbf{C}^n$-valued function, are simple and satisfy $\sum_k \log|\zeta_k^j| \geq -M$ for some constant $M$ independent of $j$. In this form **the existence of the sequence is necessary and sufficient for $z_0$ to lie in $\widehat{\gamma}$.** This is Theorem 1 below.

Theorem 2 is applied to produce interesting examples related to the conjecture above. We construct a simple closed curve $\gamma \subset \mathbf{P}^2$, which is real analytic at all but one point $p$, and has the following properties. There exists a projective line $L$ through $p$, a point $q \in L$, and a proper complex analytic subvariety $W \subset \mathbf{P}^2 - L$ such that

$$\widehat{\gamma} \ = \ W \cup \{p, q\} \qquad \text{and} \qquad \overline{\{\widehat{\gamma}\}} \ = \ W \cup L.$$

Thus, if real analyticity breaks down at a single point, the conjecture fails. Moreover, if this happens, then $\widehat{\gamma}$ may not be closed.

Related examples with other interesting properties are also given

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## The Main Theorems.

Consider a connected closed curve $\gamma$ lying in complex projective $n$-space $\mathbf{P}^n$. Write $\mathbf{P}^n = \mathbf{C}^n \cup H$, where $H$ denotes the hyperplane at infinity, and assume that $\gamma \subset \mathbf{C}^n$. We wish to characterize those points $z_0 \in \mathbf{C}^n$ which lie in the projective hull $\widehat{\gamma}$ of $\gamma$ in terms of analytic disks. We shall make use of the work of Lárusson and Sigurdsson [LS$_1$], who give a formula for the Siciak-Zahariuta extremal function $V_X$ of a connected open subset $X$ of $\mathbf{C}^n$.

For each $r > 0$ let $K_r$ denote the open tube of radius $r$ around $\gamma$, i.e.,

$$K_r \;=\; \{z \in \mathbf{C}^n : \mathrm{dist}(z, \gamma) < r\}$$

We fix a point $z_0 \in \mathbf{C}^n - \gamma$. Thus $z_0 \notin K_r$ for $r$ sufficiently small.

Let $\{f_r\}_r$ be a family of analytic maps of the unit disk $\Delta$ into $\mathbf{P}^n$, indexed by numbers $r > 0$ converging to zero. We consider the following four conditions on the family $\{f_r\}_r$:

For all $r$

(i) $f_r(\partial\Delta) \;\subset\; K_r$,

(ii) $f_r(0) \;=\; z_0$,

(iii) There exists a number $M > 0$ such that if $\zeta_1^{(r)}, ..., \zeta_{N_r}^{(r)}$ are the poles of $f_r$ in $\Delta$ (i.e., $f_r(\zeta_j^{(r)}) \in H$ for $j = 1, ..., N_r$), then we have

$$\sum_{j=1}^{N_r} \log|\zeta_j^{(r)}| \;\geq\; -M.$$

(iv) The poles are simple, that is, the $\mathbf{C}^n$-valued function $(\zeta - \zeta_j^{(r)}) f_r(\zeta)$ is holomorphic in a neighborhood of $\zeta_j^{(r)}$ for each $j$.

Theorem 1. *The point $z_0$ lies in $\widehat{\gamma}$ if and only if there exists a family $\{f_r\}_r$ of analytic maps satisfying (i) – (iv).*

**Proof. (Sufficiency)**. Assume there exists a family $\{f_r\}_r$ which satisfies (i) – (iv). For given $r$, let $\zeta_1^{(r)}, ..., \zeta_{N_r}^{(r)}$ be the poles of $f_r$. Put

$$B_r(\zeta) \;\equiv\; \prod_{j=1}^{N_r} \left( \frac{\zeta - \zeta_j^{(r)}}{1 - \overline{\zeta}_j^{(r)} \zeta} \right) \qquad \text{for} \quad \zeta \in \Delta.$$

**Claim:**

$$|B_r(0)| \;\geq\; e^{-M} \quad \text{for all } r.$$

**Proof of Claim.** Since $B_r(0) = \prod_{j=1}^{N_r}(-\zeta_j^{(r)})$ we have

$$\log|B_r(0)| \ = \ \sum_{j=1}^{N_r} \log|\zeta_j^{(r)}|.$$

By (iii) the right hand side is $\geq -M$. Hence $|B_r(0)| \ \geq \ e^{-M}$ as claimed.

For each $r$ we now define a function $G_r$ by setting

$$G_r \ \equiv \ f_r \cdot B_r. \tag{1}$$

Note that $G_r$ has no poles on $\Delta$ and $|B_r|$ is of unit length on $\partial\Delta$. Also since $f_r(\partial\Delta) \subset K_r$, there exists a constant $c_1$ such that $|G_r(\zeta)| \leq c_1$ on $\partial\Delta$ for all $r$. Hence, by the maximum principle $|G_r(\zeta)| \leq c_1$ on $\Delta$ for all $r$. Thus $\{G_r\}_r$ is a normal family on the interior of $\Delta$, and so there exists a sequence $\{G_{r_j}\}_{r_j}$, $r_j \to 0$ as $j \to \infty$, converging point-wise on $\mathrm{Int}\Delta$ to a holomorphic function $G$. Furthermore, $\{B_r\}_r$ is a normal family on the interior of $\Delta$, and we may assume without loss of generality that $B_{r_j} \ \to B$, a holomorphic function on $\mathrm{Int}\Delta$. The functions $B$ and $G$ lie in $H^\infty(\Delta)$. Moreover, by Claim 1 we have $|B(0)| \geq e^{-M}$, so $B$ is not identically zero.

We put $Z =$ the zero set of $B$ on $\mathrm{Int}\Delta$. Then $Z$ is a countable discrete subset of $\mathrm{Int}\Delta$.

Fix a point $a \in \mathrm{Int}\Delta \setminus Z$. For each $r$, $f_r(a) = G_r(a)/B_r(a)$ and as $r_j \to 0$, we have $B_{r_j}(a) \to B(a)$. By choice of $a$, $B(a) \neq 0$, so $\lim_{j\to\infty} f_{r_j}(a)$ exists and equals $G(a)/B(a)$. We define

$$f \ \equiv \ \frac{G}{B}.$$

Then $f$ is holomorphic on $\mathrm{Int}\Delta \setminus Z$ and has possible poles at the points of $Z$. Since $f_r(0) = z_0 \ \forall \ r$, we have

$$f(0) = z_0.$$

This brings us to the following.

Theorem 2. *For each $f$ constructed as above, we have*

$$f(\mathrm{Int}\Delta) \ \subset \ \widehat{\gamma}.$$

*In particular, $f(0) = z_0 \in \widehat{\gamma}$.*

**Proof of Theorem 2.** We begin with the following.

Proposition 1. *For each point $a \in \mathrm{Int}\Delta \setminus Z$ we have*

$$f(a) \in \widehat{\gamma}.$$

**Proof.** Fix a polynomial $P$ on $\mathbf{C}^n$ with degree $d$ Assume that $\|P\|_\gamma \leq 1$. Then for all $r$ sufficiently small, $\|P\|_{K_r} \leq 2$

Fix such an $r$ and let $\zeta_1^{(r)}, ..., \zeta_{N_r}^{(r)}$ be as above. The map $\zeta \mapsto P(f_r(\zeta))$ is holomorphic on $\mathrm{Int}\Delta \setminus \bigcup_{j=1}^{N_r}\{\zeta_j^{(r)}\}$. Write $P$ as

$$P(w) \ = \ \sum_{|\alpha|\le d} c_\alpha w_1^{\alpha_1} \cdots w_n^{\alpha_n}.$$

and $f_r(\zeta) = (w_1(\zeta), ..., w_n(\zeta))$ so that

$$P(f_r(\zeta)) \ = \ \sum_{|\alpha|\le d} c_\alpha w_1^{\alpha_1}(\zeta) \cdots w_n^{\alpha_n}(\zeta).$$

Then by assumption (iv) for $\zeta$ near $\zeta_j^{(r)}$

$$P(f_r(\zeta)) \ = \ \left(\zeta - \zeta_j^{(r)}\right)^{-d_j} \left(a_j + b_j(\zeta - \zeta_j^{(r)}) + c_j(\zeta - \zeta_j^{(r)})^2 + \cdots\right)$$

where $0 \le d_j \le d$ and $a_j \ne 0$. Hence,

$$\log|P(f_r(\zeta))| \ = \ -d_j \cdot \log\left|\zeta - \zeta_j^{(r)}\right| + h_j(\zeta) \tag{2}$$

where $h_j$ is harmonic near $\zeta_j^{(r)}$. Also

$$\log\left|\frac{\zeta - \zeta_j^{(r)}}{1 - \overline{\zeta}_j^{(r)}\zeta}\right| \ = \ \log\left|\zeta - \zeta_j^{(r)}\right| + k_j(\zeta)$$

where $k_j$ is harmonic on $\Delta \setminus \zeta_j^{(r)}$. We define

$$\chi_r(\zeta) \ = \ \log|P(f_r(\zeta))| + d \cdot \log|B_r(\zeta)| \qquad \text{for } \zeta \in \mathrm{Int}\Delta. \tag{3}$$

On $\mathrm{Int}\Delta \setminus \bigcup_{j=1}^{N_r}\{\zeta_j^{(r)}\}$ the function $\chi_r(\zeta)$ is subharmonic.

Fix $j = j_0$. For $\zeta$ near $\zeta_{j_0}^{(r)}$, (2) and (3) give

$$\chi_r(\zeta) \ = \ -d_j \cdot \log\left|\zeta - \zeta_{j_0}^{(r)}\right| + h_{j_0}(\zeta) + d \cdot \left\{\log\left|\frac{\zeta - \zeta_{j_0}^{(r)}}{1 - \overline{\zeta}_{j_0}^{(r)}\zeta}\right| + \sum_{j \ne j_0} \log\left|\frac{\zeta - \zeta_j^{(r)}}{1 - \overline{\zeta}_j^{(r)}\zeta}\right|\right\}.$$

Thus for $\zeta$ near $\zeta_{j_0}^{(r)}$,

$$\chi_r(\zeta) \ = \ (d - d_{j_0}) \cdot \log\left|\zeta - \zeta_{j_0}^{(r)}\right| + H_{j_0}(\zeta)$$

where $H_{j_0}$ is subharmonic there. Since $d \ge d_{j_0}$, the function $\chi_r$ is subharmonic in a neighborhood of $\zeta_{j_0}^{(r)}$.

Since this holds for all $j_0$, $\chi_r$ is subharmonic on $\mathrm{Int}\Delta$. Also $\chi_r = \log|P(f_r)|$ on $\partial\Delta$ since $|B_r| = 1$ there. Therefore, for $\zeta \in \partial\Delta$ we have

$$\chi_r(\zeta) \;=\; \log|P(f_r(\zeta))| \;\leq \log\|P\|_{K_r} \;\leq\; \log 2$$

since $f_r(\partial\Delta) \subset K_r$ and $\|P\|_{K_r} \leq 2$. By the maximum principle for subharmonic functions on $\Delta$ we have that

$$\chi_r(\zeta) \;\leq\; \log 2 \quad \text{for } \zeta \in \Delta. \tag{4}$$

Fix $a \in \mathrm{Int}\Delta \setminus Z$. Then by (3), $\chi_r(a) \;=\; \log|P(f_r(a))| + d \cdot \log|B_r(a)|$, and so

$$\log|P(f_r(a))| + d \cdot \log|B_r(a)| \;\leq\; \log 2.$$

Letting $r \to 0$, we get $\log|P(f(a))| + d \cdot \log|B(a)| \;\leq\; \log 2$ and therefore

$$|P(f(a))| \;\leq\; 2 \cdot \left|\frac{1}{B(a)}\right|^d. \tag{5}$$

Now this holds for all polynomials $P$ with degree $\leq d$ and $\|P\|_\gamma \leq 1$. Hence, $f(a) \in \widehat{\gamma}$ and Proposition 1 is proved. ∎

To complete the proof of Theorem 2 we must show that for each pole $\zeta_0$ of $f$ we have $f(\zeta_0) \in \widehat{\gamma}$. Fix a pole $\zeta_0$ and choose a small closed disk $D$ about $\zeta_0$ so that $f$ is regular on $D \setminus \{\zeta_0\}$. Let $\gamma_0 = f(\partial D)$. Then by [HL$_2$, Prop. 2.3] we have

$$f(\zeta_0) \in \widehat{\gamma}_0$$

since $f(\zeta_0)$ lies on an analytic disk in $\mathbf{P}^n$ with boundary $\gamma_0$. This means that there is a constant $C_0 > 0$ such that for every section $\mathcal{P} \in H^0(\mathbf{P}^n, \mathcal{O}(d))$ we have

$$\|\mathcal{P}(f(\zeta_0))\| \;\leq\; C_0^d \|\mathcal{P}\|_{\gamma_0}. \tag{6}$$

Now in the affine chart $\mathbf{C}^n \subset \mathbf{P}^n$ each such $\mathcal{P}$ corresponds to a polynomial $P$ of degree $\leq d$, and one has that for $z \in \mathbf{C}^n$, $\|\mathcal{P}\|_z = (1 + \|z\|^2)^{-\frac{d}{2}}|P(z)|$. It then follows from (5) above that there is a constant $\kappa > 0$ such that for all each $a \in \partial D$

$$\|\mathcal{P}(f(a))\| \;\leq\; \left|\frac{\kappa}{B(a)}\right|^d \|\mathcal{P}\|_\gamma. \tag{7}$$

Combining (6) and (7) gives a new constant $C > 0$ such that

$$\|\mathcal{P}(f(\zeta_0))\| \;\leq\; C^d \|\mathcal{P}\|_\gamma,$$

which proves that $f(\zeta_0) \in \widehat{\gamma}$ and establishes Theorem 2 and the sufficiency of conditions (i) – (iv). ∎

**Proof. (Necessity)**. Assume $z_0 \in \widehat{\gamma}$. We must provide a family $\{f_r\}_r$ of maps satisfying conditions (i) – (iv). The sets $K_r$ are defined as before.

Definition 1. Let $E$ be a set in $\mathbf{C}^n$ and denote by $\mathcal{L}_E$ the set functions $u$ in the Lelong class which satisfy $u \leq 0$ on $E$.

Definition 2. The Siciak-Zahariuta extremal function for $E$ is given by

$$V_E(z) \ = \ \sup\{u(z) : u \in \mathcal{L}_E\}.$$

Since $z_0 \in \widehat{\gamma}$, we know from [HL$_2$, p.6] that there exists a constant $K$ such that

$$u(z_0) \ \leq \ K \qquad \text{for all} \ \ u \in \mathcal{L}_E.$$

Fix $r$ and choose $u \in \mathcal{L}_{K_r}$. Then $u$ is of Lelong class and $u \leq 0$ on $K_r$. In particular $u \leq 0$ on $\gamma$. Hence $u \in \mathcal{L}_\gamma$ and therefore $u(z_0) \leq K$. From Definition 2 it follows that $V_{K_r}(z_0) \leq K$. Hence

$$-V_{K_r}(z_0) \ \geq \ -K \tag{8}$$

We now appeal to a result of Lárusson and Sigurdsson on page 178 of [LS$_1$]. (Recall that $H = \mathbf{P}^n \setminus \mathbf{C}^n$ is the hyperplane at infinity.)

Theorem (Lárusson - Sigurdsson). *Let $X$ be a connected open subset of $\mathbf{C}^n$. Then for each $z \in \mathbf{C}^n$,*

$$-V_X(z) \ = \ \sup_f \left( \sum_{f(\zeta)\in H} \log|\zeta| \right)$$

*taken over all analytic maps $f : \Delta \to \mathbf{P}^n$ with $f(\partial\Delta) \subset X$ and $f(0) = z_0$.*

Because of (8) this theorem provides us with an analytic map $f_r : \Delta \to \mathbf{P}^n$ with $f_r(\partial\Delta) \subset K_r$ and $f_r(0) = z_0$ such that

$$\sum_{f_r(\zeta)\in H} \log|\zeta| \ > \ -K - 1. \tag{9}$$

Putting $M = K+1$, we see that $f_r$ satisfies condition (iii). (Note: $\{\zeta : f_r(\zeta) \in H\} = \{\zeta_1^{(r)}, ..., \zeta_{N_r}^{(r)}\}$.) Standard transversality theory implies that for an open dense subset of $G_{z_0} \equiv \{g \in \mathrm{PGL}(n+1, \mathbf{C}) : g(z_0) = z_0\}$ the map $g \circ f_r$ is transversal to $H$, that is, $g \circ f_r$ satisfies condition (iv). (Since the group $G_{z_0}$ acts transitively on $\mathbf{P}^n - \{z_0\}$, one can use, for example, Sard's Theorem for families as in [HL$_1$].) Choosing $g$ sufficiently close to the identity we may assume that $g \circ f_r(\partial\Delta) \subset K_r$ and that (9) holds with $f_r$ replaced by $g \circ f_r$. Choosing this approximation we see that $f_r' = g \circ f_r$ satisfies all the conditions (i) – (iv).

So we have constructed the desired family $\{f_r\}_r$, and necessity is established. This completes the proof of Theorem 1. ■

Note 1. The function $f$ appearing in Theorem 2 could be constant ($\equiv z_0$), but if it is not, then we obtain a non-trivial analytic disk through $z_0$ which lies entirely in the projective hull $\widehat{\gamma}$.

**The Examples.**

**Example 1.** We shall use Theorem 2 to construct a closed curve $\gamma_0 \subset \mathbf{C}^2$, which is real analytic at all but one point, and whose projective hull contains a "large" Riemann surface $\Sigma$. In particular, $\Sigma$ will be the image of a holomorphic map of the open disk Int$\Delta$ which takes boundary values continuously (in fact, analytically) on $\gamma_0$ at all but one point. We shall then show that the closure of $\widehat{\gamma}_0$ in $\mathbf{P}^2$ is exactly the union of a projective line $L$ and a proper complex analytic (but not algebraic) subvariety $W \subset \mathbf{P}^2 \setminus L$ which extends $\Sigma$.

To produce $\Sigma$ we shall construct a family of holomorphic maps $f_n : \Delta \to \mathbf{P}^2$, $n = 1, 2, ...$ satisfying conditions (i) – (iv) for a sequence $r_n \searrow 0$ and with $z_0 = (0, 0)$.

To begin choose a sequence of numbers $\{\epsilon_j\}$ , $0 < \epsilon_j < 1$, such that $\sum_{j=1}^{\infty} \epsilon_j < \infty$. Put $a_j = 1 - \epsilon_j$, $j = 1, 2, ....$ Next choose a sequence of positive numbers $\{c_j\}$ such that $\sum_{j=1}^{\infty} \frac{c_j}{\epsilon_j} < \infty$. For $n = 1, 2, ...$ we put

$$\omega_n(\zeta) \ = \ \sum_{j=1}^{n} \frac{c_j}{\zeta - a_j} + \sum_{j=1}^{n} \frac{c_j}{a_j}.$$

Let $f_n(\zeta) = (\zeta, \omega_n(\zeta))$ for $\zeta \in \Delta$. Then $\{f_n\}$ is a sequence of holomorphic maps $\Delta \to \mathbf{P}^2$ such that $f_n$ has the poles $\zeta_j^{(n)} = a_j$, $j = 1, ..., n$. Put

$$\omega(\zeta) \ = \ \sum_{j=1}^{\infty} \frac{c_j}{\zeta - a_j} + \sum_{j=1}^{\infty} \frac{c_j}{a_j} \qquad \text{for } |\zeta| = 1.$$

For any $\zeta$ with $|\zeta| \geq 1$, we have $|\zeta - a_j| \geq 1 - a_j$, so

$$\left| \frac{c_j}{\zeta - a_j} \right| \ \leq \ \frac{c_j}{1 - a_j} \ = \ \frac{c_j}{\epsilon_j}$$

and so

$$\sum_{j=1}^{\infty} \left| \frac{c_j}{\zeta - a_j} \right| \ \leq \ \sum_{j=1}^{\infty} \frac{c_j}{\epsilon_j},$$

and by our hypothesis the right hand side converges. Thus the series defining $\omega(\zeta)$ converges absolutely on $|\zeta| = 1$. In fact it converges absolutely and uniformly for $|\zeta| \geq 1$.

We define $\gamma_0$ to be the graph of the function $\omega$ over the curve $|\zeta| = 1$ in $\mathbf{C}^2$.

Fix a point $\zeta$ with $|\zeta| = 1$ and fix $n$. The point $(\zeta, \omega(\zeta))$ lies on $\gamma_0$. Hence,

$$\begin{aligned}
\operatorname{dist}(f_n(\zeta), \gamma_0) \ &\leq \ |\omega(\zeta) - \omega_n(\zeta)| \\
&= \left| \left( \sum_{j=1}^{\infty} \frac{c_j}{\zeta - a_j} + \sum_{j=1}^{\infty} \frac{c_j}{a_j} \right) - \left( \sum_{j=1}^{n} \frac{c_j}{\zeta - a_j} + \sum_{j=1}^{n} \frac{c_j}{a_j} \right) \right| \\
&= \left| \left( \sum_{j=n+1}^{\infty} \frac{c_j}{\zeta - a_j} + \sum_{j=n+1}^{\infty} \frac{c_j}{a_j} \right) \right| \ \leq \ \sum_{j=n+1}^{\infty} \frac{c_j}{1 - a_j} + \sum_{j=n+1}^{\infty} \frac{c_j}{a_j}.
\end{aligned}$$

Fix $r$. We recall that the set $K_r$ is the tube around $\gamma_0$ of radius $r$. In view of the preceding, $\mathrm{dist}(f_n(\partial\Delta), \gamma_0)$ becomes arbitrarily small for all $n$ large enough. So the sequence $\{f_n\}$ satisfies condition (i) for a suitable sequence of numbers $r_n \searrow 0$.

Next observe that $f_n(0) = (0, \omega_n(0)) = (0, 0)$ for all $n$. Hence, the sequence $\{f_n\}$ satisfies condition (ii) with $z_0 = (0, 0)$.

Finally, fix $n$ and note that $\zeta_j^{(n)} = a_j$, $j = 1, ..., n$ are exactly the poles of the map $f_n$. Now we have

$$\log|\zeta_j^{(n)}| \;=\; \log a_j \;=\; \log(1-\epsilon_j) \;\sim\; -\epsilon_j, \quad j = 1, 2, 3, ...$$

so

$$\sum_{j=1}^{n} \log|\zeta_j^{(n)}| \;\sim\; -\sum_{j=1}^{n} \epsilon_j.$$

Now $\sum_{j=1}^{n} \epsilon_j \leq \sum_{j=1}^{\infty} \epsilon_j \equiv M < \infty$ for all $n$, and so $-\sum_{j=1}^{n} \epsilon_j \geq -M$ for all $n$. Thus, for some $M'$ we have

$$\sum_{j=1}^{n} \log|\zeta_j^{(n)}| \;\geq\; -M' \qquad \text{for all} \;\; n,$$

and the sequence satisfies condition (iii). Condition (iv) is straightforward to verify, and we are done with the construction.

Fix a point $\zeta$ in $\Delta \setminus \bigcup_{j=1}^{\infty} a_j$. Then $f_n(\zeta) = (\zeta, \omega_n(\zeta))$ and as $n \to \infty$,

$$f_n(\zeta) \;\to\; (\zeta, \omega(\zeta)),$$

where

$$\omega(\zeta) \;=\; \sum_{j=1}^{\infty} \frac{c_j}{\zeta - a_j} + \sum_{j=1}^{\infty} \frac{c_j}{a_j}. \tag{10}$$

It is easily verified that this series converges uniformly in $\zeta$ on compact subsets of $\mathrm{Int}\Delta \setminus \bigcup_{j=1}^{\infty} a_j$. In fact it converges uniformly on compact subsets of the domain $\mathbf{C} \setminus \{1\} \cup \bigcup_{j=1}^{\infty} a_j$.

Consider the meromorphic map $f(\zeta) = (\zeta, \omega(\zeta))$ on $\mathrm{Int}\Delta$. It follows from Theorem 2 that

$$\Sigma \;\equiv\; f(\mathrm{Int}\Delta) \;\subset\; \widehat{\gamma}_0.$$

This includes all points on the graph of $\omega$ over $\mathrm{Int}\Delta \setminus \bigcup_{j=1}^{\infty} a_j$.

**NB.** The meromorphic map $f$ is in fact a holomorphic map $f : \mathrm{Int}\Delta \to \mathbf{P}^2$. However, its image passes **infinitely often** through the point $\ell \in H \cong \mathbf{P}^1$ corresponding to the "vertical" line in $\mathbf{C}^2$. In particular, the image of $f$ is not an analytic subvariety at that point.

We observed above that the function $\omega(\zeta)$ defined in (10) converges uniformly in $\mathbf{C} \setminus \mathrm{Int}\Delta$. Moreover, its graph extends across infinity to give a regularly embedded disk $\Sigma^-$ in $\mathbf{P}^2$ with boundary $\gamma_0$, taken from the "outside". Thus by [HL$_2$, Prop. 2.3] we have $\Sigma^- \subset \widehat{\gamma}_0$.

We now denote by $L \subset \mathbf{P}^2$ the projective line determined by $\zeta = 1$. Let $W$ be the closure in $\mathbf{P}^2 - L$ of the graph of $\omega$. Note that

$$W = \Sigma \cup \Sigma^- \setminus \{\ell, p\}$$

where $\ell$ is the common polar point referred to above and $p = (1, \omega(1))$. Note that $W$ is a complex analytic subvariety of dimension 1 in $\mathbf{P}^2 - L$. We have proved that $W \subseteq \widehat{\gamma}_0$.

THEOREM 3. *For appropriate choices of the sequences $\{\epsilon_j\}$ and $\{c_j\}$ one has that*

$$\widehat{\gamma}_0 \ = \ W \cup \{\ell, p\} \qquad \text{and} \qquad \overline{(\widehat{\gamma}_0)} \ = \ W \cup L$$

**Proof.** We must show that points of $\mathbf{P}^2 \setminus (W \cup \{\ell, p\})$ do not lie on $\widehat{\gamma}_0$. The second assertion then follows from the Picard Theorem applied to the essential singularity of $\omega$ at 1.

Consider the polynomial of degree $N+1$:

$$P_N(z, w) \ \equiv \left( w - \kappa - \sum_{n=1}^{N} \frac{c_n}{z - a_n} \right) \prod_{n=1}^{N} (z - a_n) \tag{11}$$

Note that

$$P_N(z, w(z)) \ = \ \left( \sum_{n=N+1}^{\infty} \frac{c_n}{z - a_n} \right) \prod_{n=1}^{N} (z - a_n)$$

For $|z| = 1$ we have the estimate that $|z - a_n| \geq \epsilon_n$. Hence we have

$$\|P_N\|_{\gamma_0} \ = \ \sup_{|z|=1} \left| \left( \sum_{n=N+1}^{\infty} \frac{c_n}{z - a_n} \right) \prod_{n=1}^{N} (z - a_n) \right| \ \leq \ \left( \sum_{n=N+1}^{\infty} \frac{c_n}{\epsilon_n} \right) 2^N \tag{12}$$

Now choose $\{c_n\}, \{\epsilon_n\}$ so that

$$\sum_{n=N+1}^{\infty} \frac{c_n}{\epsilon_n} \ < \ \left( \frac{1}{N+1} \right)^{N+1} \tag{13}$$

For example set $\epsilon_n = \frac{1}{2^n}$ and $c_n = \frac{1}{2^n} \frac{1}{2^n} \frac{1}{n^n}$

Now choose $z$ with $z \neq 1$ and $z \neq a_n$ for any $n$. Pick any $w \neq w(z)$. Consider equation (11). The first factor on the RHS converges to $w - w(z) \neq 0$. The second factor satisfies

$$\left| \prod_{n=1}^{N} (z - a_n) \right|^{\frac{1}{N+1}} \ \longrightarrow \ |z - 1| \ \neq \ 0.$$

There cannot exist a constant $C > 0$ so that

$$|P_N(z, w)|^{\frac{1}{N+1}} \ \leq C \left\{ \|P_N\|_{\gamma_0} \right\}^{\frac{1}{N+1}}$$

for all $N$ since this implies

$$0 \neq t|z-1| \leq \frac{2C}{N+1}$$

by (12) and (13).

Suppose now that $z = a_n$ for some $n$. In equation (11) move $(z - a_n)$ over to the left factor so that factor becomes regular at $a_n$. Then we have

$$P_N(a_n, w) = c_n \prod_{j \neq n}^{N} (a_n - a_j).$$

Again we see that

$$\left| \prod_{j \neq n}^{N} (a_n - a_j) \right|^{\frac{1}{N+1}} \longrightarrow |a_n - 1| \neq 0.$$

and the same contradiction results.

Suppose now that $\zeta = 1$ and $w \neq \omega(1)$. We now choose our sequence $\{c_n\}$ to converge even more rapidly so that

$$\sum_{n=N+1}^{\infty} \frac{c_n}{\epsilon_n} < \left( \frac{1}{N+1} \right)^{(N+1)(N+1)} \tag{13$'$}$$

For example set $\epsilon_n = \frac{1}{2^n}$ and $c_n = \frac{1}{2^n} \frac{1}{2^n} \frac{1}{n^{n^2}}$. Then for large $N$ one has

$$P_N(1, w) \sim (w - \omega(1)) \prod_{n=1}^{N} \epsilon_n = (w - w(1)) \prod_{n=1}^{N} \frac{1}{2^n} = \left( \frac{1}{2} \right)^{\frac{N(N+1)}{2}}$$

Comparing with $(13)'$ as above shows that $(1, w) \notin \widehat{\gamma}_0$ when $w - \omega(1) \neq 0$.

It now remains only to eliminate all points on the line $H$ at infinity except for $\ell$ and $p$. We begin with the following observation. Let $\mathcal{P}_N \in H^0(\mathbf{P}^2, \mathcal{O}(N+1))$ denote the holomorphic section corresponding to the polynomial $P_N$. Then equations (3) and (4) imply that for some constant $K$ and all $N$

$$\sup_{\gamma_0} \|\mathcal{P}_N\| \leq \left( \frac{2K}{N+1} \right)^{N+1} \tag{14}$$

where $\| \bullet \|$ denotes the standard metric in the line bundle $\mathcal{O}(N+1)$. This equation (14) can be interpreted in any coordinate chart.

We make a change of coordinates as follows. First let $s = z - 1$ and set $P_N'(s, w) = P_N(s+1, w)$. We now pass to homogeneous coordinates $(t_0, s_0, w_0)$ where the corresponding homogeneous polynomial is

$$Q_N(t_0, s_0, w_0) \equiv t_0^{N+1} P_N' \left( \frac{s_0}{t_0}, \frac{w_0}{t_0} \right).$$

Next we pass to the affine coordinate chart defined by setting $s_0 = 1$, or equivalently by dividing by $s_0$. This gives new coordinates $(t_1, w_1)$ where $t_1 = t_0/s_0$ and $w_1 = w_0/s_0$. Thus, the change of coordinates from the old chart (where $t_0 = 1$) is: $t_1 = 1/s$, $w_1 = w/s$.

For simplicity of notation we relabel these new affine coordinates as $(t, w)$. In this affine chart our polynomial is expressed in terms of $Q_N$ be setting $s_0 = 1$, that is, the polynomial is now $P''_N(t, w) = Q_N(t, 1, w)$. Calculation shows that

$$P''_N(t, w) \;=\; \left(w - \kappa t - \sum_{n=1}^{N} \frac{c_n t^2}{1 + \epsilon_n t}\right) \prod_{n=1}^{N} (1 + \epsilon_n t).$$

Now in the affine $(t, w)$ coordinates the line $L$ has become the line at infinity, and the old line at infinity $H$ corresponds to $\{t = 0\}$. The point $\ell$ lies at infinity on $H$ and the point $p$ corresponds to $(0, 0)$. Note that

$$P''_N(0, w) \;=\; w \qquad \text{and} \qquad \|\mathcal{P}''_N(0, w)\| \;=\; \left(\frac{1}{1 + |w|^2}\right)^{\frac{N+1}{2}} |w| \tag{15}$$

Now if $(0, w) \in \widehat{\gamma}_0$ for $w \neq 0$, then there would be a constant $C > 0$ such that

$$\|\mathcal{P}''_N(0, w)\|^{\frac{1}{N+1}} \;\leq\; C \left(\sup_{\gamma_0} \|\mathcal{P}''_N\|\right)^{\frac{1}{N+1}}$$

contradicting (14) and (15). ∎

Example 2. We repeat the construction above with poles clustering at all points of $\partial\Delta$. Put

$$\widetilde{\omega}_n(\zeta) \;\equiv\; \sum_{k=1}^{n} \sum_{\ell=1}^{k} \left(\frac{c_k}{\zeta - e^{\frac{2\pi i \ell}{k}} a_k}\right) - \kappa_n$$

where $\kappa_n$ is chosen so that $\widetilde{\omega}_n(0) = 0$. Let $a_k = 1 - \epsilon_k$ and choose $\epsilon_k > 0$ and $c_k > 0$ so that $\sum_k \epsilon_k < \infty$ and $\sum_k \frac{k c_k}{\epsilon_k} < \infty$. We now proceed in exact analogy with Example 1. The limit $\omega = \lim_n \omega_n$ converges absolutely on $\partial\Delta$ and its graph defines a curve $\gamma_\infty$ in $\mathbf{C}^2$. The same limit over Int$\Delta$ defines a meromorphic function whose graph lies in the projective hull $\widehat{\gamma}_\infty$ by Theorem 2. This limit also exists at all points of $\mathbf{C} \setminus \Delta$ and gives an exterior analytic disk contained in $\widehat{\gamma}_\infty$.

In this example the closure of $\widehat{\gamma}_\infty$ contains $\partial\Delta \times \mathbf{C}$, a subset if dimension 3.

Set

$$\omega(\zeta) \;=\; \sum_{n=1}^{\infty} \frac{c_n}{\zeta - a_n} + \kappa \qquad \text{where } \kappa \;=\; \sum_{n=1}^{\infty} \frac{c_n}{a_n}$$

We are considering the graph $W$ of $f(\zeta) = (\zeta, \omega(\zeta))$ for $\zeta \neq a_n$ any $n$. Our curve $\gamma_0$ is just the graph of $\omega$ above $\partial\Delta$. For rapidly converging $\{c_n\}$ the analogue of Theorem 3 will hold.

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