The Atiyah-Singer Operator and Scalar Curvature

Misha Gromov and Blaine Lawson

We have been asked to present our perspective on the Index Theorem, which puts us in the position of a blind person describing what he/she perceives of an elephant upon touching the elephant's tail. Index theory is a world of its own, as the reader can begin to see from the article of Connes and Kouneiher in the November, 2019 issue of the A.M.S. Notices. As geometers we are well aware only of what is directly related to our field. With that understood, we can begin.

Over his long career Is Singer made many revolutionary contributions to mathematics. Of all his work, perhaps the most far-reaching were his results with Atiyah on the Index Theorem. It was not just that they established a formula which generalized the Riemann-Roch-Hirzebruch Theorem. What was even more important was their discovery of a fundamental elliptic operator, which certainly paved the way to the Index Theorem, but which also engendered profound applications, which to date can be proved in no other way.

For background, recall that a manifold of dimension n is *orientable*, if its structure group (where the jacobians of coordinate changes live) can be reduced from GL_n to GL_n^+, the connected component of the identity. The manifold is *spin* (with $n \geq 3$) if the structure group GL_n^+ can be lifted to the universal covering group $\widetilde{\mathrm{GL}}_n^+$. At first this seems unimportant since $\widetilde{\mathrm{GL}}_n^+$ has no finite dimensional representations that are not pulled back from GL_n^+. However, if one restricts to maximal compact subgroups $\mathrm{SO}_n \to \mathrm{GL}_n^+$ and $\mathrm{Spin}_n \to \widetilde{\mathrm{GL}}_n^+$ (these inclusions are homotopy equivalences), then Spin_n *does* have representations which are not pulled back from SO_n.

In the 1960's many decades had passed since the fundamental work of É. Cartan and P.A.M. Dirac, but the fundamental operator on a spin manifold with a riemannian metric was still unknown. Borel and Hirzebruch had proved that on a spin manifold M of dimension $4k$ a certain topological invariant $\widehat{A}(M)$ is always an integer, and this is not always true for M's which are not spin. Atiyah had conjectured that perhaps this was because $\widehat{A}(M)$ was the index of an operator that only existed on spin manifolds. It was Singer who found that operator, and they were on their way. They showed in fact that, in the spin case, every elliptic operator was equivalent to this fundamental operator $D\!\!\!/$, twisted by a coefficient bundle, and the index formula is a natural generalization of Hirzebruch's. With this insight they were able, with more work, to also treat the non-spin case.

It is important to understand that to define the operator $D\!\!\!/$ it is necessary to have a metric on the spin manifold. Given this metric, the operator satisfies a Bochner-type identity, found by E. Schrödinger and A. Lichnerowicz:

$$D\!\!\!/^2 \ = \ \nabla^*\nabla + \frac{1}{4}\kappa,$$

where $\nabla^*\nabla \geq 0$ with kernel consisting of parallel creatures, and κ is the scalar curvature (the average of all the sectional curvatures), which is a very weak invariant. However, from this formula one sees that on a spin $4k$-manifold with $\widehat{A} \neq 0$, there does not exist a riemannian metric with $\kappa > 0$.

If one considers the general case of $D\!\!\!/_E$ of $D\!\!\!/$ tensored with a vector bundle with connection E, there is the formula

$$D\!\!\!/_E^2 \ = \ \nabla^*\nabla + \frac{1}{4}\kappa + \mathbf{R}_E$$

where $\mathbf{R}_E$ is explicitly defined in terms of the curvatures of E. This allows for interesting generalizations of the scalar curvature result above.

Atiyah and Singer went on to give a different proof of the Index Theorem, which was based on topological K-theory (= the Grothendieck group of isomorphism classes of complex vector bundles on a given space – work of Atiyah and Hirzebruch). They also found many important generalizations. When the operator is equivariant with respect to a compact Lie group G, the index lies in the representation ring of G, and this result has many applications, including new results in number theory. In a different direction, suppose we have a family of elliptic operators over a parameter space X, then there is a theorem where the index of the family is an element of the K-theory of X. Perhaps one of the most profound of these generalizations was the Index Theorem for $\mathrm{Cl}_k(\mathbf{R})$-linear operators, where $\mathrm{Cl}_k(\mathbf{R})$ is the Clifford algebra on $\mathbf{R}^k$ with its standard positive definite metric. Here the index is an element in the quotient of the Grothendieck group of $\mathrm{Cl}_k(\mathbf{R})$-modules which can be identified with $\mathrm{KO}_{-k}(\mathrm{pt})$, the real K-theory group for a point. Interestingly, these groups are torsion in certain dimensions. Now every spin n-manifold has a fundamental $\mathrm{Cl}_n(\mathbf{R})$-linear Atiyah-Singer operator with index in the group $\mathrm{KO}_{-n}(\mathrm{pt})$. This gives a ring homomorphism $\widehat{\mathcal{A}} : \Omega_*^{\mathrm{Spin}} \to \mathrm{KO}_{-*}(\mathrm{pt})$ from the spin cobordism ring, which must vanish on every cobordism class which contains a manifold which admits a metric with $\kappa > 0$. In particular, by a result of N. Hitchin, for all $n \equiv 1$ or $2 \pmod 8$, $(n \geq 8)$, *half of the exotic spheres do not carry a metric of positive scalar curvature*! From the classical geometer's point of view this was pure magic.

Somewhat later, Stephan Stolz showed that, in the world of simply-connected spin manifolds of dimension > 4, the index $\widehat{\mathcal{A}}$ is the only invariant obstructing positive scalar curvature. For manifolds with non-trivial fundamental group, there is an analogous, yet more wide-ranging and still unfinished, story.

We want to say that Is Singer brought more than great insights to mathematics. His talks were magical. His enthusiasm was totally contagious. He spent many years bringing together the worlds of mathematics and physics, and this transformed both fields. Is Singer not only opened people to wonders in mathematics, but he made them excited to be part of the wonderment. He was always inclusive, and he and Rosemary were very warm and outreaching. Is Singer made mathematics a better place.