ANALYTIC AND HOMOLOGICAL ASPECTS

of

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REESE HARVEY
JOHN ZWECK

JEFF CHEEGER

OUTLINE

Part I. The de Rham - Federer Theory.

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arise in many analytic settings.

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- MacPherson Formulas for Degeneracies of Bundle Maps

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has dense range. In fact this gives an **isomorphism**

$$\widehat{I\!H}^{n-k-1}(X) \xrightarrow{\cong} \mathrm{Hom}_{\infty}\left(\widehat{I\!H}^{k}(X), S^1\right)$$

with the **smooth** homomorphisms.

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(the historical beginning)

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As with ordinary cohomology,
This universality makes differential cohomology
useful and important.

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The apparatus also extends beyond differential cohomology.

$\overline{\partial}$-Sparks

Part I.

The de Rham-Federer Formulation of Differential Cohomology

- Devised by Harvey-L.-Zweck (2003)
- It motivates and is archetypical of the spark apparatus.
- The product is geometrically clear.

De Rham - Federer Sparks

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$$\text{for } f \in \mathcal{O}(U).$$

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Dirac Monopoles.

$L \to S^2 = \mathbf{P}^1_{\mathbf{C}}$ a hermitian holomorphic line bundle,
$f \in \Gamma_{\mathrm{hol}}(L)$

$$dd^c \log \|f\| \;=\; c_1(L) - \mathrm{Div}(f)$$

More Generally.

$L \to X$ a hermitian holomorphic line bundle,
over any complex manifold X.

$f \in \Gamma_{\text{hol}}(L)$

$$\boxed{dT \;=\; c_1(L) - \mathrm{Div}(f)}$$

where

$$T \;\equiv\; \frac{1}{2\pi i}\overline{\partial}\log\|f\|$$

is a 1-form with L^1_{loc}-coefficients.

Chern-Weil Transgressions.

$E \to X$ a real oriented vector bundle with orthogonal connection

$\sigma \in \Gamma(E)$ a smooth section with non-degenerate zeros

$$\boxed{dT \;=\; \chi\left(\Omega^E\right) - \mathrm{Zero}(\sigma)}$$

where

$\chi\left(\Omega^E\right)$ is the Chern-Euler form (the Pfaffian) and

T is an $(n-1)$-form with L^1_{loc}-coefficients.

Chern-Weil Transgressions.

$E \to X$ a C^∞ complex vector bundle
with unitary connection

$\sigma_0, ..., \sigma_k \in \Gamma(E)$ generic smooth sections

$$\boxed{dT \;=\; c_{n-k+1}\left(\Omega^E\right) - \mathbb{LD}(\sigma_0, ..., \sigma_k)}$$

where

$\mathbb{LD}(\sigma_0, ..., \sigma_k)$ is the linear dependency locus, and

T is an $(n-1)$-form with L^1_{loc}-coefficients.

Local MacPherson Formulas.

$E, F \to X$ C^∞ complex vector bundles of same rank with unitary connections

$\sigma : E \to F$ smooth bundle map.

p a U_n-invariant polynomial on $\mathfrak{u}_n$.

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p a U_n-invariant polynomial on $\mathfrak{u}_n$.

$$\boxed{dT = p\left(\Omega^F\right) - p\left(\Omega^E\right) - \sum_k \mathrm{Res}_{p,k}[\Sigma_k(\sigma)]}$$

where

T is an $(n-1)$-form with L^1_{loc}-coefficients.

Morse Theory.

$f : X \to \mathbf{R}$ a Morse function on a compact oriented manifold X.

$\varphi_t : X \to X$ a Morse-Smale flow.

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There exists a linear map

$$T : \mathcal{E}^*(X) \longrightarrow \mathcal{E}'^{*-1}(X)$$

such that

$$dT(\alpha) = \alpha - \sum_{p \in Cr(f)} n_p S_p$$

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for any closed form α with

$$n_p = \int_{U_p} \alpha \in \mathbf{Z}.$$

QUESTION:

What are these forms T?

Do they define invariants?

DE RHAM THEORY

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currents of dimension k

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Example 1. $M \subset X$ a compact oriented submanifold of dimension k.

$$[M] \in \mathcal{E}'_k(X) \qquad \text{defined by} \qquad [M](\varphi) \equiv \int_M \varphi$$

$$\partial[M] = [\partial M]$$

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Example 2. $\psi \in \mathcal{E}^{n-k}(X)$.

$$[\psi] \in \mathcal{E}'_k(X) \qquad \text{defined by} \qquad [\psi](\varphi) \equiv \int_X \psi \wedge \varphi$$

$$\partial[\psi] = (-1)^{n-k+1}[d\psi]$$

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Theorem. (de Rham).

$$H^*(\mathcal{E}^*(X)) \cong H^*(\mathcal{E}'^*(X)) \cong H^*(X; \mathbf{R})$$

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Definition. A **de Rham - Federer spark** of degree k is a current

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Theorem. *φ and R are uniquely determined by a and*

$$d\varphi = dR = 0.$$

EQUIVALENCE

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$$\mathcal{S}^k(X) \;\equiv\; \text{the group of sparks of degree } k.$$

Definition. Two sparks $a, a' \in \mathcal{S}^k(X)$ are **equivalent** if

$$a - a' \;=\; db + S \qquad \text{where}$$

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Definition. The group of **de Rham-Federer spark classes** of degree k is the quotient

$$\widehat{I\!H}^k(X) \;\equiv\; \mathcal{S}^k(X)/\left\{d\mathcal{E}'^{k-1}(X) + \mathcal{R}^k(X)\right\}$$

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Furthermore, φ *and* $[R] \in H^{k+1}(X; \mathbf{Z})$
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Note. φ has **integral periods**

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This determines homomorphisms.

$$\delta_1 : \widehat{I\!\!H}^k(X) \;\longrightarrow\; \mathcal{Z}_0^{k+1}(X)$$

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by

$$\delta_1([a]) \ = \ \varphi \qquad \text{and} \qquad \delta_2([a]) \ = \ [R]$$

SEQUENCES

Theorem. *There are short exact sequences*

$$0 \longrightarrow H^k(X, S^1) \longrightarrow \widehat{I\!H}^k(X) \xrightarrow{\delta_1} \mathcal{Z}_0^{k+1}(X) \longrightarrow 0$$

$$0 \longrightarrow \widehat{I\!H}^k_\infty(X) \longrightarrow \widehat{I\!H}^k(X) \xrightarrow{\delta_2} H^{k+1}(X, \mathbf{Z}) \longrightarrow 0$$

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where

$$\widehat{I\!H}^k_\infty(X) \equiv \mathcal{E}^k(X)/\mathcal{Z}_0^k(X)$$

are the **smooth** characters,
those represented by smooth forms.

THE EXACT GRID

$$
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \frac{H^k(X,\mathbf{R})}{H^k_{\text{free}}(X,\mathbf{Z})} & \to & \widehat{H}^k_\infty(X) & \to & d\mathcal{E}^k(X) & \to \quad 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^k(X,S^1) & \to & \widehat{H}^k(X) & \xrightarrow{\delta_1} & \mathcal{Z}^{k+1}_0(X) & \to \quad 0 \\
 & \downarrow & & \delta_2 \downarrow & & \downarrow & \\
0 \to & H^{k+1}_{\text{tor}}(X,\mathbf{Z}) & \to & H^{k+1}(X,\mathbf{Z}) & \to & H^{k+1}_{\text{free}}(X,\mathbf{Z}) & \to \quad 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}
$$

THE PRODUCT

Theorem. *Given classes*

$$\alpha \in \widehat{I\!H}^k(X) \qquad \text{and} \qquad \beta \in \widehat{I\!H}^\ell(X).$$

there exist representatives $a \in \alpha$ and $b \in \beta$ with

$$da \;=\; \phi - R \qquad \text{and} \qquad db \;=\; \psi - S$$

so that $a \wedge S$, $R \wedge b$ and $R \wedge S$ are well-defined currents and $R \wedge S$ is rectifiable.

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$$da \;=\; \phi - R \qquad \text{and} \qquad db \;=\; \psi - S$$

so that $a \wedge S$, $R \wedge b$ and $R \wedge S$ are well-defined currents and $R \wedge S$ is rectifiable.

Definition. Given a and b as above, define products

$$a * b \;\stackrel{\text{def}}{=}\; a \wedge \psi \,+\, (-1)^{k+1} R \wedge b$$
$$a \,\tilde{*}\, b \;\stackrel{\text{def}}{=}\; a \wedge S \,+\, (-1)^{k+1} \phi \wedge b$$

and note that

$$d(a * b) \;=\; d(a \,\tilde{*}\, b) \;=\; \phi \wedge \psi \,-\, R \wedge S.$$

THE PRODUCT

Proposition. *Given classes*

$$\alpha \in \widehat{I\!H}^{k}(X) \qquad \text{and} \qquad \beta \in \widehat{I\!H}^{\ell}(X).$$

there exist representatives $a \in \alpha$ *and* $b \in \beta$ *with*

$$da \ = \ \phi - R \qquad \text{and} \qquad db \ = \ \psi - S$$

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$$d(a * b) \ = \ d(a \,\tilde{*}\, b) \ = \ \phi \wedge \psi \ - \ R \wedge S.$$

Theorem. *The classes* $[a * b]$ *and* $[a \tilde{*} b]$ *in* $\widehat{I\!H}^{k+\ell+1}(X)$ *agree and are independent of the choice of representatives* $a \in \alpha$ *and* $b \in \beta$*. Setting*

$$\alpha * \beta \ \stackrel{\text{def}}{=} \ [a * b] \ = \ [a \tilde{*} b]$$

gives $\widehat{I\!H}^{*}(X)$ *the structure of a graded commutative ring with unit such that* δ_1, δ_2 *are a ring homomorphisms.*

THE ORIGINAL CHEEGER-SIMONS CHARACTERS

$$\widehat{H}^k(X, \mathbf{R}/\mathbf{Z})$$

given by

$$h \in \text{Hom}\,(\mathcal{Z}_k(X), \mathbf{R}/\mathbf{Z})$$

with

$$\delta h \;\equiv\; \phi \;(\text{mod}\mathbf{Z}) \qquad \text{for some } \phi \in \mathcal{E}^{k+1}(X)$$

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$$\Psi : \widehat{I\!H}^k(X) \xrightarrow{\cong} \widehat{H}^k(X; \mathbf{R}/\mathbf{Z})$$

induced by integration.

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Idea: Given $\alpha \in \widehat{I\!\!H}^k(X)$ and $Z \in \mathcal{Z}_k(X)$, there exists $a \in \alpha$ smooth on a neighborhood of $\operatorname{supp}(Z)$.

$$\Psi(\alpha) = \int_Z a$$

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$$\Psi(\alpha) \ = \ \int_Z a$$

Given two such representives $a, a' \in \alpha$,

$$\int_Z a \ \equiv \ \int_Z a' \ \ (\mathrm{mod} \ \ \mathbf{Z}).$$

DIFFERENT D-F SPARK COMPLEXES

In the above one could:

Replace $\mathcal{R}^k(X)$ by C^∞ singular $(n-k)$-chains

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THE VARIOUS FORMULATIONS

(i) De Rham-Federer Theories

(ii) Abelian Gerbes with Connection
mod Gauge Equivalence

(iii) Cheeger-Simons Characters
(the historical beginning)

(iv) Hypersparks

(v) Holonomy Maps

(vi) Many others

ARE ALL COVERED BY THE FOLLOWING
HOMOLOGICAL ALGEBRAIC MACHINE

HOMOLOGICAL SPARK COMPLEXES

Definition. A **homological spark complex** is a triple of cochain complexes (F^*, E^*, I^*) together with morphisms

$$I^* \xrightarrow{\ \Psi\ } F^* \supset E^*$$

such that:

(i) $\Psi(I^k) \cap E^k = \{0\}$ for $k > 0$,

(ii) $H^*(E) \cong H^*(F)$, and

(iii) $\Psi : I^0 \to F^0$ is injective.

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Definition. Given (F^*, E^*, I^*) a **spark of degree** k is a pair

$$(a, r) \in F^k \oplus I^{k+1}$$

which satisfies the **spark equation**

(i) $da = e - \Psi(r)$ for some $e \in E^{k+1}$, and

(ii) $dr = 0$.

(iii) $de = 0$ (follows from (i) and (ii))

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$$\mathcal{S}^k \equiv \mathcal{S}^k(F^*, E^*, I^*)$$

is the **group of sparks** of degree k.

EQUIVALENCE

Definition. Two sparks $(a, r), (a'r') \in \mathcal{S}^k(F^*, E^*, I^*)$ are **equivalent** if there exists a pair

$$(b, s) \in F^{k-1} \oplus I^k$$

$$\begin{aligned} &\text{(i)} \quad a - a' \;=\; db + \Psi(s) \\ &\text{(ii)} \quad r - r' \;=\; -ds. \end{aligned}$$

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The equivalence classes:

$$\widehat{I\!H}^k(F^*, E^*, I^*) = \widehat{I\!H}^k$$

are the **group of spark classes of degree** k

SEQUENCES

$$Z^k(E) = \{e \in E^k : de = 0\}$$

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$$Z_I^k(E) \;\equiv\; \{e \in Z^k(E) \;:\; [e] = \Psi_*(\rho) \quad \text{for some} \;\; \rho \in H^k(I)\}$$

where $[e]$ is the class of e in $H^k(E) \cong H^k(F)$.

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where $[e]$ is the class of e in $H^k(E) \cong H^k(F)$.

Lemma. *There exist well-defined surjective homomorphisms:*

$$\widehat{I\!H}^k \xrightarrow{\delta_1} Z_I^{k+1}(E) \qquad \text{and} \qquad \widehat{I\!H}^k \xrightarrow{\delta_2} H^{k+1}(I)$$

given on $(a, r) \in \mathcal{S}^k$ *by*

$$\delta_1(a, r) = e \qquad \text{and} \qquad \delta_2(a, r) = [r]$$

where $da = e - \Psi(r)$.

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where $da = e - \Psi(r)$.

Lemma. *Let* $\widehat{I\!H}^k_E = \ker \delta_2$. *Then*

$$\widehat{I\!H}^k_E \;=\; E^k / Z_I^k(E)$$

SEQUENCES

Definition. Associated to (F^*, E^*, I^*) is the **cone complex** (G^*, D) defined by

$$
\begin{aligned}
G^k &\equiv F^k \oplus I^{k+1} \qquad\qquad k \geq -1 \\
D(a,r) &= (da + \Psi(r), -dr)
\end{aligned}
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There is a short exact sequence of complexes

$$0 \to F^* \to G^* \to I^*(1) \to 0$$

where $I^k(1) \equiv I^{k+1}$. The morphism Ψ defines a chain map of degree 1:

$$F^* \quad \xleftarrow{\Psi} \quad I^*(1)$$

which induces the connecting homomorphisms in the associated long exact sequence in cohomology.

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Proposition. *There are two short exact sequences:*

$$0 \longrightarrow H^k(G) \longrightarrow \widehat{I\!H}^k \xrightarrow{\delta_1} Z_I^{k+1}(E) \longrightarrow 0$$

$$0 \longrightarrow \widehat{I\!H}^k_E \longrightarrow \widehat{I\!H}^k \xrightarrow{\delta_2} H^{k+1}(I) \longrightarrow 0$$

THE GRID

Consider $\Psi_* : H^k(I) \to H^k(F) \cong H^k(E)$, and define

$$H_I^k(E) \ \equiv \ \text{Image}\{\Psi_*\} \qquad \text{and} \qquad Ker^k(I) \ \equiv \ \ker\{\Psi_*\}$$

THE GRID

The exact sequences fit into a 3×3 commutative grid.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{H^k(E)}{H_I^k(E)} & \longrightarrow & \widehat{I\!H}_E^k & \longrightarrow & dE^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^k(G) & \longrightarrow & \widehat{I\!H}^k & \xrightarrow{\delta_1} & Z_I^{k+1}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \delta_2 \downarrow & & \downarrow & & \\
0 & \longrightarrow & Ker^{k+1}(I) & \longrightarrow & H^{k+1}(I) & \xrightarrow{\Psi_*} & H_I^{k+1}(E) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

IMPORTANT CONCEPT

QUASI-ISOMORPHISMS OF SPARK COMPLEXES

Definition. Two spark complexes (F^*, E^*, I^*) and $(\overline{F}^*, \overline{E}^*, \overline{I}^*)$ are **quasi-isomorphic** if there exists a commutative diagram of morphisms

$$\begin{array}{ccccc} \overline{I}^* & \xrightarrow{\overline{\Psi}} & \overline{F}^* & \supset & \overline{E}^* \\ \psi \uparrow & & \cup & & \| \\ I^* & \xrightarrow{\Psi} & F^* & \supset & E^* \end{array}$$

inducing an isomorphism

$$\psi^* : H^*(I) \xrightarrow{\cong} H^*(\overline{I})$$

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Theorem. *A quasi-isomorphism*

$$(F^*, E^*, I^*) \cong (\overline{F}^*, \overline{E}^*, \overline{I}^*)$$

induces an **isomorphism**

$$\widehat{\mathbb{H}}^*(F^*, E^*, I^*) \cong \widehat{\mathbb{H}}^*(\overline{F}^*, \overline{E}^*, \overline{I}^*)$$

and an isomorphism of the associated grids.

EXAMPLES

1. de Rham -Federer Spark Complexes.

$$
\begin{aligned}
F^* &= \mathcal{E}'^*(X) \\
E^* &= \mathcal{E}^*(X) \\
I^* &= \mathcal{I}^*(X)
\end{aligned}
$$

$$\mathcal{I}^k(X) = \{R \in \mathcal{E}'^k(X) : R \text{ and } \partial R \text{ are rectifiable}\}$$

EXAMPLES

1. de Rham -Federer Spark Complexes.

$$\begin{aligned} F^* &= \mathcal{E}'^*(X) \\ E^* &= \mathcal{E}^*(X) \\ I^* &= \mathcal{IF}^*(X) \end{aligned}$$

$$\mathcal{IF}^k(X) = \{R+\partial S \in \mathcal{E}'^k(X) : R \text{ and } S \text{ are rectifiable}\}$$

EXAMPLES

1. de Rham -Federer Spark Complexes.

$$
\begin{aligned}
F^* &= {\mathcal{E}'}^*_{L^1_{\text{loc}}}(X) \\
E^* &= \mathcal{E}^*(X) \\
I^* &= \mathcal{I}^*(X)
\end{aligned}
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EXAMPLES

1. de Rham -Federer Spark Complexes.

$$
\begin{aligned}
F^* &= {\mathcal{E}'}^*(X) \\
E^* &= \mathcal{E}^*(X) \\
I^* &= \mathcal{C}^*(X)
\end{aligned}
$$

$\mathcal{C}^*(X) =$ smooth singular chains

2. Smooth Hyperspark Complexes.

Fix an open covering $\mathcal{U} = \{U_i\}$ of X with each $U_I = U_{i_0} \cap \cdots \cap U_{i_p}$ contractible. Consider the Čech-de Rham double complex

$$\bigoplus_{p,q\geq 0} C^p(\mathcal{U}, \mathcal{E}^q) \quad \text{with} \quad D = (-1)^q \delta + d$$

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There are two **edge complexes**:

$$0 \to \mathcal{E}^q(X) \to C^0(\mathcal{U}, \mathcal{E}^q) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{E}^q)$$

$$0 \to C^p(\mathcal{U}, \mathbf{R}) \to C^p(\mathcal{U}, \mathcal{E}^0) \xrightarrow{d} C^p(\mathcal{U}, \mathcal{E}^1)$$

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$$0 \to C^p(\mathcal{U}, \mathbf{R}) \to C^p(\mathcal{U}, \mathcal{E}^0) \xrightarrow{d} C^p(\mathcal{U}, \mathcal{E}^1)$$

Define:

$$\begin{aligned} F^k &= \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}^q) \\ E^k &= \mathcal{E}^k(X) \\ I^k &= C^k(\mathcal{U}, \mathbf{Z}) \subset C^k(\mathcal{U}, \mathbf{R}) \end{aligned}$$

$$
\begin{array}{ccccccccc}
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathcal{E}^2(X) & \subset & C^0(\mathcal{U},\mathcal{E}^2) & \to & C^1(\mathcal{U},\mathcal{E}^2) & \to & C^2(\mathcal{U},\mathcal{E}^2) & \to & \cdots \\
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\end{array}
$$

The spark classes are abelian gerbes with connection
modulo gauge equivalence.

3. Hyperspark Complexes.

Almost the same:

$$\begin{aligned} F^k &= \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{E}'^q) \\ E^k &= \mathcal{E}^k(X) \\ I^k &= \bigoplus_{p+q=k} C^p(\mathcal{U}, \mathcal{I}^q_{\text{loc}}) \end{aligned}$$

This evidently contains the smooth Hyperspark complex.

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\end{aligned}
$$

This evidently contains the smooth Hyperspark complex.

It also contains the dR-F complexes.

These inclusions are quasi-isomorphisms.

4. Cheeger-Simons Spark Complex.

$$\mathcal{C}_k(X) \;\equiv\; C^\infty \text{ singular integral } k\text{-chains}$$

$$C^k_{\mathbf{R}}(X) \;\equiv\; \mathrm{Hom}\,(\mathcal{C}_k(X), \mathbf{R}) \qquad \text{and} \qquad C^k_{\mathbf{Z}}(X) \;\equiv\; \mathrm{Hom}\,(\mathcal{C}_k(X), \mathbf{Z})$$

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$$\begin{aligned} F^* &= C^*_{\mathbf{R}}(X) \\ E^* &= \mathcal{E}^*(X) \\ I^* &= C^k_{\mathbf{Z}}(X) \end{aligned}$$

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$$\widehat{I\!H}^* \cong \mathrm{DiffChar}^*$$

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$$\widehat{I\!\!H}^* \ \cong \text{DiffChar}^*$$

This is quasi-isomorphic to smooth hypersparks, therefore ...

OTHER COMPLEXES

$\overline{\partial}$-Sparks.

X a complex manifold.

Fix an integer $p > 0$.

OTHER COMPLEXES

$\overline{\partial}$-Sparks.

X a complex manifold.
Fix an integer $p > 0$.

Consider the truncated de Rham complex $(\mathcal{E}'^*(X,p), \overline{d})$ with

$$\mathcal{E}'^k(X,p) \equiv \bigoplus_{r+s=k, r<p} \mathcal{E}'^{r,s}(X) \qquad \text{and} \qquad \overline{d} \equiv \Psi \circ d$$

where

$$\Psi : \mathcal{E}'^k(X) \longrightarrow \mathcal{E}'^k(X,p)$$

is the projection $\Psi(a) = a^{0,k} + a^{1,k-1} + \ldots + a^{p-1,k-p+1}$.

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where

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is the projection $\Psi(a) = a^{0,k} + a^{1,k-1} + \ldots + a^{p-1,k-p+1}$.

Note the subcomplex

$$\mathcal{E}^k(X,p) \equiv \bigoplus_{r+s=k, r<p} \mathcal{E}^{r,s}(X)$$

of smooth forms with projection $\Psi : \mathcal{E}^*(X) \to \mathcal{E}^*(X,p)$.

OTHER COMPLEXES

$\overline{\partial}$-Sparks.

X a complex manifold.

Fix an integer $p > 0$.

Consider the truncated de Rham complex $(\mathcal{E}'^*(X,p), \overline{d})$ with

$$\mathcal{E}'^k(X,p) \equiv \bigoplus_{r+s=k, r<p} \mathcal{E}'^{r,s}(X) \qquad \text{and} \qquad \overline{d} \equiv \Psi \circ d$$

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Definition. The **$\overline{d}$-spark complex of level** p is the triple (F^*, E^*, I^*) where

$$F^k \equiv \mathcal{E}'^k(X,p)$$
$$E^k \equiv \mathcal{E}^k(X,p)$$
$$I^k \equiv \mathcal{I}^k(X)$$

with maps

$$E^* \subset F^* \qquad \text{and} \qquad \Psi : I^* \longrightarrow F^*$$

The group of associated spark classes is

$$\widehat{\mathbf{H}}^k(X,p).$$

THE GRID

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{J}_p(X) & \rightarrow & \widehat{\mathbf{H}}^{2p-1}_{\infty}(X,p) & \rightarrow & \overline{d}\mathcal{E}^{2p-1}(X,p) & \rightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & H^{2p}_{\mathcal{D}}(X,\mathbf{Z}(p))(X) & \rightarrow & \widehat{\mathbf{H}}^{2p-1}(X,p) & \rightarrow & \mathcal{Z}^{2p}_{\mathbf{Z}}(X,p) & \rightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathrm{Hdg}^{p,p}(X) & \rightarrow & H^{2p}(X;\mathbf{Z}) & \rightarrow & H^{2p}_{\mathbf{Z}}(X,p) & \rightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

POINCARÉ-PONTRJAGIN DUALITY

Recall: For a locally compact abelian topological group G, the **Pontrjagin dual** is:

$$G^* \equiv \mathrm{Hom}\,(G, S^1)$$

where Hom denotes continuous group homomorphisms.

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If $\Lambda \subset \mathbf{R}^m$ is a lattice of full rank,

$$\mathrm{Hom}\,(\mathbf{R}^m/\Lambda, S^1) = \mathrm{Hom}\,(\Lambda, \mathbf{Z})$$

$$\mathrm{Hom}\,(\Lambda, S^1) = \mathrm{Hom}\,(\Lambda, \mathbf{R})/\mathrm{Hom}\,(\Lambda, \mathbf{Z})$$

Theorem. (Poincaré-Pontrjagin Duality). *X a compact oriented manifold of dimension n. The pairing*

$$\widehat{I\!H}^{n-k-1}(X) \times \widehat{I\!H}^{k}(X) \longrightarrow S^1$$

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The resulting mapping

$$\widehat{I\!H}^{n-k-1}(X) \longrightarrow \operatorname{Hom}\left(\widehat{I\!H}^{k}(X), S^1\right)$$

has dense range. In fact this gives an **isomorphism**

$$\widehat{I\!H}^{n-k-1}(X) \xrightarrow{\cong} \operatorname{Hom}_{\infty}\left(\widehat{I\!H}^{k}(X), S^1\right)$$

with the **smooth** *homomorphisms.*

Definition of Hom_{∞}

A homomorphism $h : \widehat{I\!H}^k_\infty(X) \to S^1$ is called **smooth** if there exists a smooth form $\omega \in \mathcal{Z}_0^{n-k}(X)$ such that

$$h(\alpha) \equiv \int_X a \wedge \omega \pmod{\mathbf{Z}}$$

for $a \in \alpha$. This is independent of the choice of $a \in \alpha$.

The **smooth Pontrjagin dual** of $\widehat{I\!H}^k(X)$ is the subgroup

$$\mathrm{Hom}_{\infty}(\widehat{I\!H}^k(X),\, S^1) \subset \mathrm{Hom}\,(\widehat{I\!H}^k(X),\, S^1)$$

of those elements whose restriction to $\widehat{I\!H}^k_\infty$ is smooth.

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$$\mathrm{Hom}\,(d\mathcal{E}^k,\mathbf{R}/\mathbf{Z}) \;=\; \mathrm{Hom}\,(d\mathcal{E}^k,\mathbf{R}) \;=\; (d\mathcal{E}^k)' \;=\; d\mathcal{E}'^{\,n-k-1}$$

Def. $$\mathrm{Hom}_{\,\infty}(d\mathcal{E}^k,\mathbf{R}/\mathbf{Z}) \;=\; d\mathcal{E}^{n-k-1}$$

A Simple Picture of Duality

There are (non-canonical) group homomorphisms:

$$\widehat{I\!\!H}^k(X) \;\cong\; d\mathcal{E}^k(X) \times \frac{H^k(X;\,\mathbf{R})}{H^k_{\mathrm{free}}(X;\,\mathbf{Z})} \times H^{k+1}(X;\,\mathbf{Z})$$

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Now apply:

$$\operatorname{Hom}\left(H^{k}(X;\,\mathbf{Z}),\,S^1\right) \;\cong\; H^{n-k}(X;\,S^1)$$

$$\operatorname{Hom}\left(\frac{H^{k}(X;\,\mathbf{R})}{H^{k}_{\text{free}}(X;\,\mathbf{Z})},\,S^1\right) \;\cong\; H^{n-k}_{\text{free}}(X;\,\mathbf{Z})$$

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We see this explicitly from

$$0 \to \frac{H^1(\Sigma;\, \mathbf{R})}{H^1(\Sigma;\, \mathbf{Z})} \to \widehat{I\!H}^1(\Sigma) \to d\mathcal{E}^1(\Sigma) \times \mathbf{Z} \to 0$$

$$0 \to \frac{H^0(\Sigma;\, \mathbf{R})}{H^0(\Sigma;\, \mathbf{Z})} \to \widehat{I\!H}^0(\Sigma) \to d\mathcal{E}^0(\Sigma) \times \mathbf{Z}^{2g} \to 0$$

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from which we deduce that

$$\widehat{I\!H}^1(\Sigma) \cong (S^1)^{2g} \times \mathbf{Z} \times d\mathcal{E}^1(\Sigma)$$

$$\widehat{I\!H}^0(\Sigma) \cong \mathbf{Z}^{2g} \times S^1 \times d\mathcal{E}^0(\Sigma)$$

Example 2. (3-manifolds).

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Using Poincaré duality this can be rewritten

$$\text{Hom}\,(H_1(M;\mathbf{Z})_{\text{free}}, S^1) \times H_1(M;\mathbf{Z})_{\text{free}} \times d\mathcal{E}^1(M) \times H_1(M;\mathbf{Z})_{\text{torsion}}$$

from which the self-duality is manifest.

Example 3. (Complex Projective Space $\mathbf{P}^n_{\mathbf{C}}$). If $X = \mathbf{P}^n_{\mathbf{C}}$, we see that

$$\frac{H^k(X;\mathbf{R})}{H^k(X;\mathbf{Z})} = \begin{cases} S^1 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

and

$$\mathcal{Z}_0^{k+1}(X) = \begin{cases} d\mathcal{E}^k(X) & \text{if } k \text{ is even} \\ \mathbf{Z} \times d\mathcal{E}^k(X) & \text{if } k \text{ is odd.} \end{cases}$$

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We find that:

k	$\widehat{I\!H}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z} \times d\mathcal{E}^1$
2	$S^1 \times d\mathcal{E}^2$
3	$\mathbf{Z} \times d\mathcal{E}^3$
$\vdots$	$\vdots$
$2n-3$	$\mathbf{Z} \times d\mathcal{E}^{2n-3}$
$2n-2$	$S^1 \times d\mathcal{E}^{2n-2}$
$2n-1$	$\mathbf{Z} \times d\mathcal{E}^{2n-1}$
$2n$	S^1

Example 4. (Real Projective Space $\mathbf{P}_{\mathbf{R}}^{2n+1}$)

For $X = \mathbf{P}_{\mathbf{R}}^{2n+1}$ we have that:

k	$\widehat{I\!H}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z}_2 \times d\mathcal{E}^1$
2	$d\mathcal{E}^2$
3	$\mathbf{Z}_2 \times d\mathcal{E}^3$
4	$d\mathcal{E}^4$
5	$\mathbf{Z}_2 \times d\mathcal{E}^5$
6	$d\mathcal{E}^6$
.	.
.	.
.	.
$2n-4$	$d\mathcal{E}^{2n-4}$
$2n-3$	$\mathbf{Z}_2 \times d\mathcal{E}^{2n-3}$
$2n-2$	$d\mathcal{E}^{2n-2}$
$2n-1$	$\mathbf{Z}_2 \times d\mathcal{E}^{2n-1}$
$2n-0$	$\mathbf{Z} \times d\mathcal{E}^{2n}$
$2n+1$	S^1

Recall that $\mathrm{Hom}\,(\mathbf{Z}_2, S^1) = \mathbf{Z}_2$.

Example 5. (Products of projective spaces).

When $X = \mathbf{P}^2_{\mathbf{C}} \times \mathbf{P}^2_{\mathbf{C}}$, we find that:

k	$\widehat{I\!H}^k(X)$
-1	$\mathbf{Z}$
0	$S^1 \times d\mathcal{E}^0$
1	$\mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^1$
2	$S^1 \times S^1 \times d\mathcal{E}^2$
3	$\mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^3$
4	$S^1 \times S^1 \times S^1 \times d\mathcal{E}^4$
5	$\mathbf{Z} \times \mathbf{Z} \times d\mathcal{E}^5$
6	$S^1 \times S^1 \times d\mathcal{E}^6$
7	$\mathbf{Z} \times d\mathcal{E}^7$
8	S^1

A Proof of Duality. Show the pairing is non-degenerate. Fix $\alpha \in \widehat{I\!H}^{n-k-1}(X)$ and assume

$$(\alpha * \beta)\,[X] \;=\; 0 \qquad \forall \beta \in \widehat{I\!H}^{k}(X).$$

We shall prove that $\alpha = 0$.

To begin suppose β is represented by a smooth form b. Then

$$(\alpha * \beta)[X] \;=\; (-1)^{n-k} \int_X \delta_1 \alpha \wedge b \;\equiv\; 0 \pmod{\mathbf{Z}}.$$

Since this holds $\forall$ k-forms b, we have $\delta_1 \alpha = 0$.
Hence, $\exists\, a \in \alpha$ with $da = R$, a rectifiable cycle whose integral homology class $[R]$ is torsion.

We show $[R] = 0$. Choose a class $u \in H^{k+1}(X;\mathbf{Z})_{\rm tor}$ of order m, and let T be a rectifiable cycle with $[T] = u$. Since $m[T] = 0$ there exists a rectifiable current S of degree k with $mT = dS$. Set $\beta = [\frac{1}{m}S] \in \widehat{I\!H}^{k}(X)$ and note that $\delta_1 \beta = 0$. Now

$$\begin{aligned}
(\alpha * \beta)[X] \;&=\; (-1)^{n-k}\,(d_2 a \wedge b)\,[X] \;=\; (-1)^{n-k}(R \wedge \tfrac{1}{m}S)[X] \\
&=\; (-1)^{n-k}\tfrac{1}{m}\{\text{intersection number of } R \text{ with } S\} \\
&=\; (-1)^{n-k} Lk([R], u) \\
&\equiv\; 0 \pmod{\mathbf{Z}}
\end{aligned}$$

where Lk denotes the Seifert-deRham linking number. Since this holds for all u, the non-degeneracy of this linking pairing on torsion cycles implies that $[R] = 0$ in $H^{n-k}(X;\mathbf{Z})$. Hence $\delta_2 \alpha = 0$ and so after adding an exact current we may assume that a is a smooth d-closed form of degree $n-k-1$.

Now choose any class $v \in H^{k+1}(X; \mathbf{Z}) \cong H_{n-k-1}(X; \mathbf{Z})$ and let S be a rectifiable cycle in v. Let ϕ be a smooth d-closed form representing $v \otimes \mathbf{R}$ and choose a current b with $db = \phi - S$. Let $\beta = [b] \in \widehat{I\!\!H}^k(X)$. Then

$$(\alpha * [b])[X] \;=\; (a \wedge d_2 b)[X] \;\equiv\; \int_S a \;\equiv\; 0 \pmod{\mathbf{Z}}.$$

Hence, $[a] = 0$ in $H^{n-k-1}(X; \mathbf{R})/H^{n-k-1}_{\text{free}}(X; \mathbf{Z}) = \ker \delta$, and so $\alpha = 0$ as claimed.

By the commutativity of the $*$-product we may interchange α and β above and conclude that the pairing is non-degenerate as asserted.

Finally we recall that $\widehat{I\!\!H}^k_\infty(X) \subset \widehat{I\!\!H}^k(X)$ is a closed subgroup with discrete quotient, and $\widehat{I\!\!H}^k_\infty(X) \cong \mathcal{E}^k(X)/\mathcal{Z}^k_0(X) \cong d\mathcal{E}^k(X) \oplus \{H^k(X; \mathbf{R})/H^k_{\text{free}}(X; \mathbf{Z})\}$. Recall that

$$\operatorname{Hom}(d\mathcal{E}^k(X), S^1) \;=\; d\mathcal{E}'^{\,n-k-1}(X).$$

It follows that the image of the duality homomorphism consists exactly of the smooth homomorphisms. ■