Jim Simons Work
on the
Theory of Minimal Varieties

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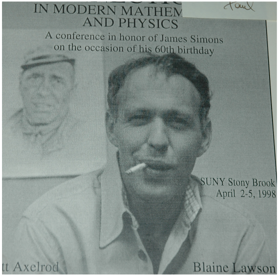

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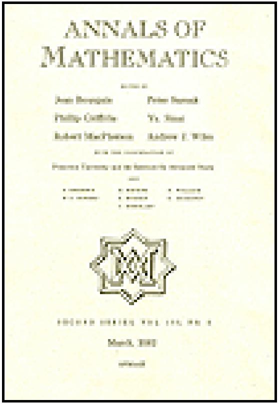

Minimal varieties in riemannian manifolds

By James Simons

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0. Introduction

Our object in this paper is twofold. First, we give a basic exposition of immersed minimal varieties in a riemannian manifold. The principal result of this general investigation is the derivation of the linear elliptic second order equation satisfied by the second fundamental form of any minimal variety in any ambient manifold (cf. Theorem 4.2.1).

Second, we apply these general results in a more detailed study of minimal varieties in the sphere and in euclidean space. This study includes an estimation of a lower bound for the index and the nullity of a non-totally geodesic closed minimal variety immersed in S^n; a theorem which generalizes to arbitrary co-dimensions the theorem of De Giorgi [8] concerning the image

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- Established complete interior regularity for minimizing hypersurfaces
  in dimensions ≤ 7.
- Established the Bernstein Conjecture
  in dimensions ≤ 8.
- Produced the example which eventually showed that
  both of the above theorems were sharp.

The Classical Theory of Minimal Surfaces

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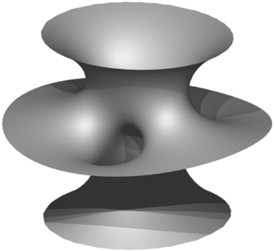
Riemann, Weierstrauss

The Idea: Consider a smooth surface in Euclidean 3-space $\Sigma \subset \mathbf{E}^3$.



The Classical Theory of Minimal Surfaces

Riemann, Weierstrauss

The Idea: Consider a smooth surface in Euclidean 3-space $\Sigma \subset \mathbf{E}^3$.

Suppose that for every deformation Σ_t, ($\Sigma_0 = \Sigma$) in the interior, the area satisfies

$$A(\Sigma_t) \;\geq\; A(\Sigma)\,.$$

The Classical Theory of Minimal Surfaces

Riemann, Weierstrauss

Then for all such deformations:

$$\frac{d}{dt} A(\Sigma_t)\Big|_{t=0} = 0$$

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Then for all such deformations:

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If in addition

$$\frac{d^2}{dt^2} A(\Sigma_t)\Big|_{t=0} \geq 0 \qquad \Sigma \text{ is called a } \textbf{Stable Minimal Surface}.$$

Observation:

These conditions

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Make sense

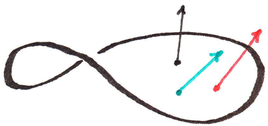
- in arbitrary dimensions and codimensions,
- in general riemannian manifolds,
- and for quite general objects Σ

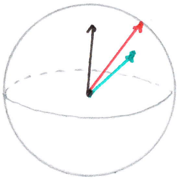
A Geometric Characterization in $\mathbf{R}^3$

THE GAUSS MAP

$$N : \Sigma \longrightarrow S^2$$

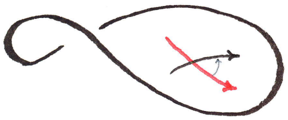
The Gauss map associates to each point $x \in \Sigma$, the normal vector $N(x)$ to Σ at x, i.e., the vector perpendicular to the tangent plane to Σ at x.

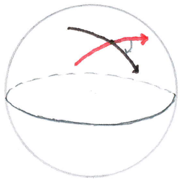




Σ is **minimal** if and only if the Gauss map is (anti)-**conformal**.

Angles are preserved (but direction is reversed).





Complex Anaysis Enters the Picture

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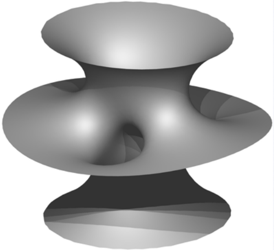
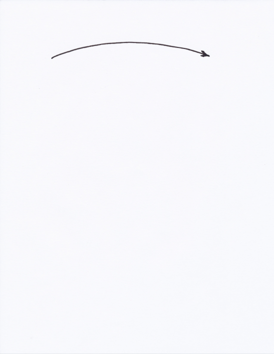
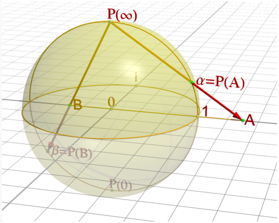
Take Stereographic Projection







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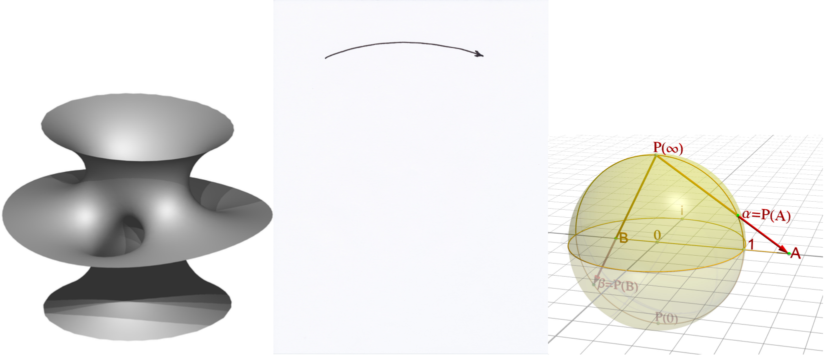

Complex Analysis is

a rich and deep subject

with many beautiful results.

Elementary Question

Suppose our surface Σ is the graph of a function $z = f(x, y)$
over a domain D in the (x, y)-plane

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When is this graph a minimal surface?

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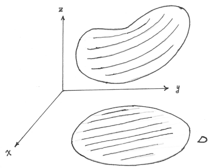
ANSWER: It must satisfy the differential equation

$$\boxed{(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} \;=\; 0.}$$



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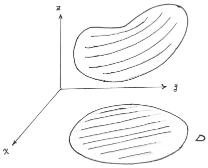
ANSWER: It must satisfy the differential equation

$$\boxed{(1 + |\nabla f|^2)\Delta f - (\nabla f)^t \mathbf{H}(f)\nabla f \;=\; 0.}$$



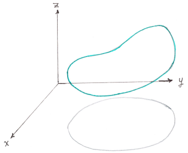
The Dirichlet Problem

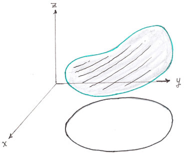
Let D be the round disk of radius R.
Let φ be an arbitrary continuous function on the boundary circle.

.

Theorem. *There exists a unique function $f(x, y)$ continuous on D and smooth in its interior, such that $f = \varphi$ on ∂D and in the interior it satisfies the minimal surface equation:*

$$(1 + |\nabla f|^2)\Delta f - (\nabla f)^t \mathbf{H}(f)\nabla f \; = \; 0$$





The Bernstein Theorem

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One would expect to produce many functions $f(x, y)$
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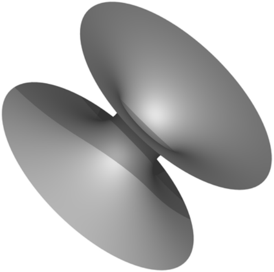
Surprise!!

The Bernstein Theorem (1918). *Any solution of the minimal surface equation which is defined for all (x, y) in the plane must be is linear, i.e., its graph is an affine 2-plane.*

This is a beautiful and astonishing result.

If we remove a tiny disk from the plane,
there is a function defined everywhere outside that disk
whose graph is a minimal surface.



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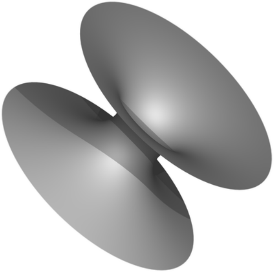

This is also true if we remove a half-line from the plane,

For the Classical Proof of this Theorem

we return to the **Gauss Map**.

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Notice: If

$$\Sigma = \{(x, y, f(x, y)) : (x, y) \in D\}$$

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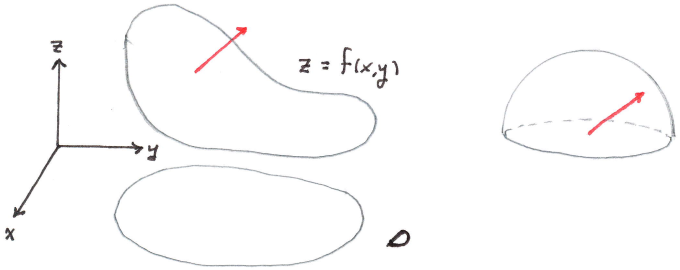

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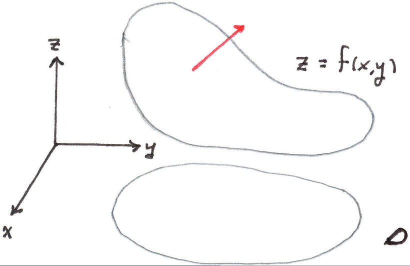

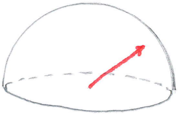

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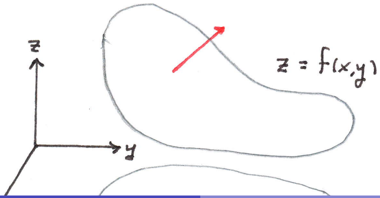
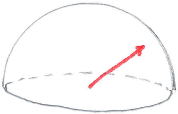
The proof is given by showing that Σ must have the conformal type of **C**.

Then the Gauss map becomes a bounded entire function and must be constant.





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Restrict to codimension one.

The Bernstein Conjecture.

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Let

$$f : \mathbf{R}^n \longrightarrow \mathbf{R}$$

be a solution of the minimal surface equation

$$(1 + |\nabla f|^2)\Delta f - (\nabla f)^t \mathbf{H}(f) \nabla f \;=\; 0.$$

defined over the entire space $\mathbf{R}^n$.

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Then f must be linear.

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Let

$$B \subset \mathbf{R}^n$$

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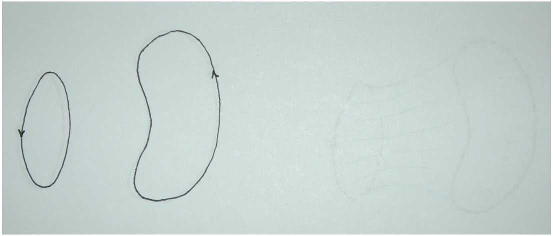


The Plateau Problem – in $\mathbf{R}^n$.

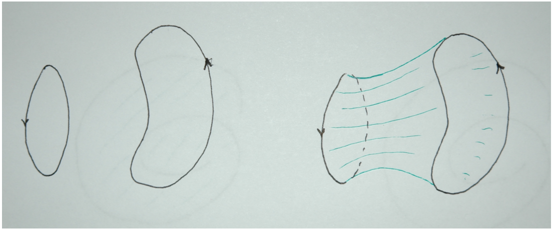
Problem: Find a p-dimensional "submanifold" Σ with "boundary" B

The Plateau Problem – in $\mathbf{R}^n$.

Problem: Find a p-dimensional "submanifold" Σ with "boundary" B such that

$$\mathcal{H}^p(\Sigma) \;\leq\; \mathcal{H}^p(\Sigma')$$

for all such Σ' with boundary B.



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- What do we mean by “submanifold”?
- What do we mean by “boundary”?
- Do solutions exist?
- How regular are the solutions?

The Plateau Problem – Classical Results.

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Douglas and Rado 1930

The Plateau Problem – Classical Results.

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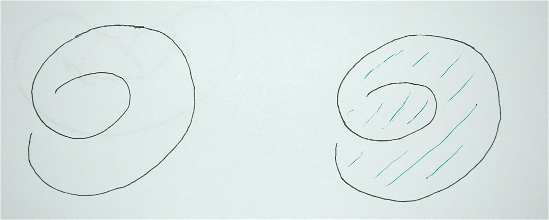
Considered a simple closed curve $\Gamma \subset \mathbf{R}^n$



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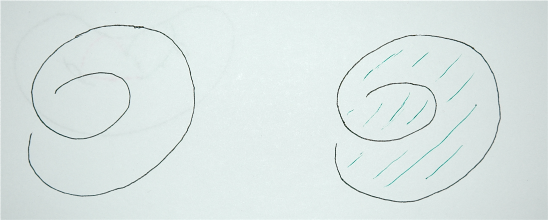

and tried to minimize area among all continuous maps of the disk

$$\psi : \Delta \to \mathbf{R}^n$$

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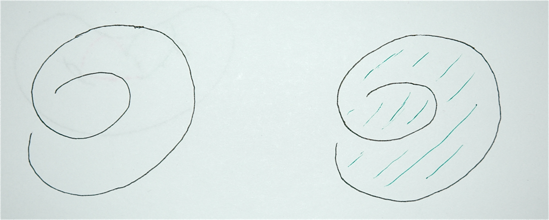

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$$\psi : \Delta \rightarrow \mathbf{R}^n$$

having first derivatives in L^2 and mapping

$$\psi : \partial\Delta \rightarrow \Gamma \qquad \text{(monotonically)}$$

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such that $[B] \to 0$ under the induced map

$$H_p(B, \Lambda) \;\to\; H_p(\Sigma, \Lambda)$$

on Čech homology with coefficients in Λ (say, $\mathbf{Z}$ or $\mathbf{Z}_2$),

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on Čech homology with coefficients in Λ (say, $\mathbf{Z}$ or $\mathbf{Z}_2$), and then

he minimized $\mathcal{H}^p(\Sigma)$ in this class.

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A **complete solution** to the **unoriented Plateau Problem** in $\mathbf{R}^3$.

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i.e.,

an $\mathcal{H}^p$-measurable field of unit simple p-vectors $\vec{E}$ with

$$\vec{E}_x \cong T_x E \qquad \text{for } \mathcal{H}^p\text{-a.a.} x \in E$$

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The integral

$$\int_E \alpha \;\equiv\; \int_E \alpha\left(\vec{E}_x\right)\, d\mathcal{H}^p(x)$$

is well defined.

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$$(\partial[E])(\alpha) \equiv [E](d\alpha)$$

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and the **mass** of T

$$M(T) \equiv \sum_{j=1}^{\infty} n_j \mathcal{H}^p(E_j) < \infty.$$

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$$\partial : \mathcal{I}_p(X) \to \mathcal{I}_{p-1}(X) \qquad \text{and} \qquad \partial^2 = 0.$$

Homology.

For any manifold X we have a chain complex

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This remains true for much more general spaces X.

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Given a compact set $K \subset X$ and $c > 0$, let

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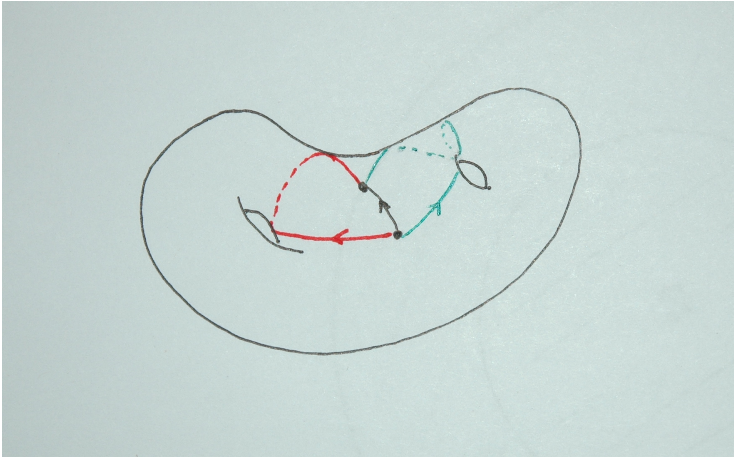
$$M(T) \quad = \quad \inf_{R \in \mathcal{I}_{p+1}(X)} M(T + \partial R)$$

Picture.



Consequence.

Let X be a compact riemannian manifold.

Corollary (Federer-Fleming)

Every homology class $u \in H_p(X; \mathbf{Z})$

contains an integral current of least mass.

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The result holds in general riemannian 3-manifolds.

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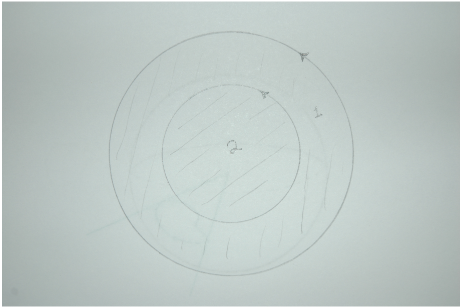

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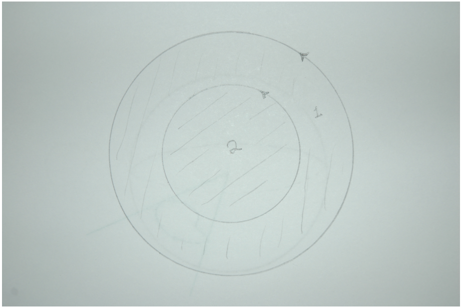

So T has the form of a locally finite sum

$$T = \sum_k n_k [M_k].$$

Higher Dimensions and Codimensions??.

Complete regularity fails
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That is, for every open set $\Omega \subset\subset X$

$$M([V]) \ \leq \ M([V] + \partial S) \qquad \forall\ S \in \mathcal{I}_{2p+1}(\Omega)$$

Regularity in Codimension-One.

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Theorem. (Almgren-De Giorgi).

The Bernstein conjecture holds for minimal graphs

$$\Gamma = \{(x, f(x)) \in \mathbf{R}^{n+1} : x \in \mathbf{R}^n\}$$

for $n \leq 4$**.**

The Proof – Revolves Around Cones.

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$$S^n \equiv \{x \in \mathbf{R}^{n+1} : \|x\| = 1\}.$$

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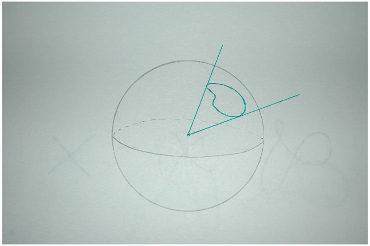

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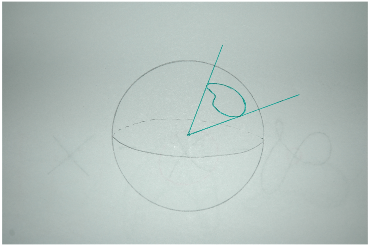

This concept extends naturally to currents.

Proposition.

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QUESTION:

$$C(M) \text{ is } \begin{cases} \textit{minimizing} \\ \textit{stable} \end{cases} \text{ in } \mathbf{R}^{n+1} \qquad \Longleftrightarrow \qquad M \text{ is ??? in } S^n$$

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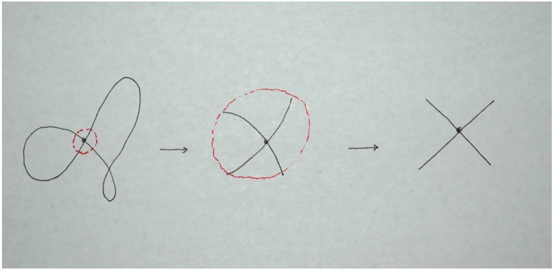

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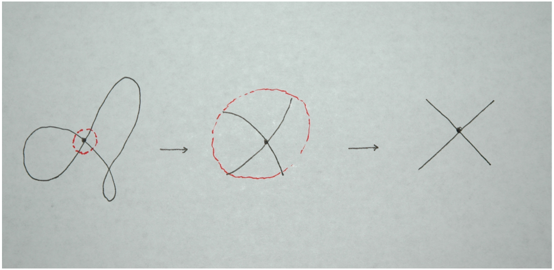

IDEA: Consider sequences of dilations.

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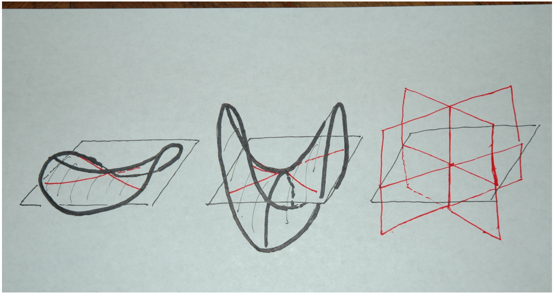

IDEA: Consider sequences of contractions.

How Do Cones Enter?

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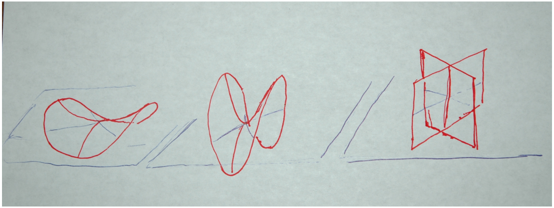

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One can now apply induction on dimension

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FURTHERMORE,

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$$C(S^3 \times S^3) \equiv \{(x,y) \in \mathbf{R}^4 \times \mathbf{R}^4 : |x| = |y|\} \subset \mathbf{R}^8$$

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The **mean curvature vector field** is the normal vector field along M given by

$$H \equiv \text{trace} B.$$

The First Variational Formula Let $\varphi_t : M \to X$ be a normal deformation of M with derivative V at $t = 0$. Then

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Corollary (Using Harvey-Shiffman).

Every stable integral current in $\mathbf{P}^n(\mathbf{C})$ is an algebraic cycle.