REMOVABLE SINGULARITIES FOR NONLINEAR SUBEQUATIONS

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Abstract

Let F be a fully nonlinear second-order partial differential subequation of degenerate elliptic type on a manifold X. We study the question: Which closed subsets $E \subset X$ have the property that every F-subharmonic function (subsolution) on $X - E$, which is locally bounded across E, extends to an F-subharmonic function on X. We also study the related question for F-harmonic functions (solutions) which are continuous across E. The main result asserts that if there exists a convex cone subequation M such that $F + M \subset F$, then any closed set E which is M-polar has these properties. *M-polar* means that $E = \{\psi = -\infty\}$ where ψ is M-subharmonic on X and smooth outside of E. Many examples and generalizations are given. These include removable singularity results for *all branches* of the complex and quaternionic Monge-Ampère equations, and a general removable singularity result for the harmonics of geometrically defined subequations.

For pure second-order subequations in $\mathbb{R}^n$ with monotonicity cone M the *Riesz characteristic* $p = p_M$ is introduced, and extension theorems are proved for any closed singular set E of locally finite Hausdorff $(p-2)$-measure. This applies for example to branches of the equation $\sigma_k(D^2u) = 0$ (k^{th} elementary function) where $p_M = n/k$, and its complex and quaternionic counterparts where $p_M = \frac{2n}{k}$ and $p_M = \frac{4n}{k}$ respectively.

For convex cone subequations themselves, several removable singularity theorems are proved, independent of the results above.

[1] Partially supported by the N.S.F.

Date: March 5, 2014.
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1. Introduction

Some Historical Background (Linear Equations). Removable singularity results for solutions to *linear* partial differential equations have a long history with a cornucopia of results. The general set-up involves two things: first a closed subset E of a manifold X which, since most results are local, might as well be an open subset of $\mathbb{R}^n$, and second, a solution u of the PDE in the complement $X - E$. The conclusion is generally the same, namely, that u extends to a solution on X. However, two different types of hypotheses are required, one on the "growth" of u across E, and the other on the "size" of the exceptional set E.

The first classical result of this kind was the Riemann Removable Singularity Theorem for holomorphic functions on the punctured disk satisfying $f(z) = o(|z|^{-1})$. In 1956 Bochner generalized Riemann's result to solutions of *linear* PDE's which satisfy the "growth hypothesis" $f(x) = o(\mathrm{dist}(x, E)^{-q})$. The remarkable thing about Bochner's removable singularity theorem is that the only information needed about the differential equation is its order m. The "size" hypothesis on E is that in dimension $n-m-q$ the set E should be (in some sense) locally finite. The majority of removable singularity results are of this kind – only depending on the order m of the equation. (See [24], [25] for more detailed results.)

The main exception is where the "growth" assumption on u states that u is locally bounded across E. If $E = \{0\}$ in $\mathbb{R}^2$, then the harmonic functions extend, while the fundamental solution for the wave equation, namely the characteristic function of the forward light cone, does not extend, even though it is bounded. In the bounded harmonic case, capacity is the right notion of "size", with capacity zero for E giving a necessary and sufficient condition for removability.

The purpose of this paper is to prove removable singularity results for fully non-linear second-order equations which are "elliptic" but can be very degenerate. We begin with a removable singularity result of a very general nature for subharmonics (or subsolutions) which are locally bounded across the singular set E. This theorem is then applied to yield the appropriate result, again of a very general nature, for the harmonics (or viscosity solutions). These results are then applied – worked out in detail and sometimes generalized – in a number of special cases, including all the branches of the complex and quaternionic Monge-Ampère equations, and also all the geometrically defined subequations.

We also obtain extension results across closed sets of locally finite Hausdorff k-measure for computable values of k. Here again there are many interesting applications. These are the first such results for general non-uniformly elliptic equations.

The central concept here is that of a subequation F (see Section 2) **which is M-monotone** for some convex cone subequation M, that is, $F + M = F$ point-wise on X. A subset $E \subset X$ is called C^∞ *M-polar* if $E = \{\psi = -\infty\}$ for an M-subharmonic function ψ which is smooth in $X - E$. The first main result is the following.

THEOREM 6.1. *Suppose F is a subequation on X which is M-monotone. If $E \subset X$ is a closed subset which is locally C^∞ M-polar, then E is removable for F-subharmonic functions which are locally bounded above across E.*

The hypothesis that the function be locally bounded above across E is automatic in certain cases (see Lemma 6.2).

Theorem 6.1 gives a corresponding result for F-harmonics. These are functions u such that u is a subsolution (subharmonic) for F and $-u$ is a subsolution for the dual subequation $\widetilde{F}$ (see (7.1)).

THEOREM 7.1. *Suppose F is a subequation which is M-monotone and E is a closed set with no interior, which is locally C^∞ M-polar. Then for $u \in C(X)$,*

$$u \text{ is } F\text{-harmonic on } X - E \qquad \Rightarrow \qquad u \text{ is } F\text{-harmonic on } X.$$

These two theorems have a number of applications, explored in Sections 8-11.

When M is a convex cone subequation, one has $M + M \subset M$ and $\widetilde{M} + M \subset \widetilde{M}$. Applying Theorem 6.1 with $F = M$ should be viewed as a weak result. In general there are plenty of removable sets for M-subharmonics which are not M-polar (See Section 12). However, applying Theorem 6.1 with $F = \widetilde{M}$ is a much better result since $\widetilde{M} \supset M$ is much larger than M and highly non-convex. See Remark 7.2 and Corollary 7.3

A basic and illuminating case is that of classical plurisubharmonic functions in $\mathbb{C}^n$. Here M is the convex cone subequation $\mathcal{P}^{\mathbb{C}}$ defined by requiring the complex hermitian hessian $(D^2u)_{\mathbb{C}} \cong (\partial^2 u/\partial z_i \partial \bar{z}_j)$ to be ≥ 0. The $\mathcal{P}^{\mathbb{C}}$-polar sets are the classical pluripolar sets, and the theorems above apply to prove new removable singularity results for **all branches** of the complex Monge-Ampère equation (Theorem 8.1). For the principal branch, the $\mathcal{P}^{\mathbb{C}}$-harmonic functions are just the continuous maximal functions, i.e., the continuous plurisubharmonic functions u satisfying $\det_{\mathbb{C}}(D^2u) = 0$, and there are extendability results due to Bedford and Taylor [5], [6], [7].

Another important case involves the p-plurisubharmonic functions, which are discussed in Sections 9, 10 and 11. When p is an integer, these are the functions which become classically subharmonic when restricted to minimal

p-dimensional submanifolds. They are studied extensively in [20]. The corresponding convex cone subequation $\mathcal{P}_p$ is a monotonicity cone for a large collection of interesting subequations and their duals (see Section 11). For appropriate integers p these include the Lagrangian and Special Lagrangian subequations in $\mathbb{C}^n$, the associative and coassociative subequations in $\mathbb{R}^7$, the Cayley subequations in $\mathbb{R}^8$, and their analogues on riemannian manifolds (see [15]). The cases where p is not an integer also have important applications, and lead to the concept of Riesz characteristics, discussed below.

The $\mathcal{P}_p$-polar sets are studied in Section 10 using Riesz potentials, and the following is proved.

Theorem 10.5. *Any closed subset $E \subset \mathbb{R}^n$ with locally finite Hausdorff $(p-2)$-measure, is locally C^∞ $\mathcal{P}_p$-polar. More generally, any closed subset $E \subset \mathbb{R}^n$ with $(p-2)$-capacity zero is locally C^∞ $\mathcal{P}_p$-polar. Therefore, if a subequation F is $\mathcal{P}_p$-monotone, then any such set is removable for F-subharmonics and F-harmonics, as in the theorems above.*

Any subequation $F = \mathcal{P}(\mathbb{G})$ which is defined geometrically by a closed subset $\mathbb{G} \subset G(p, \mathbb{R}^n)$ of the Grassmannian of p-planes (see (11.1)) is $\mathcal{P}_p$-monotone and the theorem applies. It gives extension theorems for $\mathbb{G}$-harmonics and $\mathbb{G}$-plurisubharmonics, and dually $\mathbb{G}$-plurisubharmonics. This includes the specific geometric cases cited above. For example, for Lagrangian or Special Lagrangian subequations in $\mathbb{C}^n$ closed sets of locally finite Hausdorff $n-2$-measure are removable.

More generally let $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ be a pure second-order subequation with monotonicity cone M. If $\mathcal{P}_p \subset M$, then F is $\mathcal{P}_p$-monotone, and Theorem 10.5 applies. This leads to the question of determining the largest $\mathcal{P}_p \subset M$. An easily computable invariant, called the **Riesz characteristic** p_M **of** M, provides the answer (see Definition 11.3 and Theorem 11.4).

Theorem 10.5 then gives the following (cf. Theorem 11.6).

Theorem A. *Suppose $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a subequation in $\mathbb{R}^n$ with monotonicity cone $M \subset \mathrm{Sym}^2(\mathbb{R}^n)$. Let $p = p_M$ be the Riesz characteristic of M. Then any closed subset E of locally finite Hausdorff $(p-2)$-measure (or, more generally, of $(p-2)$-capacity zero) is removable for F-subharmonics and F-harmonics, as in the theorems above.*

A large set of interesting subequations to which these results apply is the family defined by homogeneous polynomials P on $\mathrm{Sym}^2(\mathbb{R}^n)$ which are Gårding hyperbolic with respect to the identity (cf. [11], [16], [17]). Here the convex cone subequation M corresponds to the Gårding cone, and each branch of the equation $P(D^2u) = 0$ is M-monotone.

More specifically, a homogeneous polynomial $P : \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R}$ of degree m is **Gårding hyperbolic** with respect to I if $P(I) > 0$ and for each A, all the roots of the one-variable polynomial $p_A(t) = P(A + tI)$ are real. In this case one can write $p_A(t) = \prod_i (t + \lambda_i^P(A))$ where $\lambda_1^P(A) \leq \cdots \leq \lambda_m^P(A)$ are the *ordered P-eigenvalues* of A. We assume $\lambda_i^P(A+B) \geq \lambda_i^P(A)$ whenever $B \geq 0$. Then each *branch*

$$\Lambda_i^P \ \equiv\ \{A : \lambda_i^P(A) \ \geq\ 0\}, \qquad 1 \ \leq\ i\ \leq\ m$$

is a subequation, and the principal branch

$$M \ \equiv\ \Lambda_1^P \ =\ \{A : \lambda_i^P(A) \ \geq\ 0\ \forall\, i\},$$

called the **Gårding cone**, is a **convex** cone which satisfies $\Lambda_i^P + M \subset \Lambda_i^P$, that is, it is a monotonicity cone for each branch of the equation $P(D^2u) = 0$.

A basic example is given by the k^{th} elementary symmetric function $\sigma_k(A)$ of the eigenvalues of A. The Gårding cone is $M = \Sigma_k = \{A : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\}$, and the Riesz characteristic is computed to be

$$p_{\Sigma_k} \ =\ \frac{n}{k}.$$

Note that $\Sigma_1 = \mathcal{P}$, the basic homogeneous real Monge-Ampère equation on convex functions, whose study goes back to Alexandrov [1] and Pogorelov [31], [32]. The other Gårding cones Σ_k, sometimes called *Hessian equations*, have been studied by Trudinger-Wang [34], [35], [36], Labutin [26], and others. Note however, that each Σ_k has $k-1$ other branches, which are neither convex nor uniformly elliptic, but to which Theorem A applies with $p = n/k$.

Further interesting Gårding polynomials are discussed in Section 11. One is the p-fold sum operator, which for integer p is given by $M_p(A) = \prod(\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A))$, whose principal branch is $\mathcal{P}_p$.

We note that every subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$, which is defined purely in terms of the eigenvalues of the matrices (i.e., is $\mathrm{O}(n)$-invariant), has **complex and quaternionic analogues** $F^{\mathbb{C}} \subset \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n)$ and $F^{\mathbb{H}} \subset \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{H}^n)$. The set $F^{\mathbb{C}}$ is defined by imposing the constraints of F on the n eigenvalues of the hermitian symmetric part $A_{\mathbb{C}} = \frac{1}{2}(A - JAJ)$ of A. The set $F^{\mathbb{H}}$ is defined similarly. This applies for example to the equations Σ_k above, and the special cases $\mathcal{P}^{\mathbb{C}} = \Sigma_1^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{H}} = \Sigma_1^{\mathbb{H}}$ correspond to the complex and quaternionic Monge-Ampère equations on the underlying plurisubharmonic functions. In complete generality the Riesz characteristics satisfy:

$$p_{F^{\mathbb{C}}} \ =\ 2p_F \qquad \text{and} \qquad p_{F^{\mathbb{H}}} \ =\ 4p_F$$

Note (Uniform ellipticity). There are many natural cones, which when they are monotonicity cones for a subequation F, imply that F is uniformly elliptic. One family of such cones is $\mathcal{P}(\delta) = \{A : A + \delta(\mathrm{tr}A)I \geq 0\}$. Another is the family of *Pucci cones* $\mathcal{P}_{\lambda,\Lambda}$ defined in Example 2 of Section 11. Monotonicity with respect to $\mathcal{P}_{\lambda,\Lambda}$ corresponds to the standard (λ,Λ)-uniform ellipticity. For $\mathcal{P}(\delta)$ the Riesz characteristic is $(1+\delta n)/(1+\delta)$, and for $\mathcal{P}_{\lambda,\Lambda}$ the Riesz characteristic is $p_{\lambda,\Lambda} = \frac{\lambda}{\Lambda}(n-1)+1$. Thus as the uniform ellipticity constant increases, the Hausdorff dimension of the removable sets increases. However, we point out that the main thrust of this paper is to establish removable singularity results for equations which are *not* necessarily uniformly elliptic.

Further Results – Variable Coefficients. There is a theorem similar to Theorem A above which holds for variable-coefficients (see the end of Section 11). There is also a result on general riemannian manifolds where $\mathcal{P}_p$ is replaced by its riemannian analogue, defined using the riemannian hessian. See Theorem 11.7.

After circulating an early version of this paper, we received an article [2] by Amendola, Galise and Vitolo which, in the language above, studied uniformly elliptic equations of Riesz characteristic $p_{\lambda,\Lambda} \geq 2$ (see condition (1.1) in [2]). The authors proved the extendability of the maximum principle and comparison, and also the removability of singularities across sets of $(p_{\lambda,\Lambda}-2)$-capacity zero. In particular, they established Theorem A above in the (λ,Λ)-uniformly elliptic case.

Removable singularity theorems have also been established for certain uniformly elliptic equations by Labutin (see [26], [27], [28] and references therein.) Results on the removability of isolated singularities for certain uniformly elliptic equations also appeared in papers of Armstrong, Sirakov and Smart [3], [4]. There are also some very recent results of Vitolo [37].

For convex cone subequations M themselves, a wide variety of strong removability results are obtained rather easily from the linear case using the Strong Bellman Principle [23], provided that M is "second-order complete". Using linear results from [24] (see also [12], [25]), some of these possiblities are discussed in Section 12. None of them depend on the rest of the paper. For example, we show that every closed set E of locally finite Hausdorff codimension-2 measure is removable for M-subharmonic functions which are locally bounded above across E. Several other removable singularity results, which entail various combinations of "growth" and "size", are also given in Section 12.

Results in this paper were inspired by work of Caffarelli, Li and Nirenberg [8]. In Appendix A we present an alternate proof of one of their results.

2. Subequations and Subharmonic Functions.

By a **subequation** on a manifold X we mean a closed subset of the 2-jet bundle of X, (locally, a closed subset of $X \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$ with $X^{\mathrm{open}} \subset \mathbb{R}^n$) with non-empty fibres over X satisfying the primary condition

$$F + \mathcal{P} \ \subset \ F \qquad (\textbf{Positivity})$$

where $\mathcal{P} = \{(x, 0, 0, A) : x \in X \text{ and } A \geq 0\}$. Intrinsically, $\mathcal{P}$ is the set whose fibre at x consists of 2-jets of functions with minimum value 0 at x. In [15, §§2 and 3] a subequation is assumed to satisfy a further mild topological regularity condition which is important for discussing the dual subequation $\widetilde{F}$ (see Section 7). If each fibre F_x of F is a convex cone with vertex at the origin in $\mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$, then F is called a *convex cone subequation*.

Let $\mathrm{USC}(X)$ denote the space of upper semi-continuous $[-\infty, \infty)$-valued functions u on X. Often u is not allowed to be $\equiv -\infty$ on any connected component of X. By a **(viscosity) test function** for $u \in \mathrm{USC}(X)$ at a point x we mean a C^2-function φ defined near x such that $u - \varphi$ has local maximum value 0 at x. (See [9], [10] and references therein for a discussion of the viscosity approach to differential equations.)

Definition 2.1. A function $u \in \mathrm{USC}(X)$ is ***F*-subharmonic on** X if for each $x \in X$ and each test function φ for u at x, we have

$$(\varphi(x), D_x\varphi, D_x^2\varphi) \ \in \ F_x.$$

Positivity ensures that if $u \in C^2(X)$ and $(u(x), D_x u, D_x^2 u) \in F_x$ for all x, then u is F-subharmonic on X. Each subequation F has its own potential theory, which studies the set $F(X)$ of F-subharmonic functions. Of course, each such potential theory has its own unique characteristics. However, there is a surprisingly large common core of general results. See, for example, [15, Theorem 2.6] for the beginning of a list of results that hold for all these theories. The purpose of this note is to add a removable singularity theorem to each of these potential theories.

3. Removable Sets.

We consider closed subsets E of X, and we restrict attention to upper semi-continuous functions u on $X - E$ which are locally bounded above at points of E. Each such function u has a canonical upper semi-continuous extension U across E to all of X defined as follows. If $\mathrm{Int}E = \emptyset$, set

$$U(x) \ \equiv \ \limsup_{\substack{y \to x \\ y \notin E}} u(y) \ = \ \lim_{\epsilon \searrow 0} \sup_{B(x,\epsilon) - E} u \tag{3.1}$$

When Int$E \neq \emptyset$, extend this by setting $U(x) \equiv -\infty$ on Int$E$. It is easy to see that U is the upper semi-continuous regularization of the function v, which is defined to be u on $X - E$ and $-\infty$ on E. (This fact will be used in Section 6 to complete the proof of Theorem 6.1.)

Definition 3.2. A closed set E is **removable for F-subharmonic functions locally bounded above across** E if for each function u which is F-subharmonic on $X - E$ and is bounded above in a neighborhood of each point of E, the canonical extension U is F-subharmonic on X.

The most elementary removable singularity result can be stated as follows.

Proposition 3.3. *If $U \in \mathrm{USC}(X)$ and U has no test functions at points $x \in E$, then $U \in F(X - E) \quad \Rightarrow \quad U \in F(X)$.*

For example, take $U(x) = u(x) + \lambda|x|$ with u smooth and $E = \{0\}$.

4. Monotonicity Subequations.

Given a subequation F on X, a **monotonicity subequation** for F is a convex cone subequation M on X satisfying

$$F + M \ \subset\ F \qquad\qquad (\textbf{Monotonicity})$$

where "+" denotes fibre-wise sum. In this case we say the subequation F is M**-monotone**. It follows easily from the definition of F-subharmonicity that:

$$u \in F(X) \text{ and } \psi \in M(X) \cap C^\infty(X) \qquad \Rightarrow \qquad u + \epsilon\psi \in F(X) \text{ for all } \epsilon > 0. \tag{4.1}$$

5. Polar Sets for a Subequation.

For the purposes of dealing with removable singularities it is convenient to limit attention to a restricted class of "polar sets" for a subequation.

A function $\psi \in \mathrm{USC}(X)$ is called a **polar function** for a closed set E if $E = \{x \in X : \psi(x) = -\infty\}$. If in addition ψ is smooth on $X - E$, it is called a **smooth polar function** for E. Let M be a subequation.

Definition 5.1. A closed subset $E \subset X$ is called C^∞ M**-polar** if it admits a smooth polar function which is M-subharmonic on X.

6. Polar Sets are Removable.

This is a classical result for the euclidean Laplacian. Moreover, the classical proof extends quite easily to the following very general case. Let M be a convex cone subequation.

THEOREM 6.1. *Suppose F is a subequation which is M-monotone. If E is a closed subset which is locally C^∞ M-polar, then E is removable for F-subharmonic functions which are locally bounded above across E.*

Proof. Given $u \in F(X - E)$ we must show that $U \in F(X)$. Let ψ denote a smooth M-subharmonic polar function for E. As noted above in (4.1), each $U + \epsilon\psi$ is F-subharmonic on $X - E$. Since $(U + \epsilon\psi)(x) = -\infty$ at each point $x \in E$, $U + \epsilon\psi$ cannot have a test function at such a point x. This proves that for each $\epsilon > 0$, $U + \epsilon\psi \in F(X)$. Now the set $\{\psi < 0\}$ is an open set containing E, and so we can assume that $X = \{\psi < 0\}$, i.e., $\psi < 0$ on X.

By the "families bounded above property" for F-subharmonic functions ([15, Thm. 2.6 (E)]), the upper semi-continuous regularization of the upper envelope v of the family $\{U + \epsilon\psi\}$ is F-subharmonic on X. This upper envelope v is given by

$$v(x) \;=\; \begin{cases} u(x) & \text{if } \; x \notin E \\ -\infty & \text{if } \; x \in E \end{cases}$$

since E is exactly the $-\infty$ set of ψ. Finally, as noted in Section 3, v has upper semi-continuous regularization U. ∎

The hypothesis in Theorem 6.1 of being bounded above across E can be deleted for certain subequations. Applications of this result will be left to the reader.

Lemma 6.2. *Suppose $F \subset \Lambda_p \equiv \{\lambda_p(A) \geq 0\}$ and E has $(n-p)$-dimensional Hausdorff measure zero. Then each F-subharmonic function on $X - E$ is locally bounded above across E.*

Proof. Pick a point in E which we can assume is the origin. As in Shiffman [33] we can construct a product of balls $B \times B' \subset \mathbb{R}^n$ with B a ball in $\mathbb{R}^p$ such that $E \cap (\partial B \times B') = \emptyset$ and the projection of $E \cap (B \times B')$ on B' has measure zero. By the Restriction Theorem 5.3 in [18], the function u restricted to the good slices $B \times \{y\} \subset B \times B' - E$ is a subaffine function, and hence satisfies the maximum principle on each good slice. Therefore,

$$u(x, y) \leq \sup_{x \in \partial B} u(x, y) \qquad \text{for almost every } y, \quad \text{and so}$$

$$\sup_{B \times B' - E} u \;\leq\; \sup_{\partial B \times B'} u \;<\; \infty. \qquad ∎$$

7. Harmonic Functions

Given a subequation F recall the **dual subequation**

$$\widetilde{F} \;=\; -(\sim \mathrm{Int}F) \;=\sim(-\mathrm{Int}F) \tag{7.1}$$

(see [13], [15]). Here a mild topological condition (T) on the set F is required [15, 3.2].

Definition. A function u is F**-harmonic** on X if $u \in F(X)$ and $-u \in \widetilde{F}(X)$.

For smooth functions u this means that

$$(u(x), D_x u, D_x^2 u) \in \partial F_x \qquad \text{for each } x \in X.$$

Let M be a convex cone subequation.

THEOREM 7.1. *Suppose F is a subequation which is M-monotone and E is a closed set with no interior, which is locally C^∞ M-polar. Then for $u \in C(X)$,*

$$u \text{ is } F\text{-harmonic on } X - E \qquad \Rightarrow \qquad u \text{ is } F\text{-harmonic on } X.$$

Proof. Corollary 3.5 in [15] states that

$$F + M \;\subset\; F \qquad \Longleftrightarrow \qquad \widetilde{F} + M \;\subset\; \widetilde{F} \tag{7.2}$$

for any subset M of the 2-jet bundle. Theorem 6.1 applies to the F-subharmonic function u on $X - E$ proving that the canonical extension U defined by (3.1) is F-subharmonic on X. Similarly, because of (7.2), the $\widetilde{F}$-subharmonic function $v = -u$ on $X - E$ has canonical extension V which is $\widetilde{F}$-subharmonic on X. By continuity of u on X and the fact that $\mathrm{Int}E = \emptyset$, the canonical extensions $-U$ and V agree, proving that U is F-harmonic on X. ■

Remark 7.2. Note that for a convex cone subequation M the space of dual $\widetilde{M}$-subharmonics is much larger than the space of M-harmonics. To see this note that (7.2) implies that $\widetilde{M}$ is M-monotone, and since $0 \in \widetilde{M}$, it follows that $M \subset \widetilde{M}$. Because of this, the special case of Theorem 6.1, stating that M-polar sets are removable for M-subharmonics, should be considered a weak result. In general there are plenty of removable sets for M-subharmonics which are not M-polar (See Section 12). By contrast, that M-polar sets are removable for $\widetilde{M}$-subharmonics (Theorem 6.1) and for M-harmonics (Theorem 7.1) constitute much better results, which are stated as follows.

Corollary 7.3. *Suppose that M is a convex cone subequation and E is a closed set which is locally C^∞ M-polar. Then E is removable for $\widetilde{M}$-subharmonic functions which are locally bounded above across E. Moreover, if E has no interior and $u \in C(X)$, then*

$$u \text{ is } M\text{-harmonic on } X - E \qquad \Rightarrow \qquad u \text{ is } M\text{-harmonic on } X.$$

Remark 7.4. (Horizontally Subharmonic). This example illustrates the general nature of Theorems 6.1 and 7.1 and may help illuminate some of the trivialities that are involved. Suppose $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$. Define F by requiring that the horizontal Laplacian satisfy $\sum_{k=1}^p \frac{\partial^2 u}{\partial x_k^2} \geq 0$. Assume $q > 0$, i.e., F does not involve all the variables in $\mathbb{R}^n$. Choose an open set $Y \subset \mathbb{R}^q$ and consider an arbitrary function $u : Y \to [-\infty, \infty)$. Then:

$$u(y) \text{ is } F\text{-subharmonic on } \mathbb{R}^p \times Y \qquad \iff \qquad u \in \mathrm{USC}(Y)$$

$$u(y) \text{ is } F\text{-harmonic on } \mathbb{R}^p \times Y \qquad \iff \qquad u \in C(Y)$$

Note that this example is self-dual, i.e., $\widetilde{F} = F$ (if $p \geq 1$) and a convex cone subequation. It is now easy to see that the removable singularities result Theorem 6.1 must apply to general functions $u \in \mathrm{USC}(Y - E)$ yielding $U \in \mathrm{USC}(Y)$, while the hypothesis in Theorem 7.1, in addition to $u \in C(Y - E)$ must include enough information to conclude that $U \in C(Y)$.

Consider, for example, $u(y) = \sin(1/|y|)$ with $E = \{y = 0\}$. This function is F-harmonic on $\mathbb{R}^n - E$ and bounded across E but cannot be extended to an F-harmonic function on $\mathbb{R}^n$. It is also easy to see that for this convex cone subequation, the hypothesis "Int$E = \emptyset$" occuring in the second half of Corollary 7.3 cannot be dropped.

Remark 7.5.(Convex Cone Subequations which are Second-Order Complete). If M is such a subequation (see Section 12), the M-subharmonic functions are also L-subharmonic for a linear equation L with positive definite second-order part. This is enough to ensure that such a function is locally Lebesgue integrable. In particular, this excludes the possibility of a closed set E, which admits a smooth M-polar function, having any interior. Hence, in this case the hypothesis Int$E = \emptyset$ can be dropped in Theorem 7.1 and Corollary 7.3.

Here is a situation where the hypothesis of continuity on u in Theorem 7.1 can be weakened to local boundedness.

THEOREM 7.6. *Let F be a subequation on X which is M-monotone. Let $E \subset X$ be a compact M-polar set with no interior. Suppose that $E \subset \Omega^{\text{open}} \subset\subset X$ where the Dirichlet problem for F-harmonic functions on Ω is uniquely solvable for all continuous boundary functions. Then if*

$u \in \mathrm{USC}(\overline{\Omega} - E)$ is F-harmonic on $\Omega - E$ and locally bounded across E, then u extends to an F-harmonic function on Ω.

Proof. Let $U = u^*$ be the upper semi-continuous regularization of u on Ω. Let $-V = (-u)^* = -u_*$ be the upper semi-continuous regularization of $-u$ on Ω. Then $V \leq U$, and by Theorem 7.1 we have that

$$U \text{ is } F\text{-subharmonic on } \Omega \quad \text{and} \quad V \text{ is } \widetilde{F}\text{-subharmonic on } \Omega.$$

Let $\widetilde{u}$ be the unique F-harmonic extension of $u\big|_{\partial\Omega}$ to Ω. Then since $\widetilde{u}$ and $-\widetilde{u}$ are obtained by the Perron process, we have $U \leq \widetilde{u}$ and $-V \leq -\widetilde{u}$. Hence, $\widetilde{u} \leq V \leq U \leq \widetilde{u}$. ∎

8. Classical Plurisubharmonic Functions.

The classical case is that of plurisubharmonic functions and pluripolar sets in $\mathbb{C}^n$. Note that in our parlance the plurisubharmonic functions are defined by the constant coefficient, pure second-order subequation

$$\mathcal{P}^{\mathbb{C}} \;\equiv\; \{A \in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n) : A_{\mathbb{C}} \geq 0\}$$

where $A_{\mathbb{C}} \equiv \frac{1}{2}(A - JAJ)$ denotes the hermitian symmetric component of A. (Here $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ denotes the complex structure.) Thus $u \in \mathrm{USC}(X)$ is plurisubharmonic if $D^2_x\varphi \in \mathcal{P}^{\mathbb{C}}$ for each test function φ for u at points $x \in X$. Pluripolar sets are classically defined simply as subsets of $-\infty$ level sets of plurisubharmonic functions (not allowed to be identically $-\infty$ on any component of X). The pluripolar sets are very well understood.

Fix an open set $X \subset \mathbb{C}^n$ and consider the homogeneous complex Monge-Ampère equation

$$\det{}_{\mathbb{C}} \left\{(D^2u)_{\mathbb{C}}\right\} \;=\; 0$$

Corresponding to this equation there are n distinct, pure second-order subequations, or "branches", defined using the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of $A_{\mathbb{C}} \equiv\; (D^2u)_{\mathbb{C}}$ by setting

$$\boldsymbol{\Lambda}_k^{\mathbb{C}} \;\equiv\; \{\lambda_k(A_{\mathbb{C}}) \geq 0\}$$

A C^2-function u which is $\boldsymbol{\Lambda}_k^{\mathbb{C}}$-harmonic for any one of these branches satisfies the differential equation $\det_{\mathbb{C}}\{(D^2u)_{\mathbb{C}}\} = 0$, but there are n distinct notions of being $\boldsymbol{\Lambda}_k^{\mathbb{C}}$-harmonic. Only the real parts of holomorphic functions are $\boldsymbol{\Lambda}_k^{\mathbb{C}}$-harmonic for all k.

Note that $\lambda_1(A_{\mathbb{C}}) \geq 0$ is the requirement that $A_{\mathbb{C}} \geq 0$ so that $\mathcal{P}^{\mathbb{C}}$ is the first, or smallest branch $\boldsymbol{\Lambda}_1^{\mathbb{C}}$. This subequation $\boldsymbol{\Lambda}_1^{\mathbb{C}} = \mathcal{P}^{\mathbb{C}}$ is the classical homogeneous Monge-Ampère equation for plurisubhamonic functions. It is the only convex branch.

One can show that a function is $\mathcal{P}^{\mathbb{C}}$-harmonic if and only if it is continuous and maximally plurisubharmonic, ensuring that maximality is a local concept for continuous functions.

It is easy to see that the dual of the k^{th} branch $\mathbf{\Lambda}_k^{\mathbb{C}}$ is $\mathbf{\Lambda}_{n-k+1}^{\mathbb{C}}$. In particular, the dual of the smallest branch $\mathcal{P}^{\mathbb{C}} = \mathbf{\Lambda}_1^{\mathbb{C}}$ is the largest branch $\widetilde{\mathcal{P}}^{\mathbb{C}} = \mathbf{\Lambda}_n^{\mathbb{C}}$ which only requires that one eigenvalue $\lambda_{\max}$ of $\left(D^2u\right)_{\mathbb{C}}$ be ≥ 0. (This largest branch $\widetilde{\mathcal{P}}^{\mathbb{C}}$ is the complex analogue of the subequation $\widetilde{\mathcal{P}}$ whose subharmonics are the subaffine functions introduced in [13]. In fact, the $\widetilde{\mathcal{P}}^{\mathbb{C}}$ -subharmonics are characterized as being "sub" the real parts of holomorphic functions on $\mathbb{C}^n$ [18, Prop. 5.4].) One can easily show that $\mathcal{P}^{\mathbb{C}}$ is a monotonicity cone for each subequation $\mathbf{\Lambda}_k^{\mathbb{C}}$. As noted in Remark 7.5, plursubharmonic functions are locally Lebesgue integrable. Thus, pluripolar sets have measure zero and hence no interior.

THEOREM 8.1. *Let $E^{\text{closed}} \subset X^{\text{open}} \subset \mathbb{C}^n$ be a locally pluripolar set. Any function $u \in \mathbf{\Lambda}_k^{\mathbb{C}}(X-E)$ which is locally bounded above across E has a canonical extension $U \in \mathbf{\Lambda}_k^{\mathbb{C}}(X)$. Moreover, if a $\mathbf{\Lambda}_k^{\mathbb{C}}$-harmonic function on $X-E$ has a continuous extension to X, then this extension is $\mathbf{\Lambda}_k^{\mathbb{C}}$-harmonic on X.*

Proof of Theorem 8.1 If E is contained in $\{\psi = -\infty\}$ (locally) where ψ plurisubharmonic on X and smooth outside $\{\psi = -\infty\}$, and where $\{\psi = -\infty\}$ closed, then we can simply replace E by $\{\psi = -\infty\}$ and apply Theorems 6.1 and 7.1. However, for the more general assertion of Theorem 8.1 we proceed as follows.

Suppose that locally we have a plurisubharmonic function ψ with $E \subset \{\psi = -\infty\}$ (but ψ may not be smooth outside $\{\psi = -\infty\}$). The proof of Theorem 6.1 is valid if we can show that $u \in \mathbf{\Lambda}_k^{\mathbb{C}}(X-E)$ implies that $u+\epsilon\psi \in \mathbf{\Lambda}_k^{\mathbb{C}}(X-E)$. This can be reduced to the case where u and ψ are quasi-convex (see, for example, Theorem 8.2 in [13]). In the quasi-convex case u and ψ have second derivatives $D_x^2 u$ and $D_x^2\psi$ a.e., and $D_x^2(u+\epsilon\psi) = D_x^2 u + \epsilon D_x^2\psi$ a.e. By Corollary 7.5 in [13] the fact that $D_x^2(u+\epsilon\psi) \in \mathbf{\Lambda}_k^{\mathbb{C}} + \mathcal{P}^{\mathbb{C}} = \mathbf{\Lambda}_k^{\mathbb{C}}$ a.e. implies that $u+\epsilon\psi \in \mathbf{\Lambda}_k^{\mathbb{C}}(X-E)$. ∎

9. p-Plurisubharmonic Functions and p-Monotonicity.

This is our second example, and again the roots are classical. Fix p with $1 \leq p \leq n$. First we consider the case where p is an integer. The convex cone subequation $\mathcal{P}_p \equiv \mathcal{P}(G(p,\mathbb{R}^n))$ is defined in [13] by

$$\mathcal{P}_p \;\equiv\; \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}_W A \geq 0 \;\forall\, W \in G(p,\mathbb{R}^n)\}. \tag{9.1}$$

Here $G(p, \mathbb{R}^n)$ denotes the grassmannian of p-dimensional subspaces of $\mathbb{R}^n$ and $\mathrm{tr}_W A$ denotes the trace of the quadratic form A restricted to W. Perhaps the simplest definition of $A \in \mathcal{P}_p$ is given by requiring

$$\lambda_1(A) + \cdots + \lambda_p(A) \ \geq 0 \tag{9.2}$$

where $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$ are the ordered eigenvalues of A.

A function $u \in \mathrm{USC}(X)$ on an open set $X \subset \mathbb{R}^n$ is said to be **p-plurisubharmonic on** X if for each point $x \in X$ and each test function φ for u at x, we have

$$D_x^2\varphi \in \mathcal{P}_p, \qquad \text{i.e.,}\ \ \mathrm{tr}_W \left\{D_x^2\varphi\right\} \ \geq\ 0 \quad \forall\, W \in G(p, \mathbb{R}^n) \tag{9.3}$$

or equivalently

$$\lambda_1(D_x^2\varphi) + \cdots + \lambda_p(D_x^2\varphi) \ \geq 0. \tag*{(9.3)$'$}$$

The dual subequation $\widetilde{\mathcal{P}_p}$ can be defined by

$$\widetilde{\mathcal{P}_p} \ = \ \left\{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}_W \geq 0 \ \text{ for some } W \in G(p, \mathbb{R}^n)\right\}$$

or equivalently, $A \in \widetilde{\mathcal{P}_p}$ if

$$\lambda_{n-p+1}(A) + \cdots + \lambda_n(A) \ \geq\ 0.$$

Subharmonic functions for the dual subequation $\widetilde{\mathcal{P}_p}$ will be referred to as **dually p-plurisubharmonic functions**. Function which are $\mathcal{P}_p$-harmonic, will be referred to simply as **p-harmonic functions**.

The terminology *p-plurisubharmonic* is justified by the Restriction Theorem 7.3 in [18] which says that $u \in \mathrm{USC}(X)$ is p-plurisubharmonic if and only if the restriction of u to each affine p-plane (or more generally to each connected p-dimensional minimal submanifold) is subharmonic (or $\equiv -\infty$) for the classical Laplacian (Laplace-Beltrami operator) on that p-plane.

The notion of p-plurisubharmonicity extends to any real number p with $1 \leq p \leq n$, using the ordered eigenvalues as follows. Let $\bar{p} = [p]$ denote the greatest integer in p.

Definition 9.1. For $A \in \mathrm{Sym}^2(\mathbb{R}^n)$ we say that

$$A \in \mathcal{P}_p \qquad \iff \qquad \lambda_1(A) + \cdots + \lambda_{\bar{p}}(A) + (p - \bar{p})\lambda_{\bar{p}+1} \ \geq\ 0$$

To see that $\mathcal{P}_p$ is a convex cone, one shows that $\mathcal{P}_p$ is the polar of the set of symmetric forms $P_{e_1} + \cdots + P_{e_{\bar{p}}} + (p - \bar{p})P_{e_{\bar{p}+1}}$ where $e_1, ..., e_n$ are orthonormal. Here P_e denotes orthogonal projection onto the line through e.

If E is a subset of $\{u = -\infty\}$ where $u \in \mathcal{P}_p(X)$ (and u is not identically $-\infty$ on any component of X), then E is said to be **p-pluripolar**. Note that since $u \in \mathcal{P}_p(X)$ implies that $\Delta u \geq 0$, the p-pluripolar sets can not have interior. Theorems 6.1 and 7.1 lead to the following.

THEOREM 9.2. *Suppose that E is a closed p-pluripolar set in X and $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is any pure second order, constant coefficient subequation which is $\mathcal{P}_p$-monotone.*

(1) Then E is removable for F-subharmonic functions which are locally bounded above across E. In particular, dually p-plurisubharmonic functions on $X - E$ extend to dually p-plurisubharmonic functions on X if they are locally bounded above.

(2) If $u \in C(X)$ is F-harmonic on $X - E$, then u is F-harmonic on X. In particular, continuous functions on X which are p-harmonic on $X - E$ are p-harmonic on X.

Since we have not assumed ψ to be smooth outside $\{\psi = -\infty\}$, the proof of Theorem 9.2 requires an extra step which is exactly like the one given in the proof of Theorem 8.1 above, and hence is omitted.

Remark 9.3. Theorem 9.2 is vacuous if $1 \leq p < 2$ since there are no p-pluripolar sets E. However, the classical "$u + \epsilon\psi$ technique" employed in the proof can still be used with the elementary Proposition 3.3 replacing the fact that $u + \epsilon\psi$ has no test functions at a point where ψ equals $-\infty$. See, for instance, the proof given in Appendix A of a theorem of Caffarelli, Li and Nirenberg.

Before describing lots of examples of subequations F which are $\mathcal{P}_p$-monotone in Section 11, we present some sufficient conditions for a set E to be p-pluripolar.

10. p-Pluripolar Sets and Riesz Potentials.

The p^{th} **Riesz kernel/function** K_p is defined (modulo a positive normalization) by

$$K_p(x) \ = \ |x|^{2-p}, \ \ 1 \leq p < 2, \qquad K_2(x) \ = \ \log|x|, \qquad \text{and}$$
$$K_p(x) \ = \ -\frac{1}{|x|^{p-2}}, \qquad p > 2. \tag{10.1}$$

They are fundamental for the subequation $\mathcal{P}_p$.

Proposition 10.2. *The Riesz kernel K_p is p-harmonic on $\mathbb{R}^n - \{0\}$ and p-plurisubharmonic on $\mathbb{R}^n$.*

Proof. For $x \neq 0$ set $e = x/|x|$. The functions K_p have second derivatives $D_x^2 K_p$ given (modulo a positive constant multiple) by the single formula

$$\frac{1}{|x|^p}\left(P_{e^\perp} - (p-1)P_e\right) \ = \ \frac{1}{|x|^p}\left(I - pP_e\right) \tag{10.2}$$

This easily implies that K_p is p-harmonic on $\mathbb{R}^n - \{0\}$. Since K_p has no test functions at $x = 0$, it is p-plurisubharmonic on all of $\mathbb{R}^n$. ∎

Taking convex combinations of $K_p(x-y)$ via convolution yields a general class of p-plurisubharmonic functions on $\mathbb{R}^n$.

Proposition 10.3. *Suppose μ is a compactly supported non-negative measure on $\mathbb{R}^n$. Then the p^{th} Riesz potential*

$$u \equiv K_p * \mu \tag{10.3}$$

defines a p-plurisubharmonic function on $\mathbb{R}^n$ which vanishes at ∞ for $p \geq 3$ and is $L^1_{\text{loc}}(\mathbb{R}^n)$.

Proof. Replacing K_p by K_p^α, the maximum of K_p and $-\alpha$, one obtains continuous functions $u^\alpha \equiv K_p^\alpha * \mu$ which decrease down, as $\alpha \to \infty$, to the point-wise defined function $u = K_p * \mu$. Hence u is upper semi-continuous. For integer p its restriction to each affine p-plane W is an (infinite) convex combination of Δ_W-subharmonic functions, and as such is Δ_W-subharmonic. The proof for general p is left to the reader. ∎

There is an extensive literature on Riesz potentials (see Landkof [29]). A compact set $E \subset \mathbb{R}^n$ is said to be **K_p-polar** if it is contained in the $-\infty$ set of some Riesz potential $u = K_p * \mu$. In fact μ can be chosen so that $\text{supp}\mu = E$ and $\{K_p * \mu = -\infty\} = E$ (see Section 1, Chapter III in [29]). Hence the Riesz potential is smooth in the complement of E. There is also a well defined notion of the K_p-capacity of E – commonly called the $p-2$ capacity. Both of the facts:

$$E \text{ is } K_p\text{-polar} \quad \Longleftrightarrow \quad E \text{ has } K_p\text{-capacity (or } (p-2)\text{-capacity) zero,}$$

$$E \text{ has finite Hausdorff } (p-2)\text{-measure} \quad \Rightarrow \quad \text{the } K_p\text{-capacity of } E \text{ is zero.}$$

are classical (see Sections 1 and 4 of Chapter III in [29]).

Putting all this together with Proposition 10.3 we have the following.

Proposition 10.4. *Suppose E is a compact subset of $\mathbb{R}^n$. Then:*

$$\begin{aligned} &E \text{ has finite Hausdorff } p-2\text{-measure} \\ \Rightarrow\quad &E \text{ has } K_p\text{-capacity (i.e., } p-2 \text{ capacity) zero} \\ \Rightarrow\quad &E \text{ is } C^\infty\ p\text{-pluripolar.} \end{aligned}$$

Further combining with Theorem 9.2 yields the following main result.

THEOREM 10.5. *Suppose $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a $\mathcal{P}_p$-monotone subequation, and E is a closed subset of $X \subset \mathbb{R}^n$. If E has locally finite Hausdorff $(p-2)$-measure, or, more generally, if E has $p-2$ capacity zero, then E is removable for F-subharmonics and F-harmonics as described in parts (1) and (2) of Theorem 9.2.*

11. Subequations which are $\mathcal{P}_p$-Monotone.

The previous two Theorems 9.2 and 10.5 require that the subequation F be $\mathcal{P}_p$-monotone. Typically F comes equipped with a natural monotonicity cone M. The purpose of this section is to characterize when $\mathcal{P}_p \subset M$, which ensures that F is $\mathcal{P}_p$-monotone.

Geometric Subequations.

First we illustrate with a few examples, starting with a large class – the *geometric* subequations $\mathbf{M} \equiv \mathcal{P}(\mathbb{G})$ defined by fixing a closed subset $\mathbb{G} \subset G(p, \mathbb{R}^n))$ and setting

$$\mathcal{P}(\mathbb{G}) \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \mathrm{tr}_W A \geq 0 \ \ \forall\, W \in \mathbb{G}\}. \tag{11.1}$$

Since $\mathbb{G} \subset G(p, \mathbb{R}^n)$, it is obvious that $\mathcal{P}_p \equiv \mathcal{P}(G(p, \mathbb{R}^n))$ is contained in $\mathcal{P}(\mathbb{G})$. Thus

$$F \text{ is } \mathcal{P}(\mathbb{G}) \text{ monotone} \qquad \Rightarrow \qquad F \text{ is } \mathcal{P}_p \text{ monotone}. \tag{11.2}$$

This means that $\mathcal{P}(\mathbb{G})$ and its dual $\widetilde{\mathcal{P}}(\mathbb{G})$ are $\mathcal{P}_p$-monotone, and that Theorems 9.2 and 10.5 apply to $\widetilde{\mathcal{P}}(\mathbb{G})$. These convex cone subequations have been studied in [13] and [14], and in the more general riemannian setting in [15]. Here it is convenient to replace the term $\mathcal{P}(\mathbb{G})$-*subharmonic* by $\mathbb{G}$-*plurisubharmonic* , and the term $\mathcal{P}(\mathbb{G})$-*polar* by $\mathbb{G}$-*pluripolar*.

For an example, consider $\mathbb{R}^{2n} = \mathbb{C}^n$ and let $\mathbb{G} \subset G(2, \mathbb{R}^{2n})$ be the set of complex lines. Then $\mathcal{P}(\mathbb{G})$ is exactly the subequation $\mathcal{P}^{\mathbb{C}}$ discussed in Section 8.

One also has $\mathbb{G} = \mathrm{LAG} \equiv \{W \in G(n, \mathbb{R}^{2n}) : W \text{ is Lagrangian}\}$. There is a Lagrangian equation of Monge-Ampère type with many branches, each of which is $\mathcal{P}(\mathrm{LAG})$-monotone (see [13]). By Proposition 10.4 any set of locally finite Hausdorff (n-2)-measure in $\mathbb{R}^{2n}$ is n-pluripolar and therefore LAG-pluripolar.

Another large class of examples comes from calibrations (see [13]). Among the interesting examples are the special Lagrangian, associative, coassociative and Cayley calibrations. The Theorems 9.2 and 10.5 apply to each case. For example, any set $E \subset \mathbb{R}^8$ of locally finite Hausdorff 2-measure is Cayley pluripolar. It would be interesting to find polar sets of larger dimension than that given by Proposition 10.4 for these calibrations. Preliminary work indicates that they may not exist.

Remark 11.1. It should be noted however that since $\mathcal{P}(\mathbb{G})$ is a convex cone, the results of Section 12 apply to show that codimension-2 sets are removable for $\mathbb{G}$-plurisubharmonic functions which are locally bounded above (provided that $\mathbb{G}$ involves all the variables). Consequently, the new parts of Theorem 9.2 and Theorem 10.5 are part (1) for $F = \widetilde{\mathcal{P}}(\mathbb{G})$ (for dually $\mathbb{G}$-plurisubharmonic functions) and part (2) for $\mathbb{G}$-harmonics.

One can also consider quaternionic n-space $\mathbb{R}^{4n} = \mathbb{H}^n$ and define $\mathcal{P}^{\mathbb{H}}$ geometrically by the subset of quaternionic lines in $G(4, \mathbb{R}^{4n})$. However, $\mathcal{P}^{\mathbb{C},e} \subset \mathcal{P}^{\mathbb{H}}$ for each of the complex structures defined on $\mathbb{H}^n$ by left multiplication by a unit imaginary quaternion e. In particular, this implies that the pluripolar sets for each of these complex structures (e.g., the complex analytic hypersurfaces) are quaternionic polar.

Example 11.2. (Branches of the p-Fold Sum Equation). For p fixed, the convex cone subequation $\mathcal{P}_p$ naturally occurs as a monotonicity subequation for each of the following subequations. In fact, $\mathcal{P}_p$ is the first, or smallest, branch of the family of subequations

$$\mathcal{P}_p \;=\; \mathcal{P}_p^1 \;\subset\; \mathcal{P}_p^2 \;\subset\; \cdots \;\subset\; \mathcal{P}_p^N \;=\; \widetilde{\mathcal{P}}_p$$

determined by the Gårding polynomial

$$MA_p(A) \;=\; \prod_{i_1 < \cdots < i_p} \left(\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A)\right)$$

whose eigenvalues are the p-fold sums of the eigenvalues of $A \in \mathrm{Sym}^2(\mathbb{R}^n)$. None of the other branches of MA_p are convex, but (see [16] for the details)

$$\text{each branch of } MA_p \text{ is } \mathcal{P}_p \text{ monotone.} \tag{11.3}$$

The Riesz Characteristic.

The following simple observation is the basis of this subsection. Suppose $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a subequation with monotonicity cone M. Then if $\mathcal{P}_p$ is contained in M, $\mathcal{P}_p$ is also a monotonicity cone for F and Theorems 9.2 and 10.5 apply.

The cones $\mathcal{P}_p$ are nested, that is, for real numbers $p < p'$ we have $\mathcal{P}_p \subset \mathcal{P}_{p'}$. A characterization of the largest $\mathcal{P}_p$ contained in M was given by Theorem 5.1b in [20]. The statement, given below, uses the following easily computed invariant of M.

Definition 11.3. The **Riesz characteristic** p_M **of** M is defined by

$$p_M \;\equiv\; \sup\{p : I - pP_e \in M \quad \text{for all } |e| = 1\}.$$

THEOREM 11.4. $(1 < p < n)$. *Suppose $M \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a convex cone subequation. Then*

$$\mathcal{P}_p \ \subset\ M \qquad \Longleftrightarrow \qquad p\ \leq\ p_M.$$

Corollary 11.5. *The statement that $\mathcal{P}_p \subset M$ is equivalent to either of the following:*

(1) $K_p(x)$ is M-subharmonic on $\mathbb{R}^n$,

(2) $(K_p * \mu)(x)$ is M-subharmonic on $\mathbb{R}^n$ for all measures $\mu \geq 0$ with compact support.

Proof. Use (10.2) and Proposition 10.3. ∎

Because of Theorem 11.4 the Theorems 9.2 and 10.5 can be recast in a more useful form using the Riesz characteristic p_M of M.

THEOREM 11.6. *Suppose that $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a subequation with monotonicity cone $M \subset \mathrm{Sym}^2(\mathbb{R}^n)$ (a convex cone subequation). Let $E \subset X$ be a closed subset. Suppose any one of the following holds.*

(a) E is p_M-polar.

(b) E has locally finite Hausdorff $(p_M - 2)$-measure.

(c) E has $(p_M - 2)$-capacity zero.

Then:

(1) Each $u \in F(X - E)$ locally bounded above across E has a canonical extension to $U \in F(X)$.

(2) Each $u \in C(X)$ which is F-harmonic on $X - E$ is F-harmonic on X.

We conclude with several examples where this theorem applies. Specific cases are illustrated in Appendix B.

Example 1. (The δ-Uniformly Elliptic Cone). For each $\delta > 0$ we define

$$P(\delta)\ \equiv\ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : A + \delta(\mathrm{tr}A)\cdot I \geq 0\}$$

Note that

$$I - pP_e \in \mathcal{P}(\delta) \qquad \Longleftrightarrow \qquad I - pP_e + \delta(n-p) \geq 0 \qquad \Longleftrightarrow \qquad p\ \leq\ \frac{1+\delta n}{1+\delta}.$$

This proves that

$$\mathcal{P}(\delta) \ \text{ has Riesz characteristic } \ p \ = \ \frac{1+\delta n}{1+\delta}.$$

It is shown in [23, Lemma A.1] that all other O(n)-invariant convex cone subequations with Riesz characteristic p are contained in this $\mathcal{P}(\delta)$.

Example 2. (The Pucci Cone). For $0 < \lambda < \Lambda$ we define

$$\mathcal{P}_{\lambda,\Lambda} \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \lambda \mathrm{tr} A^+ + \Lambda \mathrm{tr} A^- \geq 0\},$$

where $A = A^+ + A^-$ is the decomposition into $A^+ \geq 0$ and $A^- \leq 0$. Since $I - pP_e = P_{e^\perp} - (p-1)P_e$, we have that

$$\mathcal{P}_{\lambda,\Lambda} \ \text{ has Riesz characteristic } \ \frac{\lambda}{\Lambda}(n-1)+1.$$

The condition that a subequation F be uniformly elliptic (in the standard sense) can be restated as "monotonicity" using either of the two cones $\mathcal{P}(\delta)$ or $\mathcal{P}_{\lambda,\Lambda}$. That is, F is uniformly elliptic if and only if either

$$\begin{aligned} F + \mathcal{P}(\delta) \ &\subset \ F \quad \text{for some } \ \delta > 0, \text{or} \\ F + \mathcal{P}_{\lambda,\Lambda} \ &\subset \ F \quad \text{for some } \ 0 < \lambda < \Lambda. \end{aligned}$$

(See Section 4.5 in [21] for more details.)

Example 3. (The k^{th} Elementary Symmetric Cone). This equation is often referred to as the k^{th} hessian equation. For $k = 1, ..., n$ we define

$$\Sigma_k \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\},$$

where $\sigma_j(A)$ is the j^{th} elementary symmetric function in the eigenvalues of A. One easily computes that

$$\Sigma_k \ \text{ has Riesz characteristic } \ \frac{n}{k}.$$

In all three of the examples above there is a polynomial equation on $\mathrm{Sym}^2(\mathbb{R}^n)$ which is Gårding hyperbolic with respect to the identity (see [16]), and the cone M is the Gårding cone associated with this polynomial. The Gårding polynomial determines other natural subequations called branches which are M-monotone.

Theorem 11.6 can be applied to a broad spectrum of nonlinear equations. Here are some cases related to the examples above.

Example 1$'$. If E has locally finite Hausdorff $\frac{\delta(n-2)-1}{\delta+1}$ measure, then E is removable (as in the conclusion of Theorem 11.6) for each branch of the equation $\det(A + \delta(\mathrm{tr}A)I) = 0$.

In fact, this removable singularity result holds for the **δ-elliptic regularization** $F(\delta) \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : A + \delta(\mathrm{tr}A)I \in F\}$ of *any* pure second-order subequation F, because $F(\delta)$ is always $\mathcal{P}(\delta)$-monotone. This follows from the general fact that

if M is a monotonicity cone for F, then $M(\delta)$ is a monotonicity cone for $F(\delta)$.

Example 2′. If E has locally finite Hausdorff $\frac{\lambda}{\Lambda}(n-1)-1$ measure, then E is removable (as in the conclusion of Theorem 11.6) for each branch of the Gårding equation defining $\mathcal{P}_{\lambda,\Lambda}$.

This Gårding polynomial is defined as follows. For each subset $I \subset \{1,...,n\}$ define $v(I) \in \mathbb{R}^n$ by $v(I)_i = \lambda$ if $i \in I$ and $v(I)_i = \Lambda$ if $i \notin I$. The points $v(I)$ are the vertices of the cube $[\lambda,\Lambda]^n \subset \mathbb{R}^n$. Let $\mathcal{I}$ denote the subset of $I \subset \{1,...,n\}$ such that the open segment from 0 to $v(I)$ is disjoint from the cube. Given $A \in \mathrm{Sym}^2(\mathbb{R}^n)$, let $\lambda_I(A) \equiv \sum_{i\in I}\lambda_i(A)$. Finally, set $I' \equiv \{1,...,n\} - I$. The the Gårding polynomial with Gårding cone $\mathcal{P}_{\lambda,\Lambda}$ is

$$p(A) \ \equiv\ \prod_{I\in\mathcal{I}} \left(\lambda\,\lambda_I(A) + \Lambda\,\lambda_{I'}(A)\right).$$

Note that with $\Lambda = 1+\delta$ and $\lambda = \delta$, the degree n polynomial $\det(A + \delta(\mathrm{tr}\,A)I)$ defining $\mathcal{P}(\delta)$ (as its Gårding cone) is a factor of this polynomial $p(A)$ defining $\mathcal{P}_{\lambda,\Lambda}$.

Example 3′. If E has locally finite Hausdorff $\frac{n}{k}-2$ measure, then E is removable (as in the conclusion of Theorem 11.6) for each branch of the equation $\sigma_k(A) = 0$.

Variable Coefficients.

The ideas above can be applied directly to variable coefficient subequations $F \subset \mathbf{J}^2(X) \equiv X \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}^2(\mathbb{R}^n)$ as defined in [15]. Suppose that such a subequation F on an open set $\Omega \subset \mathbb{R}^n$ is $\mathbf{P}_p$-monotone where $\mathbf{P}_p \equiv \mathbb{R} \times \mathbb{R}^n \times \mathcal{P}_p$ (i.e., $F + \mathbf{P}_p \subset F$ under fibre-wise sum).

Then any closed set $E \subset \Omega$ of locally finite Hausdorff $(p-2)$-measure is removable for F-subharmonics and F-harmonics as in Theorems 6.1 and 7.1.

Proof. Consider a Riesz potential ψ which is $-\infty$ on E and $\mathcal{P}_p$(i.e., $\mathbf{P}_p$)-subharmonic as above. Then E is C^∞ $\mathbf{P}_p$-polar. The result then follows from Theorems 6.1 and 7.1. ∎

Now on any riemannian manifold there is a natural subequation $\mathbf{P}_p^X$ which generalizes $\mathbf{P}_p$ on euclidean space. It is defined as in Definition 9.1 by using the ordered eigenvalues of the riemannian hessian (cf. §4 in [15]).

THEOREM 11.7. *Let F be a subequation on a riemannian manifold X which is $\mathbf{P}_p^X$-monotone. Then any closed subset $E \subset X$ of Hausdorff* *dimension $p' < p$ is removable for F-subharmonics and F-harmonics as Theorems 6.1 and 7.1*

Proof. Fix $x \in X$ and choose geodesic normal coordinates at x. Straightforward calculation of $D^2 K_p$ shows that for any fixed $p' < p$ there exists $\epsilon > 0$ so that $\mathrm{Hess}_x\{K_{p'}(x-y)\} \in \mathcal{P}_p$ (i.e. $K_{p'}(x-y)$ is $\mathbf{P}_p$-subharmonic) for all $x, y \in B(0,\epsilon)$. Thus if μ is a measure with support in $B(0,\epsilon)$, then $\mu * K_{p'}$ is $\mathbf{P}_p$-subharmonic on $B(0,\epsilon)$. Therefore, assuming that $E \cap B(0,\epsilon)$ has Hausdorff dimension $\leq p'$ we conclude that $E \cap B(0,\epsilon)$ is C^∞ $\mathbf{P}_p$-polar, and Theorems 6.1 and 7.1 apply. ∎

12. Removable Singularities for Convex Subequations.

The results of this section do not depend on Theorems 6.1 and 7.1.

A Theorem stated in [12, p.132], and again in [25, Thm. 1.2], for classical plurisubharmonic functons u (the case where $F = \mathcal{P}^{\mathbb{C}}$ as in Section 8) contains four removable singularity results which entail different local growth hypotheses on u across E, namely:

(a) $u \in C^\alpha(X)$, $(0 < \alpha \leq 1)$, (b) $u \in L^p_{\mathrm{loc}}(X)$, $(1 \leq p < \infty)$

(c) u locally bounded above across E, (d) no condition on u across E.

The different size hypotheses on E are:

(a) E has zero Hausdorff $2n-2+\alpha$ measure.

(b) E has locally finite Hausdorff $2n-2q$ measure, with $\frac{1}{p}+\frac{1}{q}=1$.

(c) E has locally finite Hausdorff $2n-2$ measure.

(d) E has zero Hausdorff $2n-2$ measure.

Parts (a) and (b) are due to the first author and Polking [24], [25], part (c) is due to Lelong [30], and part (d) is due to Shiffman [33].

Each of the first three results in this theorem has a straightforward generalization to subharmonics (but not harmonics) for any convex subequaton F which is second-order complete. A subequation F is *convex* if each fibre F_x is convex. It is *second-order complete* if each non-empty fibre $F_{x,r,p} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ "depends on all the variables in $\mathbb{R}^n$". This means that there does not exist a proper subspace $W \subset \mathbb{R}^n$ and a subset $F' \subset \mathrm{Sym}^2(W)$ such that

$$A \in F_{x,r,p} \quad \iff \quad A\big|_W \in F'.$$

(When F_x is convex, if one non-empty fibre $F_{x,r,p} \subset \mathrm{Sym}^2(\mathbb{R}^n)$ depends on all the variables in $\mathbb{R}^n$, then so does every other non-empty fibre $F_{x,r',p'}$.)

Throughout this section we make both of the above assumptions on F – convexity and second-order completeness – in addition to the positivity condition (P) and the negativity condition (N) (see Definition 3.8 in [15]).

THEOREM 12.1. *Suppose E is a closed subset of an open set $X \subset \mathbb{R}^n$, and u is F-subharmonic on $X - E$.*

(a) If $u \in C^\alpha(X)$ and E has Hausdorff $(n-2+\alpha)$-measure zero, then u is F-subharmonic on X.

(b) If $u \in \mathrm{L}^p_{\mathrm{loc}}(X)$ $(1 \le p < \infty)$ and E has locally finite Hausdorff $(n-2q)$-measure $(\frac{1}{q}+\frac{1}{p}=1)$, then u is F-subharmonic on X.

(c) If u is locally bounded above across E and E has Newtonian capacity zero, then the canonical extension U of u to X is F-subharmonic. In particular, if E has locally finite Hausdorff $(n-2)$-measure, then the capacity of E is zero and hence $U \in F(X)$.

The fourth part (d) of the theorem in [12], namely:

If u is plurisubharmonic on $X - E$ and E has Hausdorff $(2n-2)$-measure zero, then u has a plurisubharmonic extension to X.

has the following counterpart in p-geometry (see Sections 9 and 10 above.)

THEOREM 12.2. **($p \ge 2$).** *Suppose u is p-plurisubharmonic on $X - E$ and the Hausdorff $(n-p)$-measure of the closed set E is zero. Then u has a p-plurisubharmonic extension to X*

Proof. This is an immediate consequence of Part (c) of Theorem 12.1 and Lemma 6.2. ∎

Discussion of the Proof of Theorem 12.1. The proof is by reduction to the linear case.

Step 1. (The Strong Bellman Principle). The previous assumptions on F (together with a mild regularity assumption) imply that locally there exists a family $\mathcal{F}$ of linear (sub)equations of the form

$$Lu \;=\; \langle a(x), D_x^2 u\rangle + \langle b(x), D_x u\rangle - c(x)u \;\ge\; \lambda \tag{12.1}$$

with $a(x) > 0$ positive definite and $c(x) \ge 0$ at each point x with the property that

$$\begin{gathered}\text{A function } u \text{ is } F \text{ subharmonic} \quad \Longleftrightarrow \\ \text{locally } u \text{ satsfies } Lu \ge \lambda \text{ for all pairs } (L,\lambda) \in \mathcal{F}.\end{gathered} \tag{12.2}$$

This is Theorem 8.6 in [22].

Step 2. Show that a viscosity (L, λ)-subharmonic function u belongs to $L^1_{\rm loc}$ and satisfies $Lu \geq \lambda$ in the distributional sense.

Step 3. Prove the theorem in the linear distributional case. For solutions to $Lu = \lambda$, Part (a) is Theorem 4.4 in [24] while part (b) is Theorem 4.1(a) in [24]. For proving part (a) the modifications required to treat $Lu \geq \lambda$ are described on pages 705-706 in [25], while for part (b) the modifications for $Lu \geq \lambda$, as described on pages 132-133 in [12], go as follows.

Pick ψ smooth and compactly supported in X with $\psi \geq 0$. For convenience assume $\int \psi = 1$. One must show that $(Lu)(\psi) \geq \int \lambda\psi$ for all such ψ. Choose φ_ϵ as in Lemma 3.2 in [24]. Now $(Lu)(\psi) = (Lu)((1-\varphi_\epsilon)\psi) + (Lu)(\varphi_\epsilon\psi)$. The estimate on the derivatives of φ_ϵ provided by Lemma 3.2 in [24] show that $L^t(\varphi_\epsilon\psi)$ converges to zero in L^q, and hence by Hölder's inequality that $(Lu)(\varphi_\epsilon\psi)$ converges to zero. Meanwhile, $(Lu)((1-\varphi_\epsilon)\psi) \geq \int \lambda(1-\varphi_\epsilon)\psi$ (since $Lu \geq \lambda$ on $X - E$) implies that $(Lu)(\psi) \geq \int \lambda\psi$.

Perhaps it is worth noting that neither of the conditions L elliptic ($a(x) > 0$) or $c(x) \geq 0$ was used to conclude that $Lu \geq \lambda$ across E in the distributional sense. In fact, the only thing used about L was that it has order 2.

Part (c) in the linear distributional case is classical. If E has Newtonian capacity zero, then E is L-polar. Therefore our Theorem 6.1 applies with F defined by $Lu \geq \lambda$ and $M = L$. Of course our proof is just the classical proof of the removability of E for F-subharmonic functions.

Step 4. Show that distributional solutions u to $Lu \geq \lambda$ (they belong to $L^1_{\rm loc}$) have a pointwise canonical representative $U \in {\rm USC}(X)$ in the $L^1_{\rm loc}$-class given by

$$U(x) \ = \ {\rm ess} \varlimsup_{\rm y \to x} u(y) \ = \ \lim_{\rho\to 0} \operatorname*{ess\,sup}_{B_\rho(y)} \ u$$

The important point here is that U is independent of the operator L, and that U is (L, λ)-subharmonic in the viscosity sense. (See the Appendix in [19] and Section 9 in [22].)

Step 5. Use (12.2), this time in the reverse direction, to conclude that U is F-subharmonic on X.

Appendix A.
A Removable Singularity Theorem of Caffarelli, Li and Nirenberg

The classical "$u+\epsilon\psi$-technique", that we have used to prove Theorem 6.1, naturally lends itself to prove a recent result of Caffarelli, Li and Nirenberg

[8, Thm. 1.3]. Let $d_E(x) \equiv \text{dist}(x, E)$ denote the distance to a subset $E \subset \mathbb{R}^n$.

THEOREM A.1. (Caffarelli, Li and Nirenberg). *Suppose E is a closed submanifold of an open set X in $\mathbb{R}^n$, and that F is a general subequation on X. Consider a function $u \in F(X - E)$ which is locally bounded above across E. Let $U \in \text{USC}(X)$ be the canonical upper semi-continuous extension of u to X. If for all $\epsilon > 0$ sufficiently small*

$$U + \epsilon d_E \ \ has\ no\ test\ functions\ at\ points\ of\ E, \tag{A.1}$$

then U is F-subharmonic on X.

In [8] a function $U \in \text{USC}(X)$ satisfying (A.1) is said to be **upper conical on** E.

Proof. Note that $U + \epsilon d_E$ decreases to U as $\epsilon \searrow 0$. Therefore, by the *Decreasing Limit Property*" (see, for example, Theorem 2.6 in [15]), if each $U + \epsilon d_E$ is G-subharmonic on X for some fixed subequation G, then U is also G-subharmonic on X. Now for each $c > 0$ take G to be the enlargement F^c of our subequation F defined by

$$F^c_x \ \equiv \ \{J = (r, p, A) : \text{dist}_x(J, F_x) \leq c\}, \tag{A.2}$$

where the fibre-distance is defined so that $F^c = F + S^c$ for the constant coefficient subequation $S^c \equiv (-\infty, c] \times B_c(0) \times (\mathcal{P} - c \cdot \text{I})$. It will suffice to show that: for each $c > 0$

$$U + \epsilon d_E \in F^c(X) \ \ \text{for all } \epsilon \text{ sufficiently small}, \tag{A.3}$$

since the assertion: $U \in F^c(X) \ \forall c > 0 \ \Rightarrow \ U \in F(X)$ is a trivial consequence of the definition of subharmonicity. This uses the fact that $F = \bigcap_{c>0} F^c$ which follows from positivity (P) and negativity (N) for F.

By the hypothesis (A.1) and the trivial Proposition 3.3 it will suffice to prove

$$U + \epsilon d_E \in F^c(X - E) \ \ \text{for all } \epsilon \text{ sufficiently small}. \tag*{$(A.3)'$}$$

In fact, since the result is local, we may replace the submanifold E by the interior of a small compact subset of E and replace X by a thin normal neighborhood of E.

Given a point $x \notin E$ but near E, there exists a unique line segment from a point $x_0 \in E$ to x, of length $\delta = d_E(x)$. We write $x = x_0 + \delta\nu$. Let B denote the normal disk to E of radius δ at x_0. The tangent space at x splits as

$$T_x\mathbb{R}^n \ = \ T_x(B)^\perp \oplus T_x(\partial B) \oplus \text{span}\,\nu.$$

Let II^E_x denote the second fundamental form of E in the normal direction ν, translated along the line segment from x_0 to x and acting as a quadratic form on $T_x(B)^\perp$. Let $P_W \in \mathrm{Sym}^2(\mathbb{R}^n)$ denote orthogonal projection onto a subspace W. By direct computation the 2-jet of d_E is given by:

$$d_E(x) = \delta, \quad D_x d_E = \nu, \quad D_x^2 d_E = II_x^E + \frac{1}{\delta} P_{T_x(\partial B)}. \tag{A.4}$$

We can assume that $d_E(x) < 1$ for all $x \in X$. Note that $|D_x d_E| = 1$. Finally, $D_x^2 d_E \geq -\kappa I$ where $\kappa = \sup|\kappa(\nu')|$ taken over all principal curvatures of all normal directions to E (which is finite by our pre-compactness assumption). Thus with $C \equiv \max\{1, \kappa\}$ we have

$$\left(d_E(x), D_x d_E, D_x^2 d_E\right) \ \in \ S^C \quad \forall\, x \in X - E, \tag{A.5}$$

that is, d_E is S^C-subharmonic on $X - E$. It follows directly from Definition 2.1, first, that ϵd_E is $S^{\epsilon C}$-subharmonic on $X - E$, and second, since d_E is smooth, that $U + \epsilon d_E$ is $F^{\epsilon C}$-subharmonic on $X - E$. This proves (A.3)$'$ if $\epsilon < 1$ since $F^{\epsilon C}(X - E) \subset F^C(X - E)$ for $0 < \epsilon < 1$. ■

Remark 1.2 in [8] can be used to replace the hypothesis (A.1) on $U + \epsilon d_E$ with a hypothesis on U itself. The only way the hypothesis (A.1) on $U + \epsilon d_E$ can fail to be true at $\bar{x} \in E$ is for U to have a very large 2-jet at $\bar{x}$. More precisely:

if $U + \epsilon d_E$ has a test function φ at $\bar{x}$ and if ψ is any test function for $-\epsilon d_E$ at $\bar{x}$, then $\varphi + \psi$ is a test function for U at $\bar{x}$.

This can be restated in terms of jets as follows. For an upper semi-continuous function v defined near $\bar{x}$ we let

$$J_{\bar{x}}^+ v \ = \ \{(D_{\bar{x}}\varphi, D_{\bar{x}}^2) : \varphi \text{ is a test function for } v \text{ at } \bar{x}\}$$

denote the *upper (reduced) 2-jet* of v at $\bar{x}$.

$$(A.1) \text{ is false at } \bar{x}, \text{ i.e., } J_{\bar{x}}^+(U + \epsilon d_E) \neq \emptyset \qquad \Rightarrow$$

$$J_{\bar{x}}^+(U + \epsilon d_E) + J_{\bar{x}}^+(-\epsilon d_E) \ \subset \ J_{\bar{x}}^+(U). \tag{L}$$

Thus any condition which limits the size of the 2-jet of U at $\bar{x}$ enough to violate the "large" condition (L) will imply the hypothesis (A.1). For example when E is a point, it is an easy calculation to see that

$$J_{\bar{x}}^+(-\epsilon|x - \bar{x}|) \ = \ B_\epsilon(0) \times \mathrm{Sym}^2(\mathbb{R}^n). \tag{A.6}$$

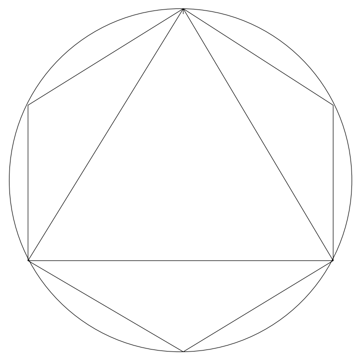
Thus if (A.1) fails when $E = \{\bar{x}\}$, then

$$B_\epsilon(p) \times \mathrm{Sym}^2(\mathbb{R}^n) \ \subset \ J_{\bar{x}}^+(U). \tag{L$'$}$$

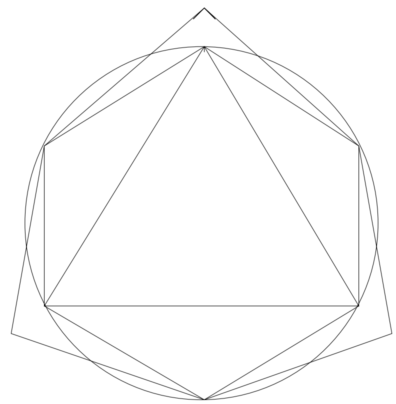
where p is the first derivative of a test function for $U + \epsilon|x - \bar{x}|$ at $\bar{x}$.

Corollary A.2. *Suppose $u \in F(X - \{\bar{x}\})$ with u bounded above across $\bar{x}$. Then the extension $U \in \mathrm{USC}(X)$ of u, given by defining $U(\bar{x}) = \overline{\lim}_{x \to \bar{x}} u(x)$, is F-subharmonic on X, unless the reduced upper 2-jet of U at $\bar{x}$ contains $B_\epsilon(p) \times \mathrm{Sym}^2(\mathbb{R}^n)$ for some $p \in \mathbb{R}^n$ and $\epsilon > 0$.*

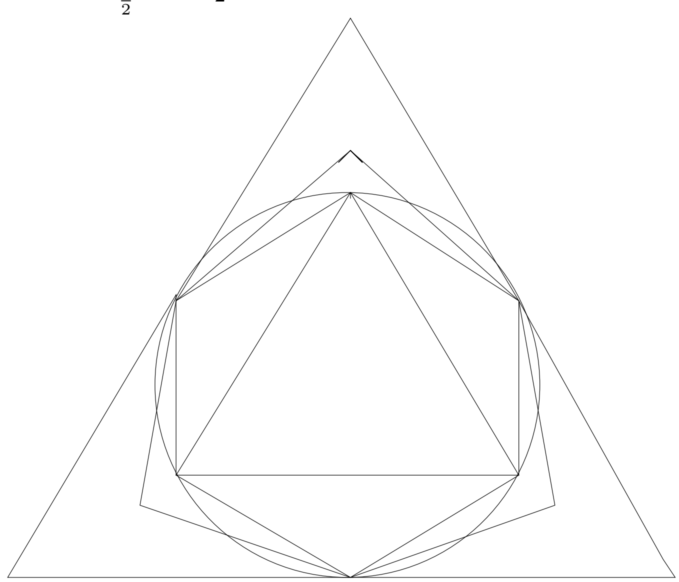
Appendix B. An Illustration: Cones with $p = \frac{3}{2}$ and $n = 3$



$\mathcal{P} \ \subset \ \mathcal{P}_{\frac{3}{2}} \ \subset \ \Sigma_2$



Pucci Cone Added



$\mathcal{P}(\delta)$ Added

These figures illustrate various convex cones discussed in the paper in the specific case of $\mathrm{Sym}^2(\mathbb{R}^3)$ and Riesz characteristic $p = \frac{3}{2}$. Since the sets are SO(3)-invariant, it suffices to represent them in eigenvalue space

$$\mathbb{R}^3 \ = \ \{(\lambda_1, \lambda_2 \lambda_3) : \lambda_k \in \mathbb{R}\}.$$

Moreover, it suffices to consider the bases of these cones in

$$\mathbb{R}^2 \ = \ \{(\lambda_1, \lambda_2 \lambda_3) \in \mathbb{R}^3 : \lambda_1 + \lambda_2 + \lambda_3 = 1\}.$$

There are two reference cones: the cone $\mathcal{P}$ whose base is the inscribed triangle, and the cone $\Delta \equiv \{\mathrm{tr} A > 0\}$ whose base is entire plane. All other cones represented have Riesz characteristic $\frac{3}{2}$. In monotone progression they are as follows.

The Inscribed Hexagon is the base of the cone $\mathcal{P}_{\frac{3}{2}}$. For slightly larger values of p, the base of the cone $\mathcal{P}_p$ is also an (irregular) hexagon. In these cases, three of the vertices coincide with the vertices of the triangle. The other three vertices are further from the origin than those of $\mathcal{P}_{3/2}$.

The Disk (which has radius $\sqrt{\frac{3}{2}}$) is the base of the second Hessian cone Σ_2, given by the conditions $\sigma_1(A) \geq 0$ and $\sigma_2(A) \geq 0$. (See Example 3 in Section 11.) By the remarks above, $p = \frac{3}{2}$ is the largest p with $\mathcal{P}_p \subset \Sigma_2$.

The Second Hexagon is the base of the Pucci cone $\mathcal{P}_{\lambda,\Lambda}$ of Riesz characteristic $\frac{3}{2}$. (See Example 2 in Section 11.) It is the smallest Pucci which contains $\mathcal{P}_{\frac{3}{2}}$. This holds if and only if $\frac{\lambda}{\Lambda} = \frac{1}{4}$. Note that the base of $\mathcal{P}$ is contained in the *interior* of the Pucci cone. This is equivalent to uniform ellipticity of the Pucci equations.

The Circumscribed Triangle is the base of the uniform ellipticity cone $\mathcal{P}(\delta)$ of Riesz characteristic $\frac{3}{2}$. (See Example 1 in Section 11.) Here $\delta = \frac{1}{3}$. The bases of the family of cones $\mathcal{P}(\delta)$ differ by homotheties from the center of the disk. The circumscribed triangle is clearly the smallest such to contain $\mathcal{P}_{\frac{3}{2}}$.

Note the three points common to the boundaries of each of the four subequations with common characteristic $p = \frac{3}{2}$. They represent $\frac{2}{3}I - P_{e_j}$, $j = 1, 2, 3$ (cf. Definition 11.3).

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