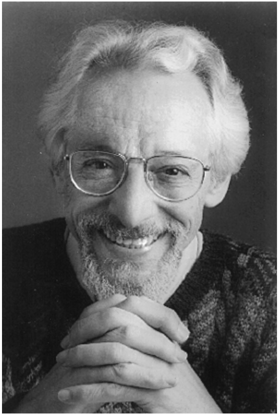
# Reflections on the Early Mathematical Life of Bob Osserman



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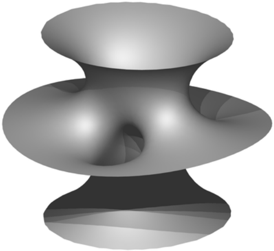

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In particular we have:

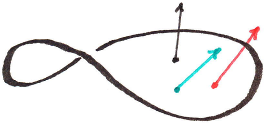
$$\frac{d}{dt} A(\Sigma_t)\Big|_{t=0} = 0.$$

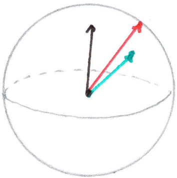
# A Geometric Characterization

**THE GAUSS MAP**

$$N : \Sigma \longrightarrow S^2$$

The Gauss map associates to each point $x \in \Sigma$, the normal vector $N(x)$ to $\Sigma$ at $x$, i.e., the vector perpendicular to the tangent plane to $\Sigma$ at $x$.



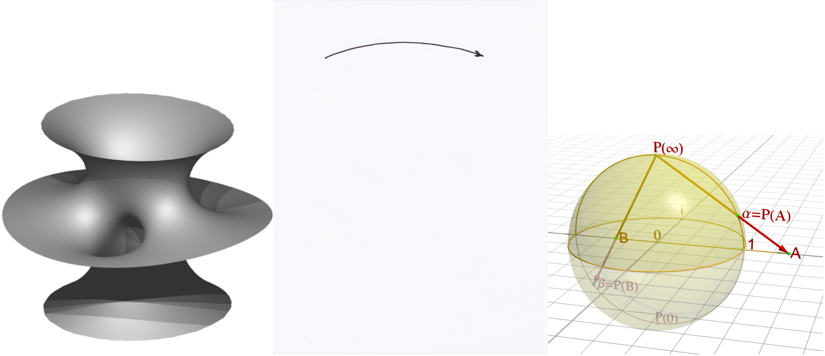


Σ is **minimal** if and only if the Gauss map is (anti)-**conformal**.

Angles are preserved (but direction is reversed).

# Complex Anaysis Enters the Picture

Take Stereographic Projection




Complex Analysis is

a rich and deep subject

with many beautiful results.

# Elementary Question

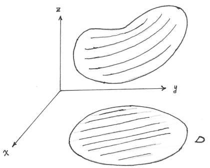
Suppose our surface $\Sigma$ is the graph of a function $z = f(x, y)$
over a domain $D$ in the $(x, y)$-plane

.

When is this graph a minimal surface?

ANSWER: It must satisfy the differential equation

$$\boxed{(1 + f_y^2)f_{xx} + (1 + f_x^2)f_{yy} - 2f_x f_y f_{xy} \ = \ 0.}$$



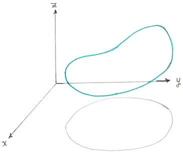
# The Dirichlet Problem

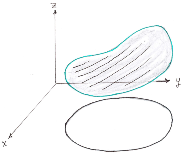
Let $D$ be the round disk of radius $R$.
Let $\varphi$ be an arbitrary continuous function on the boundary circle.

.

**Theorem.** *There exists a unique function $f(x, y)$ continuous on D and smooth in its interior, such that $f = \varphi$ on $\partial D$ and in the interior it satisfies the minimal surface equation:*

$$(1 + |\nabla f|^2)\Delta f - (\nabla f)^t \mathbf{H}(f) \nabla f \ = \ 0$$



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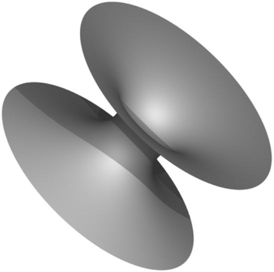
**Surprise!!**

**The Bernstein Theorem (1918).** *Any solution of the minimal surface equation which is defined for all $(x, y)$ in the plane must be is linear, i.e., its graph is an affine 2-plane.*

This is a beautiful and astonishing result.

If we remove a tiny disk from the plane,
there is a function defined everywhere outside that disk
whose graph is a minimal surface.



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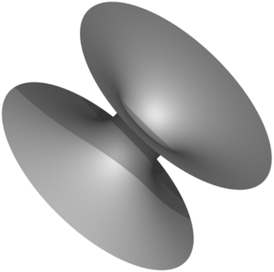


This is also true if we remove a half-line from the plane, Is this a special case – a quirk of nature?

Or – is there something deeper hidden behind this result?

# Bob Osserman

In 1959 Bob gave a wonderful geometric generalization
of Bernstein's Theorem.

In fact a significant body of his subsequent research
revolved around the mathematical questions
engendered by this initial result.

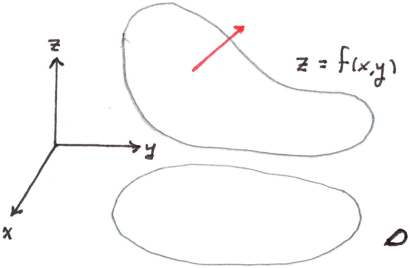
To understand his theorem
we must return to the **Gauss Map**.

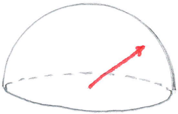
**Notice:** If

$$\Sigma = \{(x, y, f(x, y)) : (x, y) \in D\}$$

is a the graph of a function, then

the image of the Gauss map lies in the upper hemisphere



**Theorem (Osserman – 1959).**

Let $\Sigma \subset E^3$ be a complete minimal surface whose Gauss image misses a neighborhood of some point in $S^2$. Then $\Sigma$ is a plane.

Complete means there is no curve of finite length in $\Sigma$ which gets to the "end" or "boundary" of the surface (i.e., which exits every compact subset).

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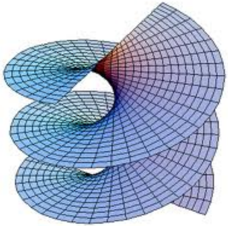

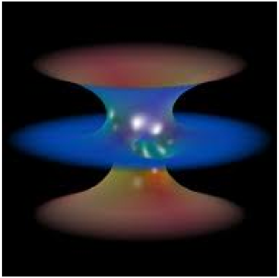

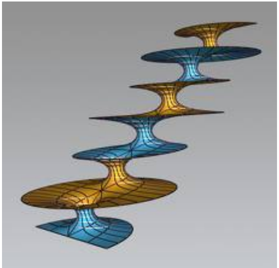

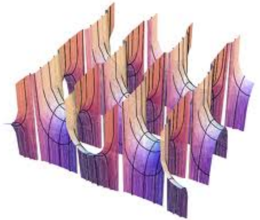

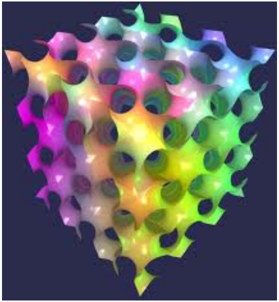

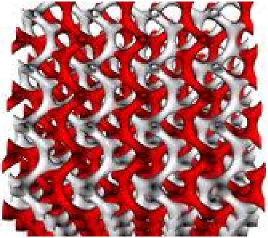

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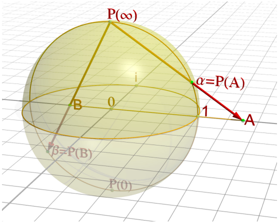
We may assume that $N$ misses a neighbothood of the north pole.

Thus after stereographic projection,




the Gauss map becomes a bounded holomorphic function. If $R \cong \mathbb{C}$, then Liouville's Theorem implies that the Gauss map must be constant, which immediately implies that $\Sigma$ is a plane.

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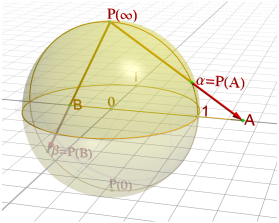


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We are reduced to the case where $R$ is conformally the unit disk.

Applying the Weierstrauss Representation Theorem and a clever complex analysis argument, Bob Osserman showed that the induced metric could not be complete.

The argument boils down to the following assertion. Let $f : \Delta \to \mathbb{C}$ be a holomorphic function which is nowhere zero. Then the riemannian metric

$$ds^2 = |f(z)|^2|dz|^2$$

is not complete. To see this set

$$F(z) \equiv \int_0^z f(\zeta)\, d\zeta.$$

By Liouville's Theorem there is a maximal disk $\{|w| < r\}$ of finite radius on which the inverse function $G(w) = F^{-1}(w)$ is defined (with $G(0) = 0$), and there is a point $w_0$ with $|w_0| = r$ such that $G$ cannot be analytically continued into any neighborhood of $w_0$.

Now the path $\Gamma = G(\{tw_0 : 0 \leq t < 1\})$ has finite length ($\leq r$) and must go to the boundary in $\Delta$.

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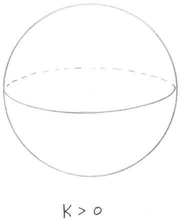
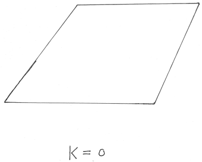
**Theorem (Osserman – 1961).**

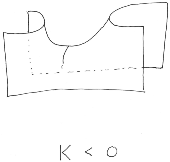
Let $\Sigma \subset E^3$ be a complete minimal surface whose Gauss image misses a set of positive logarithmic capacity in $S^2$. Then $\Sigma$ is a plane.

# Gauss Curvature

Gauss curvature is a function $K : \Sigma \to \mathbb{R}$ which is characterized in several ways.




On a minimal surface $K \leq 0$.

**Theorem (Osserman – 1961).**

Let $\Sigma \subset E^3$ be a complete minimal surface whose Gauss image misses a set of logarithmic capacity zero in $S^2$. Then $\Sigma$ is a plane.

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Let $\Sigma \subset E^3$ be a complete minimal surface.

1. Suppose $\int_\Sigma K\, dA = -\infty$, then the normals assume every direction infinitely often except for a set of logarithmic capacity zero.

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# Conjectures of Nirenberg

Prior to Bob's first work on minimal surfaces, Louis Nirenberg made the following conjectures. They were based (I imagine) on taking seriously the analogy between the Gauss map of a complete minimal surface in $\mathbb{E}^3$ and a holomorphic function on $\mathbb{C}$.

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Conjecture 1 (Liouville-type theorem). If the Gauss map of a complete minimal surface in $\mathbb{E}^3$ misses a neighborhood of some point, the surface is a plane.

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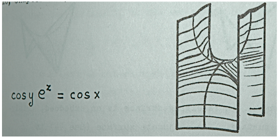
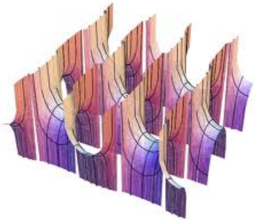
Conjecture 2 (Picard-type theorem). If the Gauss map of a complete minimal surface in $\mathbb{E}^3$ misses three points, the surface is a plane.

(Proved to be wrong by Bob!)


cos y e^z = cos x



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**Theorem.(Osserman and Mo – 1990).**

If the Gauss map of a complete minimal surface assumes five distinct values only a finite number of times, then the surface has finite total curvature.

# Minimal Surfaces in $\mathbb{E}^n$

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**Theorem. (Osserman – 1964).**

Suppose all the normals to $\Sigma$ (at all points) make an angle $\geq \alpha > 0$ with respect to some fixed direction. Fix $x \in \Sigma$ and let $d$ be the distance from $x$ to the boundary of $\Sigma$. Then the Gauss curvature of $\Sigma$ at $x$ satisfies

$$|K(x)| \leq \frac{16(n-1)}{d^2 \sin^4 \alpha}$$

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Suppose all the normals to $\Sigma$ (at all points) make an angle $\geq \alpha > 0$ with respect to some fixed direction. Fix $x \in \Sigma$ and let $d$ be the distance from $x$ to the boundary of $\Sigma$. Then the Gauss curvature of $\Sigma$ at $x$ satisfies

$$|K(x)| \leq \frac{16(n-1)}{d^2 \sin^4 \alpha}$$

**Corollary** If $\Sigma$ is complete and all the normals to $\Sigma$ make an angle $\geq \alpha > 0$ with respect to some fixed direction, then $\Sigma$ is a plane.

# Minimal Surfaces in $\mathbb{E}^n$ – Another View

Let $\Sigma \subset \mathbb{E}^n$ be a minimal surface. Choosing isothermal coordinates $z = x + iy$ gives a local conformal parameterization

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$\Phi$ is the Generalized Gauss Mapping.

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Let $\Sigma \subset \mathbb{E}^n$ be a complete minimal surface. Suppose the Gauss image $\Phi(\Sigma)$ misses a neighborhood of some hyperplane $\mathbb{P}^{n-2} \subset \mathbb{P}^{n-1}$.

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This was followed by a long collaboration of Hoffman and Osserman who obtained beautiful generalizations of many of the results I have discussed to more general surfaces in $\mathbb{E}^n$

# Graphs in $\mathbb{E}^n$

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with smooth boundary.

Consider the graph

$$\Gamma(f) \;=\; \{(x, f(x)) : x \in \Omega\} \;=\; \mathbb{E}^{p+q}$$

of a smooth function

$$f : \Omega \to \mathbb{E}^q.$$

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For functions $f$ which are only Lipschitz these equations can be replaced by the requirement that the the area of the graph be stationary.

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Given a continuous function

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**Theorem. (Allard – 1975)**

Boundary regularity holds for solutions to the Dirichlet Problem.

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**Theorem. (L and O – 1977)**

Regularity fails for solutions to the Dirichlet problem. There are Lipschitz solutions which are not $C^1$.

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What happens in between?

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**Theorem. (L and 0)**

The cone on the Hopf map

$$f(x) \ = \ \|x\| \varphi \left( \frac{x}{\|x\|} \right) \qquad for \|x\| \le 1$$

is the solution to the Dirichlet problem on the unit 4-disk $D$ for boundary values $\varphi$ on $\partial D = S^3$.

# Reflections on the Early Mathematical Life of Bob Osserman