POTENTIAL THEORY FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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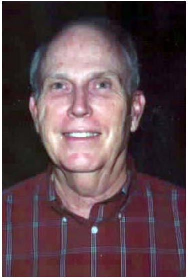
with Reese Harvey



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there is an associated “Potential Theory”

based on the functions which satisfy the condition:

$$f(D^2u) \geq 0$$

in a generalized sense.

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The Theory of Subharmonic Functions

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The upper semi-continuous functions which are "sub the harmonics":

$$u \;\leq\; h \quad \text{on} \quad \partial K \qquad \Rightarrow \qquad u \;\leq\; h \quad \text{on} \quad K$$

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3. **The Complex Monge-Ampère Equation:**

$$\det_{\mathbb{C}}\left\{(D^2u)_{\mathbb{C}}\right\} = 0 \quad \text{and} \quad (D^2u)_{\mathbb{C}} \geq 0.$$

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Poincaré, Oka, Grauert, Lelong, Hörmander, etc.

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$$F \;\equiv\; \mathcal{P}^{\mathbb{C}} \;\equiv\; \{A \in \mathrm{Sym}^2_{\mathbb{R}}(\mathbb{C}^n) : A_{\mathbb{C}} \geq 0\}.$$

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Obective: To extend this to a larger class of functions u,

called the ***F*-subharmonic functions** $F(X)$,

and to develop an accompanying potential theory for this class.

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must imply

$$D^2_{x_0}\varphi \in F.$$

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Note:

If φ satisfies the conditions above for u at a point x_0,
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Conclusion: We must require $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ to satisfy the condition

$$\mathbf{F} + \mathcal{P} \;\subset\; \mathbf{F}$$

where $\mathcal{P} \equiv \;\{A : A \geq 0\}$.

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A C^2-function on a domain $X \subset \mathbb{R}^n$ is ***F*-subharmonic** if

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We want to **extend these notions to upper semi-continuous functions.**

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- Note that
$$F \cap -\widetilde{F} \;=\; \partial F$$

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$$u \in C^2(X) \text{ is } F\text{-harmonic} \qquad \Longleftrightarrow \qquad D_x^2 u \in \partial F \quad \forall\, x \in X.$$

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The homogeneous real Monge-Ampère Equation

$D^2 u \geq 0$ **and** $\det(D^2 u) = 0$.

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Let $\Omega \subset\subset \mathbb{R}^n$ *be a domain whose boundary* $\partial\Omega$ *is strictly* F *and* $\widetilde{F}$*-convex.*

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Let $\Omega \subset\subset \mathbb{R}^n$ be a domain whose boundary $\partial\Omega$ is strictly F and $\widetilde{F}$-convex. Then for each $\varphi \in C(\partial\Omega)$, there exists a unique $u \in C(\overline{\Omega})$ such that

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(2) $u\big|_{\partial\Omega} = \varphi.$

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The subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a cone which is invariant under a subgroup

$$G \subset \mathrm{O}(n)$$

which acts transitively on the sphere

$$S^{n-1} \subset \mathbb{R}^n.$$

Eigenvalue Equations

If $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is $\mathrm{O}(n)$-invariant,

then it is completely determined by a condition

on the **eigenvalues** of $A \in \mathrm{Sym}^2(\mathbb{R}^n)$.

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Any such F has a **complex and quaternionic counterpart**

denoted $\qquad F^{\mathbb{C}}$ and $F^{\mathbb{H}}$

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These subequations are U(n) and Sp(n) invariant.

$$\mathbb{C}^n = (\mathbb{R}^{2n}, J)$$

The **complex hermitian symmetric part** of $A \in \mathrm{Sym}^2(\mathbb{R}^{2n})$ is

$$A_{\mathbb{C}} \;=\; \tfrac{1}{2}(A - JAJ).$$

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be the **ordered eigenvalues** of A.

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It also has **complex and quaternionic counterparts.**

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A Particular Instance

Valentino Tosatti and Ben Weinkove

made use of

$\mathcal{P}^{\mathbb{C}}_{n-1}$-subharmonic and $\mathcal{P}^{\mathbb{C}}_{n-1}$-harmonic functions

in their recent work on the Monge-Ampère Equations

and the Calabi-Yau problem on general complex manifolds.

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The Pucci equation: For $0 < \lambda < \Lambda$

$$\mathcal{P}_{\lambda,\Lambda} \ \equiv \ \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \lambda \mathrm{tr} A^+ + \Lambda \mathrm{tr} A^- \geq 0\}, \qquad p \ = \ \frac{\lambda}{\Lambda}(n-1) + 1.$$

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is an increasing radial F-subharmonic function iff $\psi(t)$ satisfies the one-variable subequation

$$\psi''(t) + \frac{(p-1)}{t}\psi'(t) \;\geq\; 0 \quad \text{and} \quad \psi'(t) \;\geq\; 0.$$

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where $C \geq 0$, $k \in \mathbb{R}$, and $K_p(t)$ is the p^{th} Riesz function defined on $0 < t < \infty$ by

$$K_p(t) \;=\; \begin{cases} t^{2-p} & \textit{if} \;\; 1 \leq p \; < 2 \\ \log t & \textit{if} \;\; p = 2 \\ -\frac{1}{t^{p-2}} & \textit{if} \;\; 2 < p < \infty. \end{cases}$$

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where $K = K_p$ *is the* p^{th} *Riesz function.*

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When $F = \mathcal{P}^{\mathbb{C}}$

$$u(z) \;=\; \log|f(z)| \quad \text{with } f \text{ holomorphic}$$

is plurisubharmonic.

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$$\Theta^{\Psi}(u,0) \;=\; \lim_{\substack{r,t\to 0\\ t>r>0}} \frac{\Psi(t)-\Psi(r)}{K(t)-K(r)}$$

exists and defines the **Ψ-density**, $0 \le \Theta^{\Psi} < \infty$, *of u at 0.*

$\Psi = M, S$ **or** V

Densities

In fact,

$$\Theta^{\Psi}(u,0) \;=\; \lim_{r\downarrow 0} \frac{\Psi(r)}{K(r)}.$$

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$$\Theta^{\Psi}(u,0) \;=\; \lim_{r\downarrow 0}\, \frac{\Psi(r)}{K(r)}.$$

(When $1 \le p < 2$, we must normalize so that $\Psi(0) = 0$.)

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Corollary. *For each $c > 0$ the set*

$$E_c \equiv \{x : \Theta^{\Psi}(u,x) \geq c\} \text{ is closed.}$$

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Question: Are there analogous results for other subequations?

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This result is essentially sharp.

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(3) There exists constants c, C*. so that for each* j*,*

$$\Theta_j K_p(|x - x_j|) + c \;\le\; H(x) \;\le\; \Theta_j K_p(|x - x_j|) + C$$

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Note: One has $u_r \in F(B_{R/r})$ and $B_{R/r}$ expands to $\mathbb{R}^n$ as $r \downarrow 0$.

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THEOREM. *If* $\mathbb{G}$ *has the transitivity property, then* **strong uniqueness holds** *for* $F(\mathbb{G})$.

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In the convex case uniqueness of tangents holds, which is classical.

In the complex case, uniqueness fails. This is due to **Kiselman.**

MANY QUESTIONS REMAIN OPEN.