Chapter I

Minimal Submanifolds

We begin with a study of the differential geometry of submanifolds of riemannian spaces. Our ultimate aim in this discussion is to derive and explain the formulas of first and second variation of the area integral. We have included a brief discussion of certain fundamental geometric concepts. A detailed treatment of these basics, given in much the same language, can be found in Helgason [1], Hicks [1] or the paper of Simons [1].

§1. Connections. Let M be an m-dimensional differentiable manifold, and denote by $\mathfrak{X}_M$ the space of smooth vector fields on M. (We assume everything to be class C^∞, however, in general C^2 is enough.) Let $E \longrightarrow M$ be a smooth vector bundle over M. E may be considered as a generalized product of M with the vector space $\mathbb{R}^k$, and thus the smooth sections $C^\infty(E)$ can be considered as generalized $\mathbb{R}^k$-valued functions. In this light it is natural to look for a way to differentiate the "functions" $C^\infty(E)$ with respect to vector fields on M. For the trivial bundle $M \times \mathbb{R}^k$, differentiation is canonical. In general, however, there are many equally acceptable rules for the differentiation of sections, and each such rule is called a connection.

Definition 1.1. A connection on E is a rule which assigns to each $X \in \mathfrak{X}_M$ a linear map $\nabla_X : C^\infty(E) \longrightarrow C^\infty(E)$ such that

$$\text{(i)}\quad \nabla_X(f\,\sigma) = (Xf\,)\sigma + f\nabla_X\sigma$$

$$\text{(ii)}\quad \nabla_{(f\;X+g\,Y)}\sigma = f\,\nabla_X\sigma + g\nabla_y\sigma$$

for all $\sigma \in C^\infty(E)$, $X, Y \in \mathfrak{X}_M$ and for all C^∞ functions f and g on M.

When $E = T(M)$, the tangent bundle of M, ∇ is called a <u>connection on</u> M.

We recall some elementary facts concerning connections.

A. For $p \in M$, $X \in \mathfrak{X}_M$ and $\sigma \in C^\infty(E)$, the value $(\nabla_X\sigma)_p$ depends only on X_p and the values of σ along any curve $\gamma(t)$ with $\gamma(0) = 0$ and $\frac{d\gamma}{dt}(0) = X_p$.

B. Given a connection ∇ on E, we obtain in a natural manner a connection on its dual bundle E^* by requiring that ∇ commute with contraction $E_p^* \otimes E_p \longrightarrow \mathbb{R}$ in each fibre. In particular, for $\sigma^* \in C^\infty(E^*)$, $\nabla_X\sigma^*$ is defined by the formula:

$$X(\sigma^*(\sigma)) = (\nabla_X\sigma^*)(\sigma) + \sigma^*(\nabla_X\sigma),$$

where $X \in \mathfrak{X}_M$ and $\sigma \in C^\infty(E)$.

C. Let E' and E'' be bundles with connections ∇' and ∇'' respectively. We can define a connection ∇ on $E' \oplus E''$ and $E' \otimes E'$ by setting

$$\nabla(\sigma'\oplus\sigma'') = \nabla'\sigma' \oplus \nabla''\sigma'' \text{ and}$$

$$\nabla(\sigma'\otimes\sigma'') = (\nabla'\sigma') \otimes \sigma'' + \sigma' \oplus (\nabla''\sigma'').$$

In particular, a connection ∇ can be defined on $\mathrm{Hom}(E', E'') \cong (E')^* \otimes E''$ by setting

$$(\nabla A)(\sigma) = \nabla''(A(\sigma')) - A(\nabla'\sigma').$$

Briefly, we can write

$$\nabla A = [\nabla, A].$$

We recall that the fundamental invariant of a connection is the <u>curvature</u> R, defined for $X, Y \in \mathfrak{X}_M$ as the map $R_{X,Y} : C^\infty(E) \longrightarrow C^\infty(E)$ given by

$$R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

R is a tensor field (i. e., it depends only on the values of its arguments at the point in question) which measures the lack of commutativity of second derivatives in the connection.

In the case $E = T(M)$ there is a second important tensor associated to a connection, the <u>torsion</u> T, defined for $X, Y \in \mathfrak{X}_M$ as the vector field

$$T_{X,Y} = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]}.$$

Suppose now that E is a smooth vector bundle over M equipped with a riemannian inner product; that is, we are given $g \in C^\infty(E^* \otimes E^*)$ where for each $p \in M$, g is a positive definite inner product on E_p. Then a connection ∇ on E is called <u>riemannian</u> if $\nabla g = 0$, i. e., if

$$X \cdot g(\sigma,\tau) = g(\nabla_X\sigma,\tau) + g(\sigma, \nabla_X\tau)$$

for all $X \in \mathfrak{X}_M$ and all $\sigma, \tau \in C^\infty(E)$.

Of course, when $T(M)$ has an inner product structure, M is called a riemannian manifold and we recall the following fundamental lemma.

Lemma 1. 2. There exists a unique torsion-zero riemannian connection, called the Levi-Civita connection, on any riemannian manifold.

From this point on, all manifolds will be riemannian and will be equipped with the Levi-Civita connection. Given a vector bundle E over M with connection $\tilde{\nabla}$ we can then define an invariant second derivative as follows. For $X, Y \in \mathfrak{X}_M$ and $\sigma \in C^\infty(E)$, we set

$$\tilde{\nabla}_{X,Y}\sigma = \tilde{\nabla}_X\tilde{\nabla}_Y\sigma - \tilde{\nabla}_{\nabla_X Y}\sigma .$$

This derivative depends only on the values of X and Y at the point in question, i. e., it is a tensor in these variables. Of course the arguments X and Y can not be freely interchanged, in fact,

$$\nabla_{X,Y} - \nabla_{Y,X} = R_{X,Y} \tag{1.1}$$

§2. The geometry of submanifolds. Let $\bar{M}$ be a riemannian $\bar{m}$-manifold with metric $\bar{g}(\cdot,\cdot)$ (which we shall also denote as $\langle \cdot, \cdot \rangle$) and connection $\bar{\nabla}$. Let $M \subset \bar{M}$ be riemannian submanifold which for convenience, and without loss of generality, we assume to be properly embedded in $\bar{M}$. Then there is an orthogonal splitting

$$T(\bar{M})\Big|_M = T(M) \oplus N(M)$$

where $N(M)$ is the normal bundle of M in $\bar{M}$; and $\bar{\nabla}$ induces natural riemannian connections ∇ in $T(M)$ and $N(M)$ by setting

$$\nabla_X Y = (\bar{\nabla}_X Y)^T$$

$$\nabla_X \nu = (\bar{\nabla}_X \nu)^N$$

for $X, Y \in \mathfrak{X}_M$ and $\nu \in C^\infty(N(M))$. Here $(\ )^T$ and $(\ )^N$ denote orthogonal projection on $T(M)$ and $N(M)$. Note that this definition uses Fact A above for $\bar{\nabla}$. Note also that ∇ on $T(M)$ is torsion-free and, thus, is the Levi-Civita connection on M.

We now have two connections $\bar{\nabla}$ and $\nabla \oplus \nabla$ on $T \oplus N$. Their difference is a tensor of fundamental importance in the geometry of M in $\bar{M}$. We split the difference tensor into tangent and normal components and write

$$B_{X,Y} = (\bar{\nabla}_X Y)^N$$

$$A^\nu(X) = (\bar{\nabla}_X \nu)^T$$

Then we have

<u>Lemma</u> 1.3.

(1) B <u>is symmetric</u>; i.e., $B_{X,Y} = B_{Y,X}$.

(2) <u>The two pieces are "adjoint"</u>:

$$\langle A^\nu(X), Y\rangle = - \langle \nu, B_{X,Y}\rangle .$$

Proof.

(1) $$(\bar{\nabla}_X Y)^N = (\bar{\nabla}_Y X + [X, Y])^N$$

$$= (\bar{\nabla}_Y X)^N .$$

(2) $$\langle A^{\nu} X, Y\rangle = \langle \bar{\nabla}_X \nu, Y\rangle$$

$$= X\langle \nu, Y\rangle - \langle \nu, \bar{\nabla}_X Y\rangle$$

$$= - \langle \nu, B_{X, Y}\rangle .$$

Definition 1.4. The symmetric, N(M)-valued bilinear form B defined above is called the second fundamental form of M in $\bar{M}$. The normal vector field

$$K = \text{trace}\,(B)$$

is called the mean curvature vector field of M in $\bar{M}$.

Note that K is an invariant of the pair $M \subset \bar{M}$; That is to say that an isometry of $\bar{M}$ which maps M onto M must preserve K.

We shall show in the next section that K can be interpreted essentially as (minus) the gradient of the area function on the space of immersions of M into $\bar{M}$. In view of this, we make the following

Definition 1. 5. M is called a minimal submanifold if and only if $K \equiv 0$.

§3. The first variational formula. Our purpose here is to interpret the mean curvature vector field K of $M \subset \bar{M}$ in terms of the behavior of the area of M under deformations.

Theorem 1.1. Let M be a compact submanifold of $\bar{M}$ with boundary ∂M. Suppose that E is a vector field on $\bar{M}$ such that $E|\partial M \equiv 0$, and let φ_t denote the flow generated by E in a neighborhood of M in $\bar{M}$. Then, setting $\mathcal{V}(t) = \text{volume}(\varphi_t(M))$, we have

$$\left.\frac{d\mathcal{V}}{dt}\right|_{t=0} = -\int_M \langle K, E\rangle \, dV .$$

Proof. We shall give a proof which works in a much more general situation. Let us consider φ_t as giving an immersion of M into $\bar{M}$, and let us denote by dV_t the volume element of the metric induced on M by φ_t. Then

$$\mathcal{V}(t) = \int_M dV_t,$$

and so

$$\frac{d^k\mathcal{V}}{dt^k} = \int_M \frac{d^k}{dt^k}(dV_t) . \tag{1.2}$$

The question then is how to express dV_t. Let us fix a point $p \in M$ and choose a basis $e_1, \ldots, e_m$ of $T_p(M)$ with the property that $e_1, \ldots, e_m$ are orthonormal at $t = 0$. Let $\omega_1, \ldots, \omega_m$ be the dual basis of 1-forms. Then the metric at time t at p has the form $ds_t^2 = \Sigma g_{ij}(t)\omega_i \otimes \omega_j$, where

$$g_{ij}(t) = \langle (\varphi_t)_* e_i, (\varphi_t)_* e_j \rangle .$$

It follows that

$$dV_t = \sqrt{\mathcal{G}(t)}\,\omega_1 \wedge \cdots \wedge \omega_m$$

where $\mathcal{G}(t) = \det((g_{ij}(t)))$.

This formula can be written more simply as follows. Recall that if V is a vector space with inner product $\langle \cdot , \cdot \rangle$, then there exists a natural inner product on $\Lambda^m V$ given on simple vectors by

$$\langle v_1 \wedge \cdots \wedge v_m, w_1 \wedge \cdots \wedge w_m \rangle = \det((\langle v_i, w_j \rangle)).$$

Thus, if we set $\xi = e_1 \wedge \cdots \wedge e_m$ ($\xi^* = \omega_1 \wedge \cdots \wedge \omega_m = dV_0$), then

(1.3) $$dV_t = \|(\varphi_t)_* \xi\| \, dV_0.$$

For the moment then, we can forget the manifold M and consider the action of the flow φ_t on $\Lambda^m T(\overline{M})$. Let $\xi = e_1 \wedge \cdots \wedge e_m \in \Lambda^m T_p(\overline{M})$ be any simple vector of unit length and consider

$$\mathcal{G}(t) = \|(\varphi_t)_* \xi\|^2 = (\varphi_t^* \bar{g})(\xi, \xi)$$

where $\bar{g}$ denotes the metric tensor extended to $\Lambda^m T(\overline{M})$. Then for each $k \geq 0$,

(1.4) $$\mathcal{G}^{(k)}(0) = (\mathcal{L}_E^k \bar{g})(\xi, \xi)$$

where $\mathcal{L}_E$ denotes Lie derivative with respect to E.

We now want to express these derivatives in terms of basic geometric objects. The first of these is the following. For $E \in \mathfrak{X}_M$, we define a tens

$\mathcal{A}^E \in \mathrm{Hom}(T(\overline{M}), T(\overline{M}))$ by

$$\mathcal{A}^E(X) = \overline{\nabla}_X E.$$

(See Fact A.) $\mathcal{A}^E$ then extends naturally as a derivation to the entire tensor algebra of $\overline{M}$. In particular, $\mathcal{A}^E$ extends to $\Lambda^m T(\overline{M})$ by defining

$$\mathcal{A}^E(v_1 \wedge \cdots \wedge v_m) = \sum_{j=1}^{m} v_1 \wedge \cdots \wedge \mathcal{A}^E(v_j) \wedge \cdots \wedge v_m.$$

We make some observations concerning $\mathcal{A}^E$.

1. $\mathcal{A}^E$ is antisymmetric if E is a Killing vector field.

2. $\mathcal{A}^E$ is symmetric if E is a gradient vector field (or more generally if the one-form $\omega(X) = \langle X, E \rangle$ is closed).

3. Since $\overline{\nabla}_E X - \overline{\nabla}_X E = [E, X]$, we have

$$\overline{\nabla}_E - \mathcal{A}^E = \mathcal{L}_E, \tag{1.5}$$

where this equation is valid on the entire tensor algebra of $\overline{M}$. (Note that all three terms in (1.5) extend as derivations. Of course,

$$\mathcal{L}_E = \overline{\nabla}_E \text{ and } \mathcal{A}^E = 0 \text{ on functions.)}$$

We are now ready to prove the theorem. By formula (1.5) and the fact that $\overline{\nabla} g = 0$, we have

$$\begin{aligned}
\mathcal{G}'(0) &= (\mathcal{L}_E \overline{g})(\xi, \xi) \\
&= (\overline{\nabla}_E - \mathcal{A}^E)(\overline{g})(\xi, \xi) \\
&= -(\mathcal{A}^E \overline{g})(\xi, \xi) \\
&= \overline{g}(\mathcal{A}^E \xi, \xi) + \overline{g}(\xi, \mathcal{A}^E \xi) \\
&= 2\langle \mathcal{A}^E(\xi), \xi \rangle.
\end{aligned}$$

This already establishes the following intermediate result.

Theorem 1.1'. Let ξ be the field of unit m-vectors on M (defined up to sign) such that at any $p \in M$, $\xi_p = e_1 \wedge \cdots \wedge e_m$ for some orthonormal basis $e_1, \ldots, e_m$ of $T_p(M)$. Then under the assumptions of Theorem 1,

$$\frac{d\mathcal{V}}{dt}(0) = \int_M \langle \mathcal{a}^E(\xi), \xi \rangle \, dV.$$

To complete the proof of Theorem 1.1 we observe that at any $p \in M$,

$$\begin{aligned}
\langle \mathcal{a}^E(\xi), \xi \rangle &= \Big\langle \sum_{j=1}^{m} e_1 \wedge \cdots \wedge \bar{\nabla}_{e_j} E \wedge \cdots \wedge e_m, e_1 \wedge \cdots \wedge e_m \Big\rangle \\
&= \sum_j \langle \bar{\nabla}_{e_j} E, e_j \rangle \\
&= \sum_j \langle \bar{\nabla}_{e_j} E^N, e_j \rangle + \sum_j \langle \bar{\nabla}_{e_j} E^T, e_j \rangle \\
&= \sum_j \big(\underbrace{e_j \langle E^N, e_j \rangle}_{0} - \langle E^N, \bar{\nabla}_{e_j} e_j \rangle\big) + \operatorname{div} E^T \\
&= -\langle E^N, K \rangle + \operatorname{div} E^T.
\end{aligned}$$

Since $\int_M \operatorname{div} E^T = 0$, the result follows from Theorem 1.1'.

Note that in Theorem 1.1 we need only suppose that $E^T | \partial M = 0$. Furthermore, the same proof goes through for noncompact manifolds M provided $E|M$ has compact support, and $\mathcal{V}$ is redefined as the volume of a compact neighborhood of supp(E) in M.

Observe that by Theorem 1, a given immersion $F : M \longrightarrow \overline{M}$ is minimal if ond only if F is a critical point of the volume function

$$\mathcal{V} : \mathrm{Imm}^{\infty}_{\partial M}(M, \overline{M}) \longrightarrow \mathbb{R}^+$$

in the space of immersions of M into $\overline{M}$, which are fixed on the boundary.

<u>Examples of minimal submanifolds.</u>

1. Let $F : M \longrightarrow \mathbb{R}^n$ be an immersion of an m-manifold into euclidean n-space. Then F is minimal if and only if $\nabla^2 F = 0$. (∇^2 is the laplacian of M defined at $p \in M$ as

$$\nabla^2 = \sum_{j=1}^{m} \nabla_{e_j, e_j}$$

where $e_1, \ldots, e_m$ are orthonormal.) This follows from the general fact that

$$\nabla^2 F = K$$

<u>Proof.</u> $\nabla^2 F = \nabla_{e_j, e_j} F = e_j e_j F - (\nabla_{e_j} e_j) F = e_j \cdot (F\, e_j) - F_*(\nabla_{e_j} e_j) =$
$(\overline{\nabla}_{F_* e_j} F_* e_j) - \nabla_{F_* e_j} F_* e_j - (\overline{\nabla}_{F_* e_j} F_* e_j)^N - K.$

In particular, F is a minimal immersion if and only if the coordinates of F are <u>harmonic</u> functions on M (in the induced metric). In the case $\dim M = 2$, a minimal surface can be defined as a harmonic, conformal immersion of a Riemann surface into $\mathbb{R}^n$.

2. If $F : M \longrightarrow \mathbb{C}^n$ is a holomorphic immersion of a complex manifold, then F is automatically minimal. This will be proven in greater generality in Chapter II.

3. Let $S^n = \{X \in \mathbb{R}^{n+1} : \|X\| = 1\}$. Then $F : M^m \longrightarrow S^n$ is minimal if and only if

$$\nabla^2 F = -mF.$$

A Theorem of Wu-Yi Hsiang [1] states that every compact homogeneous space can be minimally immersed into S^n with some invariant metric (n sufficiently large). Furthermore, every compact surface but $\mathbb{P}^2(\mathbb{R})$ can be minimally immersed into S^3 (cf. Lawson [1]).

§4. The second variational formula. We have seen that minimal immersions are critical points of the volume function. At critical points one usually considers the "Hessian" of second derivatives to determine the character of the critical point.

Let us fix M, $\overline{M}$ and E as in Theorem 1. We saw above (cf. formulas (1.2), (1.3) and (1.4)) that to compute $d^2\mathcal{V}/dt^2$ it is sufficient to calculate $\mathcal{J}''(0) = (\mathcal{L}_E\mathcal{L}_E\overline{g})(\xi, \xi)$. We want to express this in terms of the tensor $\mathfrak{a}^E$, the curvature of $\overline{M}$, etc. To do this we need the following.

Definition 1.7. For $E \in \mathfrak{X}_{\overline{M}}$ we define the tensor $\overline{\nabla}_{E,\cdot}E \in \mathrm{Hom}(T(\overline{M}), T(\overline{M}))$ by setting

$$\overline{\nabla}_{E,X}E = \overline{\nabla}_E\overline{\nabla}_X E - \overline{\nabla}_{\overline{\nabla}_E X}E$$

$$= [\overline{\nabla}_E, \mathfrak{a}^E](X)$$

$$= (\overline{\nabla}_E\,\mathfrak{a}^E)(X)$$

for $X \in \mathfrak{X}_{\overline{M}}$. As before, $\nabla_{E,\cdot}E$ extends to $\Lambda^m T(\overline{M})$ as a derivation.

Lemma 1.8.

$$\frac{1}{2}(\mathcal{L}_E\mathcal{L}_E\overline{g})(\xi, \xi) = \langle \mathfrak{a}^E\mathfrak{a}^E\xi, \xi\rangle + \|\mathfrak{a}^E\xi\|^2 + \langle \overline{\nabla}_{E,\xi}E, \xi\rangle$$

Proof. By (1.5) we have:

$$(\mathcal{L}_E \mathcal{L}_E \bar{g})(\xi, \xi)$$

$$= ((\bar{\nabla}_E - \mathcal{A}^E)(\bar{\nabla}_E - \mathcal{A}^E)\bar{g})(\xi, \xi)$$

$$= -((\bar{\nabla}_E - \mathcal{A}^E)\mathcal{A}^E \bar{g})(\xi, \xi)$$

$$= -(\bar{\nabla}_E \mathcal{A}^E \bar{g})(\xi, \xi) + (\mathcal{A}^E \mathcal{A}^E \bar{g})(\xi, \xi)$$

$$= -((\bar{\nabla}_E \mathcal{A}^E)\bar{g})(\xi, \xi) + (\mathcal{A}^E(\bar{\nabla}_E \bar{g}))(\xi, \xi)$$

$$+ 2\bar{g}(\mathcal{A}^E \mathcal{A}^E \xi, \xi) + 2\bar{g}(\mathcal{A}^E \xi, \mathcal{A}^E \xi)$$

$$= 2[\langle(\bar{\nabla}_E \mathcal{A}^E)\xi, \xi\rangle + \langle\mathcal{A}^E \mathcal{A}^E \xi, \xi\rangle + \langle\mathcal{A}^E \xi, \mathcal{A}^E \xi\rangle].$$

Q.E.D.

Putting together formulas (1.2), (1.3), (1.4) and Lemma 1.8 we have

Theorem 1.2. (The second variational formula). Under the assumptions of Theorem 1 we have

$$\left.\frac{d^2\mathcal{V}}{dt^2}\right|_{t=0} = \int_M \{-\langle\mathcal{A}^E \xi, \xi\rangle^2 + \langle\mathcal{A}^E \mathcal{A}^E \xi, \xi\rangle$$

$$+ \langle\mathcal{A}^E \xi, \mathcal{A}^E \xi\rangle + \langle\bar{\nabla}_{E, \xi} E, \xi\rangle\} \, dV$$

where ξ is the field of unit m-vectors on M representing the tangent planes of M (cf. Thm. 1.1').

Observations

1. If E is a Killing field then $\mathcal{A}^E$ is skew symmetric and

$$\bar{\bar{\nabla}}_{X,Y}E = \bar{R}_{X,E}Y$$

for all X, Y. It follows that $\bar{\nabla}_{E,\xi}E = 0$, $\langle \mathcal{A}^E\xi, \xi\rangle = 0$ and $\langle \mathcal{A}^E \mathcal{A}^E\xi, \xi\rangle = -\|\mathcal{A}^E\xi\|^2$. Thus $\frac{d\mathcal{V}}{dt}(0) = \frac{d^2\mathcal{V}}{dt^2}(0) = 0$ as expected, since E generates a 1-parameter group of isometries.

2. By the first variational formula it is sufficient to consider fields E such that $E|M$ is normal.

3. If M is minimal and E is normal, then

$$\langle \mathcal{A}^E\xi, \xi\rangle \equiv -\langle K, E\rangle \equiv 0$$

on M.

4. The term $\bar{\nabla}_{E,\xi}E$ above essentially involves curvature. Indeed, for $X \in T_p(\bar{M})$ we have

$$\begin{aligned}
\bar{\nabla}_{E,X}E &= \bar{\nabla}_{X,E}E + \bar{R}_{E,X}E \\
&= \bar{\nabla}_X\bar{\nabla}_E E - \bar{\nabla}_{\bar{\nabla}_X E}E + \bar{R}_{E,X}E \\
&= \mathcal{A}^{\mathcal{A}^E(E)}(X) - (\mathcal{A}^E)^2(X) + \bar{R}_{E,X}E.
\end{aligned}$$

Therefore,

(1.6)

$$\langle \bar{\nabla}_{E,\xi}E, \xi\rangle = \langle \mathcal{A}^{\mathcal{A}^E(E)}(\xi), \xi\rangle - \langle (\mathcal{A}^E)^2(\xi), \xi\rangle + \langle \bar{R}_{E,\xi}E, \xi\rangle,$$

and if M is minimal the integral of the first term on the right is zero by Theorem 1.1'. We point out that $(\mathfrak{A}^E)^2$, extended as a derivation, does not equal the product of the derivations $\mathfrak{A}^E \cdot \mathfrak{A}^E$. We shall always indicate the former by $(\mathfrak{A}^E)^2$ and the latter by $\mathfrak{A}^E \cdot \mathfrak{A}^E$.

Combining Theorem 1.2 and the above observations gives

Theorem 1.2'. If M is minimal and $E|M$ is normal, then

$$(1.7) \quad \left.\frac{d^2\mathcal{V}}{dt^2}\right|_{t=0} = \int_M \{\| \mathfrak{A}^E \xi \|^2 + \langle (\mathfrak{A}^E \mathfrak{A}^E - (\mathfrak{A}^E)^2)\xi, \xi \rangle + \langle \bar{R}_{E,\xi} E, \xi \rangle \} \, dV$$

We shall now rewrite this equation, along classical lines, in terms of an operator in the normal bundle. Recall that there exists a natural riemannian connection ∇ in the normal bundle of M given by $\nabla_X \nu = (\bar{\nabla}_X \nu)^N$. We now consider the Laplacian $\nabla^2 : C^\infty(N(M)) \longrightarrow C^\infty(N(M))$ of this connection, given at $p \in M$ by the formula

$$\nabla^2 = \sum_{j=1}^{m} \nabla_{e_j, e_j}$$

where $e_1, \ldots, e_m$ is an orthonormal basis of $T_p(M)$. Let us denote by $C_0^\infty(N(M))$ the compactly supported normal vector fields which vanish on the boundary of M. This space has a natural inner product given by

$$(\nu, \mu) = \int_M \langle \nu, \mu \rangle \, dV$$

for $\nu, \mu \in C_0^\infty(N(M))$.

<u>Lemma</u> 1.9. ∇^2 is a symmetric, negative semidefinite operator on $C_0^\infty(N(M))$.

<u>Proof</u>. A straightforward calculation shows that at any point $p \in M$,

$$(\langle \nabla^2 \nu, \mu \rangle + \langle \nabla \nu, \nabla \mu \rangle)\, dV = d^* \Omega$$

where Ω is the one-form with $\Omega(X) = \langle \nabla_X \nu, \mu \rangle$ and $\langle \nabla \nu, \nabla \mu \rangle = \Sigma \langle \nabla_{e_j} \nu, \nabla_{e_j} \mu \rangle$ for $e_1, \ldots, e_m$ orthonormal. It follows that

$$\int \langle \nabla^2 \nu, \mu \rangle \, dV = - \int \langle \nabla \nu, \nabla \mu \rangle \, dV.$$

Q. E. D.

We shall now express the integrand in (1.7) in terms of ∇^2. Let us fix a point $p \in M$ and choose $e_1, \ldots, e_m$ and $\nu_1, \ldots, \nu_m$ pointwise orthonormal, local tangent and normal vector fields respectively. Then setting $\xi = e_1 \wedge \cdots \wedge e_m$ we have

$$\text{(i)} \quad \mathcal{Q}^E \xi = \sum_{j=1}^m e_1 \wedge \cdots \wedge \bar{\nabla}_{e_j} E \wedge \cdots \wedge e_m = \sum_j \sum_i e_1 \wedge \cdots \wedge \langle \bar{\nabla}_{e_j} E, e_i \rangle e_i \wedge \cdots \wedge e_m$$

$$+ \sum_j \sum_k e_1 \wedge \cdots \wedge \langle \bar{\nabla}_{e_j} E, \nu_k \rangle \nu_k \wedge \cdots \wedge e_m = \sum_j \cancelto{0}{\langle \bar{\nabla}_{e_j} E, e_j \rangle} \cdot \xi$$

$$+ \sum_j \sum_k \langle \bar{\nabla}_{e_j} E, \nu_k \rangle e_1 \wedge \cdots \wedge \nu_k \wedge \cdots \wedge e_m \,.$$

Therefore,

$$\| \mathcal{Q}^E \xi \|^2 = \sum_j \sum_k \langle \bar{\nabla}_{e_j} E, \nu_k \rangle^2 = \sum_j \| (\bar{\nabla}_{e_j} E)^N \|^2$$

$$= \| \nabla E \|^2 \equiv - \langle \nabla^2 E, E \rangle \text{ mod (terms which integrate to zero)}$$

(ii) $$(\mathcal{A}^E\mathcal{A}^E-(\mathcal{A}^E)^2)\xi = \sum_{i\neq j} e_1\wedge\cdots\wedge\bar{\nabla}_{e_i}E\wedge\cdots\wedge\bar{\nabla}_{e_j}E\wedge\cdots\wedge e_m.$$

Therefore,

$$\langle(\mathcal{A}^E\mathcal{A}^E-(\mathcal{A}^E)^2)\xi,\xi\rangle = \sum_{i,j=1}^{m}\left(\underbrace{\langle\bar{\nabla}_{e_i}E, e_i\rangle}_{0}\underbrace{\langle\bar{\nabla}_{e_j}E, e_j\rangle}_{0} - \langle\bar{\nabla}_{e_i}E, e_j\rangle\langle\bar{\nabla}_{e_j}E, e_i\rangle\right)$$

$$= -\sum_{i,j=1}^{m}\langle E, B_{e_i,e_j}\rangle^2.$$

Consider the transpose ${}^tB : N_p(M)\longrightarrow T_p(M)\otimes T_p(M)$ of $B : T_p(M)\otimes T_p(M)\longrightarrow N_p(M)$. We get $\mathcal{B} = B\circ{}^tB : N_p(M)\longrightarrow N_p(M)$ with the following property:

$$\langle\mathcal{B}(E), E\rangle = \langle{}^tB(E), {}^tB(E)\rangle = \sum_{i,j=1}^{m}\langle{}^tB(E), e_i\otimes e_j\rangle^2 = \sum_{i,j}\langle E, B_{e_i,e_j}\rangle^2.$$

(iii) $$\langle\bar{R}_{E,\xi}E,\xi\rangle = \sum_{i=1}^{m}\langle\bar{R}_{E,e_i}E, e_i\rangle = \sum\langle\bar{R}_{e_i,E}e_i, E\rangle \overset{\text{def}}{=} \langle\bar{R}(E), E\rangle.$$

We now arrive at the conclusion.

Theorem 1.2''. If M is minimal and $E|M \in C_0(N(M))$, then

(1.8) $$\left.\frac{d^2A}{dt^2}\right|_{t=0} = \int_M \langle -\nabla^2E-\mathcal{B}(E)+\bar{R}(E), E\rangle\, dV.$$

Suppose now that M is a compact submanifold with boundary, and consider the operator $\mathcal{L}$ on $C_0^\infty(N(M))$ given by

$$\mathcal{L} = -\nabla^2 - \mathcal{B}+\bar{R}.$$

We define a bilinear form $II(\cdot,\cdot)$, called the <u>index form</u>, on $C_0^\infty(N(M))$ by

$$II(\nu,\mu) = \int_M \langle \mathcal{L}\nu, \mu \rangle \, dV.$$

$\mathcal{L}$ is a symmetric, strongly elliptic operator. Therefore $II(\cdot,\cdot)$ is a symmetric bilinear form which can be diagonalized on $C_0^\infty(N(M))$ with finite dimensional eigenspaces E_{λ_i} and eigenvalues

$$\lambda_1 < \lambda_2 < \lambda_3 < \ldots \longrightarrow \infty.$$

In analogy with standard Morse theory we can then define

$$\text{Index}\,(M) = \dim\Big(\bigoplus_{\lambda<0} E_\lambda\Big)$$

$$\text{Nullity}\,(M) = \dim(E_0)$$

Note that a vector field ν lies in E_0 if and only if $\mathcal{L}(\nu) \equiv 0$. Such a field is called a <u>Jacobi field</u>.

Of course, in the case of geodesics (dimM=1), formula (1.8) reduces to the well known:

$$\text{(1.8')} \qquad II(\nu,\mu) = \int_\gamma \Big\langle -\frac{D^2\nu}{ds^2} + \bar{R}_{\dot\gamma\,\nu}\dot\gamma, \cdot \Big\rangle \, ds$$

where s = arc length, $\dot\gamma$ is the velocity vector field of the geodesic γ, and D/ds = covariant differentiation along γ. (Note, $\mathcal{B} \equiv 0$.)

Formula (1.8'), and its form involving boundary terms for $\nu \not\equiv 0$ on ∂M, together with the general theory above, have been used by Morse, Synge, Bott and others to do some of the most fundamental work in geometry. (See Milnor [1], for example.)

One of the basic theorems in the Morse Theory of geodesics has been generalized by Smale and Simons to general minimal submanifolds as follows.

Let $M \subset \overline{M}$ be a compact minimal submanifold with boundary $\partial M \neq \phi$, and consider a smooth contraction of M, that is, a smooth map $F : M \times \mathbb{R}^{+} \longrightarrow M$ such that:

(i) $f_t = F(\cdot, t) : M \longrightarrow M$ is a diffeomorphism of M onto an open subset of M for all $t \geqq 0$; and $f_0 = \text{identity}$.

(ii) $f_t(M) \leqq f_s(M)$ if $t \geqq s$.

Given $\varepsilon > 0$, F is said to be of ε-type if $\mathcal{V}(f_t(M)) < \varepsilon$ for all t sufficiently large.

Given such a contraction we define II_t to be the index form of the minimal submanifold $f_t(M)$, and set $N_t = \text{Nullity}(f_t(M))$. It follows from the general theory of the laplacian that there exists an $\varepsilon_0 > 0$ (depending on M) such that $\mathcal{V}(f_t(M)) < \varepsilon_0$ implies $II_t > 0$.

Theorem 1.3. (Morse, Smale [1], Simons [1].) *For any contraction of ε_0-type we have*

$$\text{Index}(M) = \sum_{t>0} N_t .$$

In particular, there are a finite number of $t_i > 0$ such that $N_{t_i} > 0$. The boundaries $\partial f_{t_i}(M)$ are called conjugate boundaries.

Unfortunately for $\dim M > 1$ it has not yet been possible to use this theorem to study the topology of $\mathrm{Imm}^{\infty}_{\partial M}(M, \overline{M})$. Our purpose in these notes is to instead generalize the variational techniques due to Synge.

We leave as an interesting exercise the computation of the index and nullity of the totally geodesic subspheres of S^n.