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MORSE THEORY AND STOKES' THEOREM

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Abstract

We present a new, intrinsic approach to Morse Theory which has interesting applications in geometry. We show that a Morse function f on a manifold determines a submanifold T of the product $X \times X$, and that (in the sense that Stokes theorem is valid) T has boundary consisting of the diagonal $\Delta \subset X \times X$ and a sum

$$P = \sum_{p \in Cr(f)} U_p \times S_p$$

where S_p and U_p are the stable and unstable manifolds at the critical point p. In the language of currents,

$$\partial T = \Delta - P. \text{(Stokes Theorem)}$$

This current (or kernel) equation on $X \times X$ is equivalent to an operator equation

$$d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P}, \text{((Chain Homotopy))}$$

where $\mathbf{P}$ is a chain map onto the finite complex of currents $\mathcal{S}_f$ spanned by (integration over) the stable manifolds of f. The operator $\mathbf{P}$ can be defd on an exterior form α by

$$\mathbf{P}(\alpha) = \lim_{t \to \infty} \varphi_t^* \alpha$$

where φ_t is a gradient flow for f. The de Rham differential in the complex $\mathcal{S}_f$ is easily computed in terms of the flow lines. The chain homotopy equation also holds on certain integral chain complexes. Poincaré duality over $\mathbb{Z}$ follows from time-reversal in our operator equations. The method has many generalizations and applications. Residue theorems are established for functions with critical manifolds of higher dimension. The methods apply immediately to equivariant cohomology. Cup product formulas and a Lefschetz-type theorem are proved for the Thom-Smale Complex. Other applications include a new proof of the Carrell-Lieberman Theorem and a proof of a local version of the MacPherson Formula for characteristic classes and bundle maps.

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0. Introduction

In this paper we present an approach to Morse Theory which is stronger than the classical theory and has some interesting applications. It leads to formulas relating characteristic forms and singularities, and it unifies a body of results on holomorphic actions. It applies directly to the equivariant case. It also has the virtue of fitting neatly into the modern theory of invariants arising from topological quantum field theory.

This work resulted from addressing the following.

Question. Consider a flow $\varphi_t : X \to X$ generated by a smooth vector field on a compact manifold X. Under what cicumstances does the limit

$$\alpha_\infty \equiv \lim_{t\to\infty} \varphi_t^* \alpha$$

exist for a given smooth differential form α on X?

We do not demand that α_∞ be smooth. Even so, one would expect the answer to be "rarely, if ever". However, we shall prove that for generic gradient dynamical systems, this limit does exist and has a beautiful, simple structure.

In fact, we shall show that setting

$$\mathbf{P}(\alpha) \stackrel{\text{def}}{=} \lim_{t\to\infty} \varphi^* \alpha \tag{0.1}$$

defs a continuous operator of degree 0

$$\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$$

from smooth forms to generalized forms, i.e., currents. This operator is chain homotopic to the inclusion $\mathbf{I} : \mathcal{E}^*(X) \hookrightarrow \mathcal{D}'^*(X)$, that is, there exists a continuous operator

$$\mathbf{T} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$$

of degree -1 such that

$$d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P}. \tag{0.2}$$

By de Rham [12], $\mathbf{I}$ induces an isomorphism in cohomology. Hence so does $\mathbf{P}$.

The existence of $\mathbf{P}$ and $\mathbf{T}$ satisfying (0.1) and (0.2) is established for any flow of finite volume. This concept, which is central to our paper, is introduced in §2. A flow φ_t is said to have **finite volume** if the graph of the relation $x \overset{o}{\to\prec} y$, defined by the forward motion of the flow, has finite $(n+1)$-dimensional volume in $X \times X$, where $n = \dim(X)$. Any flow whose space-time graph

$$\mathcal{T}_\varphi \equiv \{(t, \varphi_{\frac{1}{t}}(x), x) \ : \ 0 < t \leq 1 \ \text{ and } \ x \in X\} \subset \mathbf{R} \times X \times X$$

is of finite volume has this property.

Consider now a Morse function $f : X \to \mathbf{R}$ with (finite) critical set $Cr(f)$. Suppose there is a riemannian metric on X for which the gradient flow φ_t of f has the following properties:

1. φ_t is of finite volume.
2. The stable and unstable manifolds, S_p, U_p for $p \in Cr(f)$, are of finite volume in X.
3. $p \prec q \ \Rightarrow \ \lambda_p < \lambda_q$ for all $p, q \in Cr(f)$, where λ_p denotes the index of p and where $\prec$ is the closure of the relation $\overset{o}{\rightarrow}\!\prec$.

Note that $p \prec q$ means there is a piecewise flow line connecting p in forward time to q.

We shall prove in §14that such metrics always exist. In fact they are dense in the set of all metrics which are canonically flat in some neighborhood of $Cr(f)$, and conjecturally they are dense in all metrics.

Under the hypotheses (1)–(3) the operator $\mathbf{P}$ is shown to have the following simple form:

$$\mathbf{P}(\alpha) = \sum_{p \in Cr(f)} r_p(\alpha)[S_p] \tag{0.3}$$

for all $\alpha \in \mathcal{E}^*(X)$, where

$$r_p(\alpha) = \begin{cases} \int_{U_p} \alpha & \text{if } \deg \alpha = \lambda_p \\ 0 & \text{otherwise} \end{cases}$$

and where $[S_p]$ denotes the current defined by integration over S_p. Note that the image of $\mathbf{P}$ is the finite dimensional vector subspace

$$\mathcal{S}_f = \operatorname{span}_{\mathbf{R}} \Big\{ [S_p] \Big\}_{p \in Cr(f)}.$$

It follows from (0.2) that $\mathcal{S}_f$ is d-invariant, i.e., that $(\mathcal{S}_f, d)$ is a complex, and furthermore that the inclusion $\mathcal{S}_f \subset \mathcal{D}'^*(X)$ induces an isomorphism

$$H^*(\mathcal{S}_f) \cong H^*_{\text{de Rham}}(X)$$

This immediately yields the classical strong Morse inequalities.

The exterior derivative restricted to $\mathcal{S}_f$ has the form

$$d[S_p] \;=\; \sum_{q\in Cr(f)} n_{pq}[S_q].$$

The constants n_{pq} are integers which are non-zero only when $\lambda_q = \lambda_p - 1$ and are computed, in the Morse-Smale case, by counting flow lines from p to q (cf. §4). This follows directly from Stokes' Theorem. One concludes that

$$\mathcal{S}_f^{\mathbf{Z}} \;=\; \text{span}_{\mathbf{Z}}\Big\{[S_p]\Big\}_{p\in Cr(f)}$$

is a finite-rank subcomplex of the integral currents $\mathcal{I}_*(X)$ whose inclusion induces an isomorphism

$$H^*\left(\mathcal{S}_f^{\mathbf{Z}}\right) \;\cong\; H^*(X;\,\mathbf{Z})$$

Poincaré duality (over $\mathbf{Z}$) is now directly deduced from time-reversal in the flow (§5).

The analogous relative theorems for a Morse exhaustion function on a non-compact manifold, including Alexander-Lefschetz duality, are proved in §7.

Our method of proof involves converting the operator equation (0.2) to a kernel equation

$$\partial T \;=\; [\Delta] - P \tag{0.4}$$

on $X\times X$, where Δ denotes the diagonal. There is a general correspondence between operators $\mathbf{K} : \mathcal{E}^*(X) \to \mathcal{D}'^*(X)$ of degree ℓ and currents K of dimension $n-\ell$ on $X\times X$ ([21], See Appendix A.) Under this transformation: $\mathbf{I}$ corresponds to $[\Delta]$, the pull-back of forms by φ_t corresponds to the graph of φ_t, and the chain homotopy $d\circ\mathbf{K}+\mathbf{K}\circ d$ corresponds to the current boundary ∂K. Thus equation (0.4) carries back directly to equation (0.2).

The current T in (0.4) is simply defined by integration over the graph of the relation $\overset{o}{\to}\!\prec$. If the space-time graph $\mathcal{T}_\varphi$ has finite volume, then $T = \text{pr}_*\mathcal{T}_\varphi$ where $\text{pr} : \mathbf{R}\times X\times X \to X\times X$ is the projection. Our finite-volume assumption on T implies that T is a rectifiable current and therefore that its current boundary is flat in the sense of Federer [14]. Applying the Federer Support Lemma, we conclude that

$$P \;=\; \sum_{p\in Cr(f)} [U_p]\times[S_p]. \tag{0.5}$$

Thus, T provides a homology between the diagonal and the sum of Künneth currents in $X \times X$ given by products of the unstable and stable manifolds of the flow. In other words, our Morse function gives a canonical chain approximation to the diagonal *together with an explicit homology.* This "transgression current" $T = T_f$ plays a role in defining more subtle invariants of manifolds and knots.

The entire procedure outlined above can be applied to functions with non-degenerate critical manifolds, i.e., functions of Bott-type. Suppose f is such a function and F_j, $j = 1, ..., \nu$ are the connected components of the critical set. In this case the kernel P in (0.4) is written as a sum of fibre-products

$$P \ = \ \sum_{j=1}^{\nu} [U_j \times_{F_j} S_j]. \tag{0.6}$$

of the stable and unstable manifolds of the flow. Here the subcomplex *image*($\mathbf{P}$) is not finite-dimensional. However, for each smooth form α we have

$$\mathbf{P}(\alpha) \ = \ \sum_{j=1}^{\nu} \mathrm{Res}_j(\alpha)[S_j]$$

where $\mathrm{Res}_j(\alpha)$ is a smooth, integrable *residue form* on the manifold S_j computed directly in terms of α.

This approach works nicely in the case of holomorphic $\mathbf{C}^*$-actions with fixed-points on Kähler manifolds. Here there is an underlying function of Bott-type. One finds a complex analogue of the current T and replaces equation (0.4) with a $\partial\bar{\partial}$-equation. General results of Sommese imply that T and all of the stable and unstable manifolds of the flow are subvarieties of finite-volume. One retrieves, in particular, classical results of Bialynicki-Birula [5] and of Carrell-Lieberman-Sommese [9], [10]. The approach also fits directly into MacPherson's Grassmann graph construction and Gillet-Soulé's construction of transgression classes appearing in the refined Riemann-Roch Theorem [17].

The arguments apply literally without change to the case of equivariant cohomology. It yields rapid calculations in certain cases and has been used by J. Latschev to derive a spectral sequence associated to functions in the equivariant case.

The method can be applied to derive an equation of forms and currents which relates the singularities of a smooth bundle map $A : E \to F$ to characteristic forms of E and F. At the level of cohomology this retrieves a formula of MacPherson [25], [26]. This work, which is discussed briefly in §9, began in [18] and then inspired the Morse Theory presented here.

Much has been written about assigning topological invariants of manifolds and knots to "Feynman graphs". Primary invariants of this type, such as those discussed in [3] and [4], can be constructed using operators **P**. The invariants of Kontsevich and Vasseliev (cf. [24], [8]) involve the currents **T**.

It should be remarked that while the Morse Theory due to Ed Witten [34] involves the de Rham complex, it is distinctly different from the approach presented here. Witten considers the conjugates d_t of exterior differentiation d by the function e^{-tf} for $t \geq 0$ and examines the asymptotics of the associated Hodge laplacians. Elliptic operators do not enter the story in our approach. Moreover, a crucial simplifying component of our approach (namely, the calculus of [21] for the operator d) is not available for other operators such as the Laplacian or d_t. It would be interesting to find a more direct connection between the two theories.

The idea of using the stable manifolds of a generic gradient flow to give a cell structure to a manifold goes back to R. Thom [33] (See also [30] and [29].) F. Laudenbach was the first to consider the stable currents as de Rham currents [23]. He studied Morse-Smale flows and computed the boundary operator in the Thom-Smale complex by using Stokes' Theorem as we do here.

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1. Finite volume flows

Let X be a compact smooth manifold of dimension n, and let $\varphi_t : X \to X$ be the flow generated by a smooth vector field V on X. Consider the operator $\mathbf{P}_t : \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ on the space of smooth differential forms which is given by pull-back

$$\mathbf{P}_t(\alpha) \equiv \varphi_t^*(\alpha).$$

We will exhibit a chain homotopy operator $\mathbf{T}_t : \mathcal{E}^k(X) \to \mathcal{E}^{k-1}(X)$ satisfying

$$\textbf{(A)} \qquad d \circ \mathbf{T}_t + \mathbf{T}_t \circ d = \mathbf{I} - \mathbf{P}_t,$$

and show that under a "finite volume" condition it is possible to take the limit as $t \to \infty$. We thereby obtain operators

$$\textbf{(B)} \qquad \mathbf{T} = \lim_{t\to\infty} \mathbf{T}_t \quad \text{and} \quad \mathbf{P} = \lim_{t\to\infty} \mathbf{P}_t$$

satisfying:

$$\textbf{(C)} \qquad d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P}.$$

Now operator equations such as (**A**) and (**C**) are difficult to solve directly, but as we shall see in §3, they can be converted into current equations on the product manifold $X \times X$ which are much more tractable. For example, equation **(A)** becomes the current equation

$$\text{(A)} \qquad \partial T_t = [\Delta] - P_t \qquad \text{on } X \times X$$

where $\Delta \subset X \times X$ is the diagonal and

$$P_t = [\text{graph}(\varphi_t)] = \{(\varphi_t(x), x) \,:\, x \in X\}$$

is the (reverse) graph of the diffeomorphism φ_t. The remaining conditions are that

$$(B) \qquad T = \lim_{t\to\infty} T_t \quad \text{and} \qquad P \equiv \lim_{t\to\infty} P_t$$

exist and satisfy

$$(C) \qquad \partial T = [\Delta] - P \qquad \text{on } X \times X.$$

Here T_t, T, and P are currents on $X \times X$ yet to be determined. Details of the above correspondence are discussed in §3. The remainder of this section is devoted to solving the second set of equations (A), (B), (C).

Equation (A) is particularly easy to solve. Consider the family of compact manifolds with boundary:

$$(1.1) \qquad \mathcal{T}_t \equiv \{(s, \varphi_s(x), x) \,:\, 0 \le s \le t \text{ and } x \in X\}$$

contained in $\mathbf{R} \times X \times X$. Obviously, $\mathcal{T}_t$ has the two boundary components $\{0\} \times \Delta$ and $\{t\} \times P_t$. Assume X is oriented (this condition will be dropped later) and orient $\mathcal{T}_t$ so that

$$(1.2) \qquad \partial \mathcal{T}_t = \{0\} \times \Delta - \{t\} \times P_t.$$

Let $\text{pr} : \mathbf{R} \times X \times X \to X \times X$ denote projection and set

$$(1.3) \qquad T_t \equiv (\text{pr})_*(\mathcal{T}_t).$$

Since ∂ commutes with $(\text{pr})_*$, the push-forward of (1.2) by pr_* gives equation (A).

The current T_t can be equivalently defined by

$$(1.3') \qquad T_t \;=\; \Phi_*([0,t]\times X)$$

where $\Phi : \mathbf{R}\times X \to X\times X$ is the smooth mapping given by $\Phi(s,x) = (\varphi_s(x), x)$. This mapping is an immersion exactly on the subset $\mathbf{R}\times(X - Z(V))$ where $Z(V) = \{x\in X \,:\, V(x)=0\}$. Thus if we fix a riemannian metric g on X, then $\Phi^*(g\times g)$ is a symmetric positive semi-definite tensor whose associated volume is >0 exactly on the subset $\mathbf{R}\times(X-Z(V))$.

This brings us to one of the central concepts of the paper.

Definition 1.1. A flow φ_t on X is called a **finite volume flow** if $\mathbf{R}^+\times(X-Z(V))$ has finite volume with respect to the metric induced by the immersion Φ. (This concept is independent of the choice of riemannian metric on X.)

Theorem 1.2. *Let φ_t be a finite volume flow on a compact manifold X. Then both the limits*

$$(B) \qquad P \;\equiv\; \lim_{t\to\infty}[\text{graph}\,\varphi_t] \qquad \textit{and} \qquad T \;\equiv\; \lim_{t\to\infty} T_t$$

exist as currents, and by taking the boundary of T we obtain the equation of currents

$$(C) \qquad \partial T = [\Delta] - P \qquad \textit{on} \;\; X\times X.$$

relating P to the diagonal Δ in $X\times X$.

Proof. Since φ_t is a finite-volume flow, the current $T\equiv \Phi_*((0,\infty)\times X)$ is the limit in the mass norm of the currents $T_t = \Phi_*((0,t)\times X)$ as $t\to\infty$. The continuity of the boundary operator and equation (A) imply the existence of $\lim_{t\to\infty} P_t$ and also establish equation (C). q.e.d.

Remark 1.3. Since $(\varphi_t(x), x) = (y, \varphi_{-t}(y))$ if $y=\varphi_t(x)$, it follows that

$$T^* = \Phi_*((-\infty,0)\times X)$$

is also a well-defined current for a finite-volume flow. It corresponds to the push-forward of T under the flip $(y,x)\mapsto(x,y)$ on $X\times X$.

Remark 1.4. The immersion $\Phi : \mathbf{R}\times(X-Z(V))\to X\times X$ is an embedding outside the subset $\mathbf{R}\times Per(V)$ where

$$Per(V) = \{x\in X \,:\, \varphi_t(x) = x \;\; \text{for some } t>0\,\}$$

are the non-trivial periodic points of the flow. Thus, if $Per(V)$ has measure zero, then T_t is given by integration over the embedded finite-volume submanifold $\Phi(R_t)$, where $R_t = (0,t) \times (X - Z(V) \cup Per(V))$. If furthermore the flow has finite volume, then T is given by integration over the embedded, finite-volume submanifold $\Phi(R_\infty)$.

There is evidence that any flow with periodic points cannot have finite volume. Now a gradient flow never has periodic points, and such flows are generically of finite volume (§14.). However, finite-volume flows are much more general than gradient flows. For a first example, note that any flow with fixed points on S^1 has finite volume.

Remark 1.5. If we define a relation on $X \times X$ by setting $x \overset{o}{\to} \prec y$ if $y = \varphi_t(x)$ for some $0 \le t < \infty$, then T is just the (reversed) graph of this relation. This relation is always transitive and reflexive, and it is antisymmetric if and only if φ_t has no periodic orbits (i.e., $\overset{o}{\to} \prec$ is a partial ordering precisely when φ_t has no periodic orbits).

Remark 1.6. A standard method for showing that a given flow is finite volume can be outlined as follows. Pick a coordinate change $t \mapsto \rho$ which sends $+\infty$ to 0 and $[t_0, \infty]$ to $[0, \rho_0]$. Then show that

(1.4)
$$\mathcal{T} \;\equiv\; \{(\rho, \varphi_{t(\rho)}(x), x) \;:\; 0 < \rho < \rho_0\} \;\text{ has finite volume in }\; \mathbf{R} \times X \times X.$$

Pushing forward to $X \times X$ then yields the current T with finite mass.

Perhaps the most natural such coordinate change is $r = 1/t$. Another natural choice (if the flow is considered multiplicatively) is $s = e^{-t}$. Of course finite volume in the r coordinate insures finite volume in the s coordinate since $r \mapsto s = e^{-1/r}$ is a C^∞-map.

Many interesting flows can be seen to be finite volume as follows.

Proposition 1.7. *If X is analytic and $\mathcal{T} \subset \mathbf{R} \times X \times X$ is contained in a real analytic subvariety of dimension $n+1$, then φ_t is a finite volume flow.*

Proof. The manifold points of a real analytic subvariety have (locally) finite volume. q.e.d.

A flow need not be a gradient flow to be a finite volume flow.

Example 1.8. (The standard degenerate flow on S^n) Consider the translational flow $\varphi_t(y) = y + tu$ on $\mathbf{R}^n$ where $u \in \mathbf{R}^n$ is a unit vector. We can identify $\mathbf{R}^n$ with $S^n - \{\infty\}$ so that φ_t extends to S^n as a finite volume flow. To do this choose coordinates $x = y/|y|^2$ on $\mathbf{R}^n \cong S^n - \{0\}$. Then

$$\varphi_t(x) \;=\; \frac{x + t|x|^2 u}{|u + tx|^2} \tag{1.5}$$

(The vector field $V = \dot{\to}\ \varphi_t\big|_{t=0}$ is given by $V(y) = u$ on $\mathbf{R}^n = S^n - \{\infty\}$, and by $V(x) = |x|^2 u - 2\langle x, u\rangle x$ on $\mathbf{R}^n = S^n - \{0\}$.)

The flow φ_t is finite volume flow on S^n. To see this let $r = 1/t$ and note that

$$\mathcal{T} = \{(r, z, x)\ :\ z = \varphi_{1/r}(x),\ 0 < r < \infty \ \text{ and }\ x \in \mathbf{R}^n = S^n - \{0\}\}$$

is defined by

$$z|ru + x|^2 = r(rx + |x|^2 u)$$

so that Proposition 1.7 is applicable. Note that ∞ $(x = 0)$ is the only zero of V. Although V is not a gradient vector field, it is the limit of gradient vector fields.

Our next problem is to explicitly compute the current P and its associated operator $\mathbf{P}$ under additional assumptions on the flow. We shall show that when V is a "good" gradient vector field for a Morse function, the operator $\mathbf{P}$ is projection onto the finite complex of currents spanned by the stable manifolds of the flow. Furthermore, for any given Morse function the "good" gradients are generic (cf. §14).

2. Morse-Stokes gradients; axioms

Let $f \in C^\infty(X)$ be a Morse function on a compact n-manifold X, and let $\mathrm{Cr}(f)$ denote the (finite) set of critical points of f. Recall that f is a **Morse function** if its Hessian at each critical point is non-degenerate. The standard Morse Lemma asserts that in a neighborhood of each $p \in \mathrm{Cr}(f)$ of index λ, there exist **canonical local coordinates** $(u_1, ..., u_\lambda, v_1, ..., v_{n-\lambda})$ for $|u| < r$, $|v| < r$ with $(u(p), v(p)) = (0, 0)$ such that

$$f(u, v) = f(p) - |u|^2 + |v|^2. \tag{2.1}$$

Fix a riemannian metric on X and let φ_t denote the flow associated to ∇f. We assume our metric has the form $|du|^2 + |dv|^2$ in some canonical coordinate system (u, v) about each $p \in \mathrm{Cr}(f)$. Therefore, in these coordinates the gradient flow is given by

$$\varphi_t(u, v) = (e^{-t}u, e^t v)$$

Metrics with this property will be called **canonically flat near** $\mathrm{Cr}(f)$.

Now to each $p \in \mathrm{Cr}(f)$ are associated the **stable** and **unstable manifolds** of the flow, defined respectively by

(2.2)

$$S_p = \{x \in X : \lim_{t\to\infty} \varphi_t(x) = p\} \qquad \text{and} \qquad U_p = \{x \in X : \lim_{t\to-\infty} \varphi_t(x) = p\}.$$

For coordinates (u, v) at p, chosen as above, we consider the disks

$$S_p(\epsilon) = \{(u,0) : |u| < \epsilon\} \qquad \text{and} \qquad U_p(\epsilon) = \{(0,v) : |v| < \epsilon\}$$

and observe that

$$S_p = \bigcup_{-\infty<t<0} \varphi_t\left(S_p(\epsilon)\right) \qquad \text{and} \qquad U_p = \bigcup_{0<t<+\infty} \varphi_t\left(U_p(\epsilon)\right). \tag{2.3}$$

Hence, S_p and U_p are submanifolds (but not closed subsets) of X with

$$\dim S_p = \lambda_p \qquad \text{and} \qquad \dim U_p = n - \lambda_p \tag{2.4}$$

where λ_p is the index of the critical point p. For each p we choose an orientation on U_p. This gives an orientation on the normal bundle of S_p via the splitting $T_pX = T_pU_p \oplus T_pS_p$. We thereby obtain a kernel $[U_p] \times [S_p]$ on $X \times X$ (See A.9.).

The flow φ_t induces a partial ordering on X by setting $x \prec y$ if there is a continuous path consisting of a finite number of foward-time orbits, which begins with x and ends with y. This is the closure of the partial ordering of Remark 1.5.

Definition 2.1. The gradient flow of a smooth function f on a riemannian manifold X is called **Morse-Stokes** if

1. f is a Morse function.
2. The flow is a finite-volume flow.
3. Each of the stable and unstable manifolds S_p and U_p for $p \in \mathrm{Cr}(f)$ has finite volume.
4. (iv) If $p \prec q$ and $p \neq q$ then $\lambda_p < \lambda_q$, for all p, $q \in \mathrm{Cr}(f)$.

Remark 2.2. In Section 14 we shall prove that if the gradient flow of f is Morse-Smale, then it is Morse-Stokes. Furthermore, for any Morse function f on a compact manifold X there exist riemannnian metrics on X for which the gradient flow of f is Morse-Stokes. These metrics are constructed to be canonically flat near $Cr(f)$ and are dense in all metrics with this property.

Theorem 2.3. *Let $f \in C^\infty(X)$ be a Morse function on a compact riemannian manifold whose gradient flow is Morse-Stokes. Then there is an equation of integral currents*

$$\partial T \;=\; [\Delta] - P \tag{2.5}$$

on $X \times X$, where T is an embedded submanifold of finite volume, $\Delta \subset X \times X$ is the diagonal, and where

$$P \;=\; \sum_{p \in Cr(f)} [U_p] \times [S_p] \tag{2.6}$$

Proof. For each critical point $p \in \mathrm{Cr}(f)$ we define

$$\widetilde{U}_p \stackrel{\mathrm{def}}{=} \bigcup_{p \prec q} U_q = \{x \in X : p \prec x\}.$$

Lemma 2.4. *Let $f \in C^\infty(X)$ be any Morse function whose gradient flow is of finite volume, and let $\operatorname{spt} P \subset X \times X$ denote the support of the current P defined in Theorem 1.2. Then*

$$\operatorname{spt} P \subset \bigcup_{p \in \mathrm{Cr}(f)} \widetilde{U}_p \times S_p.$$

Proof. Since $P = \lim_{t\to\infty} P_t$ and $P_t = \{(\varphi_t(x), x) \,:\, x \in X\}$, it is clear that $(y, x) \in \operatorname{spt} P$ only if there exist sequences $x_i \to x$ in X and $s_i \to \infty$ in $\mathbb{R}$ such that $y_i \equiv \varphi_{s_i}(x_i) \to y$. Let $L(x_i, y_i)$ denote the oriented flow line from x_i to y_i. Since the lengths of these lines are bounded, compactness implies that a subsequence converges to a piecewise flow line $L(x, y)$ from x to y. By the continuity of the boundary operator on currents, $\partial L(x, y) = [y] - [x]$. Finally, since $s_i \to \infty$ there must be at least one critical point on $L(x, y)$, and we define $p = \lim_{s\to\infty} \varphi_s(x)$.
q.e.d.

Consider now the compact subset

$$\Sigma \;\equiv\; \bigcup_{\substack{p \prec q \\ p \neq q}} U_q \times S_p \;\subset\; X \times X.$$

Note that from (2.4) and Axiom (iv) in Definition 2.1 that

$$\dim(\Sigma) \;\leq\; n - 1. \tag{2.7}$$

Set $\Sigma' = \Sigma \cup \{(p, p) \,:\, p \in Cr(f)\}$.

Lemma 2.5. *Let $f \in C^\infty(X)$ be a Morse function and consider the embedded submanifold*

$$T \;=\; \{(y, x) \,:\, x \notin Cr(f), \text{ and } y = \varphi_t(x) \text{ for some } \; 0 < t < \infty\}.$$

of $X \times X - \Sigma'$*. Then the closure* $\overline{T}$ *of* T *is a proper* C^∞*-submanfold with boundary*

$$\partial \overline{T} = \Delta - \sum_{p \in \mathrm{Cr}(f)} U_p \times S_p$$

in $X \times X - \Sigma'$

Proof. We first show that it will suffice to prove the assertion in a neighborhood of $(p,p) \in X \times X$ for $p \in Cr(f)$. Consider $(\bar{y}, \bar{x}) \in \overline{T} - T$. If $(\bar{y}, \bar{x}) \notin \Sigma'$, then the proof of Lemma 2.4 shows that either $(\bar{y}, \bar{x}) \in \Delta$ or $(\bar{y}, \bar{x}) \in U_p \times S_p$ for some $p \in Cr(f)$. Near points $(\bar{x}, \bar{x}) \in \Delta$, $\bar{x} \notin Cr(f)$, one easily checks that $\overline{T}$ is a submanifold with boundary Δ. If $(\bar{y}, \bar{x}) \in U_p \times S_p$, then for sufficiently large $s > 0$, the diffeomorphism $\psi_s(y,x) \equiv (\varphi_{-s}(y), \varphi_s(x))$ will map $(\bar{y}, \bar{x})$ into any given neighborhood of (p,p). Note that ψ_s leaves the subset $U_p \times S_p$ invariant, and that ψ_s^{-1} maps T into T. Hence, if $\partial \overline{T} = \Delta - U_p \times S_p$, in a neighborhood of (p,p), then $\partial \overline{T} = -U_p \times S_p$ near $(\bar{y}, \bar{x})$.

Now in a neighborhood $\mathcal{O}$ of (p,p) we may choose coordinates as in (2.1) so that T consists of points $(y,x) = (\bar{u}, \bar{v}, u, v)$ with $\bar{u} = e^{-t}u$ and $\bar{v} = e^t v$ for some $0 < t < \infty$. Consequently, in $\mathcal{O}$ the set $\overline{T}$ is given by the equations

$$\bar{u} = su \quad \text{and} \quad v = s\bar{v} \qquad \text{for some} \quad 0 \le s \le 1.$$

This obviously defines a submanifold in $\mathcal{O} - \{(p,p)\}$ with boundary consisting of Δ and the set $\{\bar{u} = 0,\ v = 0\} \cong (U_p \times S_p) \cap \mathcal{O}$. q.e.d.

Lemma 2.5 has the following immediate consequence

$$\operatorname{spt}\left\{ P - \sum_{p \in \mathrm{Cr}(f)} U_p \times S_p \right\} \subset \Sigma' \tag{2.8}$$

We now apply the following elementary but important result of Federer.

Proposition 2.6. ([14, 4.1.15]). *Let* $[W]$ *be a current in* $\mathbb{R}^n$ *defined by integration over a* k*-dimensional oriented submanifold* W *of locally finite volume. Suppose* $\operatorname{spt}(d[W]) \subset \mathbb{R}^\ell$*, a linear subspace of dimension* $\ell < k - 1$*. Then* $d[W] = 0$*.*

Combining this with (2.7) and (2.8) proves (2.6) and completes the proof of Theorem 2.3. q.e.d.

Remark 2.7. In [14, 4.1.15] Federer actually proves the following general result. Let Y be a locally flat current of dimension k defined in a convex open subset U of $\mathbb{R}^n$.

(i) If the Hausdorff measure of $\operatorname{spt}(dY)$ is zero in dimension $k-1$, then $dY = 0$.

(ii) If $\operatorname{spt}(dY) \subset U \cap \mathbb{R}^{k-1}$, then $dY = c[U \cap \mathbb{R}^{k-1}]$ for some $c \in \mathbb{R}$. Moreover, if Y is locally rectifiable, then c is an integer.

This result is philosophically central to our paper.

Remark 2.8. Using the flow given in Example 1.8 and the methods above, one constructs an $(n+1)$-current T on $S^n \times S^n$ with the property that

$$\partial T \;=\; S^n \times \{*\} + \{*\} \times S^n.$$

This is a singular analogue of the form used by Bott and Taubes to study knot invariants [8].

3. The operator equations

We now explain how to pass from the current equations (A), (B), (C) to the operator equations **(A), (B), (C)** discussed in §1. This kernel calculus was introduced in [21]. There is a brief appendix on currents with definitions and notation at the end of the paper. The discussion here includes the non-orientable case.

Let X and Y be compact manifolds, and let π_Y and π_X denote projection of $Y \times X$ onto Y and X respectively. Then each partially twisted current (or **kernel**) $K \in \mathcal{D}'^*(Y \times \widetilde{X})$, determines an operator $\mathbf{K} : \mathcal{E}^*(Y) \to \mathcal{D}'^*(X)$ by the formula

$$\mathbf{K}(\alpha) = (\pi_X)_*(K \wedge \pi_Y^* \alpha). \tag{3.1}$$

The formula (3.1) can be rewritten as

$$\mathbf{K}(\alpha)(\beta) = K(\pi_Y^* \alpha \wedge \pi_X^* \beta)$$

where $\beta \in \widetilde{\mathcal{E}}^*(X)$ is a twisted form on X. This definition is motivated by the following example.

Example 3.1. Suppose $\varphi : X \to Y$ is a smooth map, and let $\mathbf{P}_\varphi(\alpha) \equiv \varphi^*(\alpha)$ be the pull-back operator on on differential forms. Now a differential form $\varphi^* \alpha$ defines a current by setting

$$(\varphi^* \alpha)(\beta) \equiv \int_X (\varphi^* \alpha) \wedge \beta \tag{3.2}$$

for all twisted forms $\beta \in \widetilde{\mathcal{E}}^*(X)$. Consider the **graph** of φ given by $\operatorname{graph}\varphi = \{(\varphi(x), x) : x \in X\} \subset Y \times X$. Integration over the graph of φ determines a kernel or (partially twisted) current on $Y \times X$, by this same formula (3.2). Namely,

$$[\operatorname{graph}\varphi](\pi_Y^*\alpha \wedge \pi_X^*\beta) \equiv \int_X \varphi^*\alpha \wedge \beta. \tag{3.3}$$

The left hand side equals

$$([\operatorname{graph}\varphi] \wedge \pi_Y^*\alpha)(\pi_X^*\beta) = \big((\pi_X)_*([\operatorname{graph}\varphi] \wedge \pi_Y^*\alpha)\big)(\beta).$$

Therefore,

$$\mathbf{P}_\varphi(\alpha) = (\pi_X)_*([\operatorname{graph}\varphi] \wedge \pi_Y^*\alpha). \tag{3.4}$$

To complete the transfer of the operator equations (**A**) and (**C**) to current equations on $X \times X$, we need the following result.

Lemma 3.2. *Suppose the operator* $\mathbf{K} : \mathcal{E}^*(Y) \to \mathcal{D}'^*(X)$ *has kernel* $K \in \mathcal{D}'^*(Y \times \widetilde{X})$. *Suppose that* $\mathbf{K}$ *lowers degree by one, or equivalently, that* $deg(K) = dimY - 1$. *Then*

$$\textit{The operator} \quad d \circ \mathbf{K} + \mathbf{K} \circ d \quad \textit{has kernel} \quad \partial K. \tag{3.5}$$

Proof. The boundary operator ∂ is the dual of exterior differentiation. That is ∂K is defined by $(\partial K)(\pi_Y^*\alpha \wedge \pi_X^*\beta) = K(d(\pi_Y^*\alpha \wedge \pi_X^*\beta))$. Also, by definition, $(\mathbf{K}(d\alpha), \beta) \equiv K(\pi_Y^*(d\alpha) \wedge \pi_X^*\beta)$, and

$$(d(\mathbf{K}(\alpha)), \beta) \equiv (-1)^{\deg\alpha}\mathbf{K}(\alpha)(d\beta) = (-1)^{\deg\alpha}K(\pi_Y^*(\alpha) \wedge \pi_X^* d\beta),$$

since $\mathbf{K}(\alpha)$ has degree equal to $\deg\alpha - 1$. q.e.d.

The results that we need are summarized in the following table.

Operators	*Kernels*
$\mathbf{I}$	$[\Delta]$
$\mathbf{P}_t \equiv \varphi_t^*$	$P_t \equiv [\operatorname{graph}\varphi_t]$
	$\mathbf{K}$
	K
$d \circ \mathbf{K} + \mathbf{K} \circ d$	∂K

Table 3.6

This completes the transfer of the operator equations **(A), (B), (C)** to the current equations (A), (B), (C). From Theorems 1.2 and 2.3 we immediately deduce the following.

Theorem 3.3. *Let $f \in C^\infty(X)$ be a Morse function on a compact riemannian manifold X whose gradient flow φ_t is Morse-Stokes. Then for every differential form $\alpha \in \mathcal{E}^k(X)$, $0 \le k \le n$, one has*

$$\mathbf{P}(\alpha) = \lim_{t\to\infty} \varphi_t^* \alpha = \sum_{p\in \mathrm{Cr}(f)} r_p(\alpha)[S_p]$$

where the "residue" $r_p(\alpha)$ of α at p is defined by

$$r_p(\alpha) = \int_{U_p} \alpha$$

if $k = n - \lambda$ and 0 otherwise. Furthermore, there is an operator $\mathbf{T}$ of degree -1 on $\mathcal{E}^*(X)$ with values in flat currents, such that

$$d\circ \mathbf{T} + \mathbf{T}\circ d \;=\; \mathbf{I} - \mathbf{P} \tag{3.7}$$

From Theorem 3.3, we see that $\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$ maps onto the finite-dimensional subspace of currents

$$\mathcal{S}_f \;\overset{\text{def}}{=}\; \mathrm{span}\big\{[S_p]\big\}_{p\in \mathrm{Cr}(f)} \tag{3.8}$$

and that

$$\mathbf{P}\circ d \;=\; d\circ \mathbf{P}. \tag{3.9}$$

This together with (3.7) implies the following.

Corollary 3.4. *The subspace $\mathcal{S}_f$ is d-invariant and is therefore a subcomplex of $\mathcal{D}'^*(X)$. The linear map*

$$\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{S}_f$$

is a map of cochain complexes which induces an isomorphism

$$\mathbf{P} : H^*_{deR}(X) \;\overset{\cong}{\longrightarrow}\; H^*(\mathcal{S}_f).$$

4. $\mathbb{Z}$-complexes, $\mathbb{Z}/p\mathbb{Z}$-complexes

We now observe that the complex $(\mathcal{S}_f, d)$ is actually defined over the integers. Consider the lattice

$$\mathcal{S}_f^{\mathbb{Z}} \stackrel{\text{def}}{=} \text{span}_{\mathbb{Z}}\{[S_p]\}_{p\in Cr(f)}$$

and note that $\mathcal{S}_f^{\mathbb{Z}}$ forms a subgroup of the integral currents $\mathcal{I}(X)$ on X.

Theorem 4.1. *The lattice $\mathcal{S}_f^{\mathbb{Z}}$ is preserved by exterior differentiation d, that is, $(\mathcal{S}_f^{\mathbb{Z}}, d)$ is a subcomplex of $(\mathcal{S}_f, d)$. Furthermore, the inclusion of complexes $(\mathcal{S}_f^{\mathbb{Z}}, d) \subset (\mathcal{I}(X), d)$ induces an isomorphism*

$$H(\mathcal{S}_f^{\mathbb{Z}}) \cong H_*(X; \mathbb{Z})$$

Proof. Corollary 3.4 implies that for any $p \in Cr(f)$ we have

$$d[S_p] = \sum_{\lambda_q=\lambda_p-1} n_{p,q}\,[S_q] \tag{4.1}$$

for real numbers $n_{p,q}$. Furthermore, since $[S_p]$ is rectifiable, we have $n_{p,q} \in \mathbb{Z}$ for all p, q by Remark 2.7, and the first assertion is proved. Now the domain of the operator $\mathbf{P}$ extends to include any C^1 chain c which is transversal to the submanifolds U_p, $p \in Cr(f)$, while the domain of $\mathbf{T}$ extends to any C^1 chain c for which $X \times c$ is transversal to T. Standard transversality arguments show that such chain groups (over $\mathbb{Z}$) compute $H_*(X;\mathbb{Z})$. The result then follows from (3.7). q.e.d.

Corollary 4.2. *Let G be a finitely generated abelian group. Then there are natural isomorphisms*

$$H(\mathcal{S}_f^{\mathbb{Z}} \otimes_{\mathbb{Z}} G) \cong H_*(X;G).$$

Part one of Theorem 4.1 has an elementary proof when the flow is **Morse-Smale**, which means by definition that S_p is transversal to U_q for all $p, q \in Cr(f)$. Suppose the flow is Morse-Smale and that $p, q \in Cr(f)$ are critical points with $\lambda_q = \lambda_p - 1$. Then $U_q \cap S_p$ is the union of a finite set of flow lines from q to p which we denote $\Gamma_{p,q}$. To each $\gamma \in \Gamma_{p,q}$ we assign an index n_γ as follows. Let $B_\epsilon \subset S_p$ be a small ball centered at p in a canonical coordinate system (cf. (2.1)), and let y be the point where γ meets ∂B_ϵ. The orientation of S_p induces an orientation on $T_y(\partial B_\epsilon)$, which is identified by flowing backward along γ with $T_q(S_q)$.

If this identification preserves orientations we set $n_\gamma = 1$, and if not, $n_\gamma = -1$. As in [23] Stokes' Theorem gives us the following (cf. [34]).

Proposition 4.3. *When the gradient flow of f is Morse-Smale, the coefficients in (4.1) are given by*

$$n_{p,q} \;=\; (-1)^{\lambda_p} \sum_{\gamma \in \Gamma_{p,q}} n_\gamma$$

Proof. Given a form α of degree $\lambda_p - 1$, we have

$$(-1)^{\lambda_p} d[S_p](\alpha) \;=\; \int_{S_p} d\alpha \;=\; \lim_{r\to\infty} \int_{dS_p(r)} \alpha$$

where $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ as in §3. It suffices to consider forms α with support near q where $\lambda_q = \lambda_p - 1$. Near such q, the set $S_p(r)$, for large r, consists of a finite number of manifolds with boundary, transversal to U_q. There is one for each $\gamma \in \Gamma_{p,q}$. As $r \to \infty$ along one such γ, $dS_p(r)$ converges to $\pm S_q$ where the sign is determined by the agreement (or not) of the orientation of $dS_p(r)$ with the chosen orientation of S_q. q.e.d.

Remark 4.4. The integers $n_{p,q}$ have a simple definition in terms of currents. Set $S_p(r) = \varphi_{-r}(S_p(\epsilon))$ and $U_q(r) = \varphi_r(U_q(\epsilon))$ (cf. (2.3)). Then for all r sufficiently large

$$n_{p,q} \;=\; (-1)^{\lambda_p} \int_X [U_q(r)] \wedge d\,[S_p(r)] \tag{4.2}$$

where the integral denotes evaluation on the fundamental class.

5. Poincaré duality

There is a simple proof of Poincaré duality in this context. Suppose that X is compact and oriented. Given two oriented submanifolds A and B of complementary dimensions in X which meet transversally in a finite number of points, we define $A \bullet B \;=\; \int_X [A] \wedge [B]$ to be the algebraic number of intersections points (counting a point ± 1 depending on orientations as usual). Let $Cr_k(f) = \{p \in Cr(f) \;:\; \lambda_p = k\}$. Then for any k we have

$$U_q \bullet S_p \;=\; \delta_{pq} \qquad \text{for all } \; p, q \in Cr_k(f). \tag{5.1}$$

This gives a formal identification

$$\mathcal{U}_f^{\mathbb{Z}} \;\stackrel{\text{def}}{=}\; \mathbb{Z} \cdot \{[U_p]\}_{p \in Cr(f)} \;\cong\; \text{Hom}\left(\mathcal{S}_f^{\mathbb{Z}}, \mathbb{Z}\right). \tag{5.2}$$

Therefore, taking the adjoint of d gives a differential δ on $\mathcal{U}_f^{\mathbb{Z}}$ with the property that $H_{n-*}(\mathcal{U}_f^{\mathbb{Z}}, \delta) \cong H^*(X;\mathbb{Z})$. On the other hand the arguments of §§1–4 (with f replaced by $-f$) show that $\mathcal{U}_f^{\mathbb{Z}}$ is d-invariant with $H_*(\mathcal{U}_f^{\mathbb{Z}}, d) \cong H_*(X;\mathbb{Z})$. However, these two differentials on $\mathcal{U}_f^{\mathbb{Z}}$ agree up to sign as we see in the next lemma.

Lemma 5.1. *One has*

$$(dU_q) \bullet S_p \;=\; (-1)^{n-k}\, U_q \bullet (dS_p) \tag{5.3}$$

for all $p \in Cr_k(f)$ and $q \in Cr_{k-1}(f)$, and for any k.

Proof. One can see directly from the definition that the integers $n_{p,q}$ are invariant (up to a global sign) under time-reversal in the flow. However, for a simple current-theoretic proof consider the 1-dimensional current $[U_q(r)]\wedge[S_p(r)]$ consisting of a finite sum of oriented line-segments in the flow lines of $\Gamma_{p,q}$ (cf. Remark 4.4). Note that

$$d\left([U_q(r)]\wedge[S_p(r)]\right) \;=\; \left(d\,[U_q(r)]\right)\wedge[S_p(r)] + (-1)^{n-k+1}[U_q(r)]\wedge\left(d\,[S_p(r)]\right)$$

and apply (4.2). q.e.d.

Corollary 5.2. (Poincaré Duality)

$$H^{n-k}(X;\mathbb{Z}) \;\cong\; H_k(X;\mathbb{Z}) \qquad \textit{for all } k.$$

Note 5.3. In our operator picture the Poincaré duality isomorphism can be realized in a nice way. Let

$$T_{\text{tot}} \;=\; T^* + T \;=\; \{(y,x)\,:\, y = \varphi_t(x) \;\text{ for some } t \in \mathbf{R}\}.$$

Then we obtain the operator equation

$$d \circ \mathbf{T}_{\text{tot}} + \mathbf{T}_{\text{tot}} \circ d \;=\; \mathbf{P} - \check{\mathbf{P}}$$

where

$$\mathbf{P} = \sum_{p\in Cr(f)} [U_p]\times[S_p] \qquad \text{and} \qquad \check{\mathbf{P}} = \sum_{p\in Cr(f)} [S_p]\times[U_p].$$

This chain homotopy induces an isomorphism $H_*(\mathcal{U}_f^{\mathbb{Z}}) \cong H_*(\mathcal{C}_f^{\mathbb{Z}})$, which after identifying $\mathcal{U}_f^{\mathbb{Z}}$ with the cochain complex via (5.1) and (5.3), gives the duality isomorphism 5.2. When X is not oriented, a parallel analysis yields Poincaré duality with mod 2 coefficients.

6. Critical submanifolds of higher dimension

The methods introduced above apply in much greater generality. As seen in §1, one only needs the flow φ_t to be of finite volume to guarantee the existence of an operator $\mathbf{P}(\alpha) = \lim_{t\to\infty} \varphi_t^*(\alpha)$ which is chain homotopic to the identity. In this and the following sections we shall examine some important examples.

Let $f : X \to \mathbb{R}$ be a smooth function whose critical set (i.e., the set where $df = 0$) is a finite disjoint union

$$Cr(f) \;=\; \coprod_{j=1}^{\nu} F_j$$

of compact submanifolds F_j in X. We assume that Hess(f) is *non-degenerate* on the normal spaces to $Cr(f)$. Then for each j, there are stable and unstable manifolds

$$S_j = \{x \in X \;:\; \lim_{t\to\infty} \varphi_t(x) \in F_j\} \qquad \text{and}$$
$$U_j = \{x \in X \;:\; \lim_{t\to-\infty} \varphi_t(x) \in F_j\}$$

with projections

$$S_j \overset{\tau_j}{\to} F_j \overset{\sigma_j}{\to} U_j. \tag{6.1}$$

where

$$\tau_j(x) = \lim_{t\to\infty} \varphi_t(x) \qquad \text{and} \qquad \sigma_j(x) = \lim_{t\to-\infty} \varphi_t(x).$$

For each j, let $n_j = \dim(F_j)$ and set $\lambda_j = \dim(S_j) - n_j$. Then $\dim(U_j) = n - \lambda_j$. For $p \in F_j$ we define $\lambda_p \equiv \lambda_j$ and $n_p \equiv n_j$.

Definition 6.1. The gradient flow φ_t of a smooth function $f \in C^\infty(X)$ on a riemannian manifold X is called a **generalized Morse-Stokes flow** if:

(i) The critical set of f consists of a finite number of submanifolds $F_1, ..., F_\nu$ on the normals of which Hess(f) is non-degenerate.

(ii) The manifolds T, and T^*, and the stable and unstable manifolds S_j, U_j for $1 \leq j \leq \nu$ are submanifolds of finite volume. Furthermore, for each j, the fibres of the projections τ_j and σ_j are of uniformly bounded volume.

(iii) $p \prec q \quad \Rightarrow \quad \lambda_p + n_p < \lambda_q \qquad \forall p, q \in Cr(f)$.

These axioms are easily verified in a number of important cases in the algebraic and analytic category. The first main result concerning such flows is the following.

Theorem 6.2. *Suppose φ_t is a gradient flow satisfying the generalized Morse-Stokes conditions 6.1 on a compact oriented manifold X. Then there is an equation of currents*

$$\partial T \;=\; [\Delta] - P$$

on $X \times X$, where T, Δ, and P are as in Theorem 1.2, and

$$P \;=\; \sum_{j=1}^{\nu} [U_j \times_{F_j} S_j] \tag{6.2}$$

where $U_j \times_{F_j} S_j \equiv \{(y,x) \in U_j \times S_j \subset X \times X \;:\; \sigma_j(y) = \tau_j(x)\}$ denotes the fibre product of the projections (6.1)

Proof. The argument follows closely the proof of Theorem 2.3. Details are omitted. q.e.d.

Janko Latschev has found a Smale-type condition which yields this result in many cases where the hypothesis of 6.1 (iii) does not hold. In particular, Latschev's condition implies only that: $p \prec q \Rightarrow \lambda_p < \lambda_q$. Details appear in [22].

As in §3, this result can be translated into operator form.

Theorem 6.3. *Let φ_t be a gradient flow satisfying the generalized Morse-Stokes Conditions on a manifold X as above. Then for all smooth forms α on X, the limit*

$$\mathbf{P}(\alpha) = \lim_{t\to\infty} \varphi_t^*(\alpha)$$

exists and defines a continuous linear operator $\mathbf{P} : \mathcal{E}^*(X) \longrightarrow \mathcal{D}'^*(X)$ with values in flat currents on X. This operator fits into a chain homotopy

$$d \circ \mathbf{T} + \mathbf{T} \circ d \;= \mathbf{I} - \mathbf{P}. \tag{6.3}$$

Furthermore, $\mathbf{P}$ is given by the formula

$$\mathbf{P}(\alpha) \;=\; \sum_{j=1}^{\nu} Res_j(\alpha)\,[S_j] \tag{6.4}$$

where

$$Res_j(\alpha) \;\equiv\; \tau_j^*\left\{(\sigma_j)_*\left(\alpha\big|_{U_j}\right)\right\} \tag{6.5}$$

Proof. This is a direct consequence of Theorem 6.2 except for the formulae (6.4)-(6.5). To see this consider the pull-back square

$$\begin{array}{ccc} U_j \times_{F_j} S_j & \xrightarrow{t_j} & U_j \\ {\scriptstyle s_j}\downarrow & & \downarrow{\scriptstyle \sigma_j} \\ S_j & \xrightarrow{\tau_j} & F_j \end{array} \tag{6.6}$$

where t_j and s_j are the obvious projections. One sees from the definitions (cf §3) that

$$\mathbf{P}(\alpha) \;=\; \sum_{j=1}^{\nu}(s_j)_*\left\{(t_j)^*\left(\alpha\big|_{U_j}\right)\right\}.$$

The commutativity of the diagram (6.6) allows us to rewrite these terms as in (6.5). q.e.d.

Corollary 6.4. *Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p \prec q$. Then the homology of X is spanned by the images of the groups $H_{\lambda_j+\ell}(\overline{S_j})$ for $j = 1, ..., \nu$ and $\ell \geq 0$.*

Proof. Under this hypothesis $\partial(U_j \times_{F_j} S_j) = 0$ for all j, and so (6.2) yields a decomposition of $\mathbf{P}$ into operators that commute with d. q.e.d.

We can make this corollary more precise. Note that $\tau_j : S_j \to F_j$ can be given the structure of a vector bundle of rank λ_j. The closure $\overline{S_j} \subset X$ is a compactification of this bundle with a complicated structure at infinity. (See [11] for example.) There is nevertheless a homomorphism $\Theta_j : H_*(F_j) \longrightarrow H_{\lambda_j+*}(\overline{S_j})$ which after pushing forward to the one-point compactification of S_j, is the Thom isomorphism. This leads to the following (cf. [2]).

Theorem 6.5. *Suppose that $\lambda_p + n_p + 1 < \lambda_q$ for all critical points $p \prec q$ and that X and all F_j and S_j are oriented. Then there is an isomorphism*

$$H_*(X) \;\cong\; \bigoplus_j H_{*-\lambda_j}(F_j)$$

This result holds without the orientation assumptions if one takes homology with appropriately twisted coefficients. Much stronger versions of Theorems 6.3 and 6.5, are found in [22]. They include an extension to integral homology groups. Latschev also derives a spectral sequence associated to any *Bott-Smale* function satisfying a natural Smale-type transversality hypothesis. One virtue of this sequence is that the differentials are explicitly computable. Assuming for simplicity that everything is oriented, the E^1-term is given by

$$E^1_{p,q} \;=\; \bigoplus_{\lambda_j=p} H_q(F_j;\mathbb{Z})$$

and $E^k_{p,q} \Rightarrow H_*(X;\mathbb{Z})$

7. The relative case

In standard Morse Theory one often studies the change in the topology as one passes from $\{x : f(x) \le a\}$ to $\{x : f(x) \le b\}$. Our approach is easily adapted to this case.

Let $f : X \to \mathbb{R}$ be a proper Morse function, where X is not necessarily compact, and suppose that X carries a metric as in 14.3. Let $a < b$ be regular values of f and consider the compact manifold with boundary

$$Z \equiv f^{-1}([a,b]).$$

On Z we define a vector field $V = (\psi \circ f)\nabla f$ where $\psi : [a,b] \to [0,1]$ is a smooth function satisfying: (i) $\psi^{-1}(0) = \{a,b\}$, (ii) ψ is linear on $[a, a{+}\epsilon]$ and $[b{-}\epsilon, b]$, (iii) $\psi \equiv 1$ on $[a{+}2\epsilon, b{-}2\epsilon]$, (iv) $f(Cr(f)) \cap Z \subset (a{+}2\epsilon, b{-}2\epsilon)$ for some small $\epsilon > 0$.

Let $\varphi_t : Z \to Z$ be the flow of V. Note that φ_t is complete and fixes the boundary ∂Z. By 14.3, the stable and unstable manifolds of each $p \in Cr(f)$ have finite volume in Z, and so also does $T \equiv \{(y,x) : y = \varphi_t(x)$ for some $t,\ 0 < t < \infty\} \subset Z \times Z$.

We decompose the boundary

$$\partial Z \;=\; f^{-1}(b) - f^{-1}(a) \;\stackrel{\text{def}}{=}\; \partial_b - \partial_a.$$

In analogy with the fibre products appearing in §6 we have the following submanifolds of $Z \times Z$:

$$\begin{aligned} S(\partial_b) \;&=\; \{(y,x) \in Z \times Z \;:\; y \in \partial_b \text{ and } \lim_{t\to\infty} \varphi_t(x) = y\} \qquad \text{and} \\ U(\partial_a) \;&=\; \{(y,x) \in Z \times Z \;:\; x \in \partial_a \text{ and } \lim_{t\to -\infty} \varphi_t(y) = x\}. \end{aligned}$$

Theorem 7.1. *On $Z \times Z$ there is an equation of integral currents*

$$\partial T \;=\; [\Delta] - \sum_{p \in Cr(f) \cap Z} [U_p] \times [S_p] - [U(\partial_a)] - [S(\partial_b)]. \tag{7.1}$$

Proof. Consider $\mathcal{T}_t \subset \mathbf{R} \times X \times X$ defined as in (1.1) and note that

$$\partial \mathcal{T}_t \;=\; \{0\} \times [\Delta_Z] - \{t\} \times [\text{graph}\, \varphi_t] - [0,t] \times [\Delta_{\partial Z}].$$

Set $T_t = \text{pr}_* \mathcal{T}_t$ and observe that since $\text{pr}_*([0,t] \times [\Delta_{\partial Z}]) = 0$,

$$\partial T_t \;=\; [\Delta] - [\text{graph}\varphi_t].$$

By hypothesis $T = \lim_{t \to \infty} T_t$ has finite volume, and a direct application of the arguments of §2 establishes (7.1). q.e.d.

Equation (7.1) can be translated to an operator equation. However, here we want the operator to act on the *relative* forms:

$$\mathcal{E}^*(Z, \partial_b) \;\equiv\; \left\{ \alpha \in \mathcal{E}^*(Z) \;:\; \alpha\big|_{\partial_b} = 0 \right\}. \tag{7.2}$$

We begin with the case where $f > a$ and so $\partial_a = \emptyset$.

Theorem 7.2. *Suppose $f : X \longrightarrow \mathbb{R}^+$ is a proper Morse function. Let $Z = f^{-1}(-\infty, b]$ where b is a regular value of f, and consider the operator*

$$\mathbf{P} : \mathcal{E}^*(Z, \partial Z) \longrightarrow \mathcal{D}'^*(Z)$$

defined by

$$\mathbf{P}(\alpha) \;=\; \lim_{t \to \infty} \varphi_t^*(\alpha) \tag{7.3}$$

where φ_t is the truncated gradient flow defined above. This operator is well defined and continuous. In fact, there exists a continuous operator $\mathbf{T} : \mathcal{E}^*(Z, \partial Z) \longrightarrow \mathcal{D}'^*(Z)$ of degree -1 with values in flat currents, such that

$$d \circ \mathbf{T} + \mathbf{T} \circ d \;= \mathbf{I} - \mathbf{P} \tag{7.4}$$

Furthermore, $\mathbf{P}$ is given by the formula

$$\mathbf{P}(\alpha) \;=\; \sum_{p \in Cr(f) \cap Z} r_p(\alpha)[S_p] \tag{7.5}$$

where $r_p(\alpha) = \int_{U_p} \alpha$. *In particular,* $\mathbf{P}$ *is a continuous chain mapping onto the finite dimensional complex*

$$\mathcal{S}_{f,Z} \stackrel{def}{=} span_{\mathbf{R}} \Big\{ [S_p] \Big\}_{p \in Cr(f) \cap Z},$$

with differential given as in 3.5, and $\mathbf{P}$ *induces an isomorphism*

$$H^*_{deR}(Z, \partial Z) \stackrel{\cong}{\rightarrow} H(\mathcal{S}_{f,Z}).$$

Proof. This is deduced exactly as are Theorems 3.3, 3.5. We need only note that the operator given by $[S(\partial_b)]$ is zero on $\mathcal{E}^*(Z, \partial Z)$. This follows directly from the definition 3.1 and the fact that $\pi_1(S(\partial_b)) = \partial_b = \partial Z$. q.e.d.

In the more general case where $\partial_a \neq \emptyset$ we compose our operators with the projection map $\pi : \mathcal{E}^*(Z)' \longrightarrow \mathcal{E}^*(Z, \partial Z)'$, (which is adjoint to the inclusion $\mathcal{E}^*(Z, \partial Z) \subset \mathcal{E}^*(Z)$). In this case the operator corresponding to $[U(\partial_a)]$ is zero. Specifically, letting $\pi_k : X \times X \rightarrow X$ denote projection onto the k^{th} factor, we have that

$$(\mathbf{U}(\partial_a)(\alpha), \beta) = ((\pi_1)_* \{\pi_2^* \alpha \wedge [U(\partial_a)]\}, \beta) = 0$$

since $\pi_1(U(\partial_a)) = \partial_a$ and $\beta\big|_{\partial_a} = 0$. We conclude the following.

Theorem 7.3. *The operators*

$$\mathbf{T}, \mathbf{P} : \mathcal{E}^*(Z, \partial_b Z) \longrightarrow \mathcal{D}'^*(Z, \partial_a Z)$$

corresponding to the currents T, P *from Theorem 7.2 satisfy the equation*

$$d \circ \mathbf{T} + \mathbf{T} \circ d = \mathbf{I} - \mathbf{P}$$

where

$$\mathbf{P} = \sum_{p \in Cr(f) \cap Z} r_p [S_p]$$

and $r_p(\alpha) = \int_{U_p} \alpha$. *Thus,* $\mathbf{P}$ *gives a continuous chain mapping of the relative deRham complex* $\mathcal{E}^*(Z, \partial_b)$ *onto the finite dimensional complex*

$$\mathcal{S}_{f,Z} \stackrel{def}{=} span_{\mathbf{R}} \{[S_p]\}_{p \in Cr(f) \cap Z}$$

with differential given as in 3.5. This induces an isomorphism

$$H^*_{deR}(Z, \partial_b Z) \stackrel{\cong}{\rightarrow} H(\mathcal{S}_{f,Z}).$$

Remark 7.4. Taking $\partial_b Z = \emptyset$ and $\partial_a Z \neq \emptyset$ in Theorem 7.3 (and interchanging the roles of a and b) gives the version of Theorem 7.2 corresponding to going backwards in time. That is, one considers

$$\mathbf{P}(\alpha) \stackrel{\text{def}}{=} \lim_{t\to-\infty} \varphi_t^*\alpha$$

and obtains a projection operator

$$\mathbf{P} : \mathcal{E}^*(Z) \longrightarrow \mathcal{S}_{f,Z} \subset \mathcal{D}'^*(Z, \partial_b)$$

where $\mathcal{S}_{f,Z}$ is a finite complex defined over $\mathbb{Z}$.

Remark 7.5. (Duality) Arguing exactly as in §5 one can retrieve the duality theorem

$$H^k(Z, \partial_b Z; \mathbb{Z}) \cong H_{n-k}(Z, \partial_a Z : \mathbb{Z}) \tag{7.6}$$

which gives, in the special case where either $\partial_a Z = 0$ or $\partial_b Z = 0$, the Lefschetz Duality Theorem.

8. Holomorphic flows and the Carrell-Lieberman-Sommese theorem

The ideas in this paper have interesting consequences in the holomorphic case. For example, given a $\mathbf{C}^*$-action φ_t on a compact Kähler manifold X, there is a complex graph

$$\begin{aligned}\mathcal{T} \stackrel{\text{def}}{=} \{(t, \varphi_t(x), x) \in \mathbf{C}^* \times X \times X \ : \ t \in \mathbf{C}^* \ \text{ and } \ x \in X\} \\ \subset \ \mathbf{P}^1(\mathbf{C}) \times X \times X\end{aligned}$$

analogous to the graphs considered above.

Theorem 8.1. ([32]) *If φ_t has fixed-points, then $\mathcal{T}$ has finite volume and its closure $\overline{\mathcal{T}}$ in $\mathbf{P}^1(\mathbf{C}) \times X \times X$ is an analytic subvariety.*

The relation of $\mathbf{C}^*$-actions to Morse-Theory is classical. One can decompose φ_t into an "angular" S^1-action and a radial flow. Averaging a Kähler metric over S^1 and applying an argument of Frankel [14], we find a function $f : X \to \mathbf{R}$ of Bott-Morse type whose gradient generates the radial action. One can now apply the methods of this paper, in particular those of §6.

Thus when φ_t has fixed-points, $\overline{\mathcal{T}}$ gives a rational equivalence between the diagonal Δ in $X \times X$ and an analytic cycle P whose components consist of fibre products of stable and unstable manifolds over components of the fixed-point set of the action.

When the fixed-points are all isolated, P becomes a sum of analytic Künneth components $P = \sum \overline{S}_p \times \widetilde{\overline{U}}_p$, and we recover the well-known fact that the cohomology of X is freely generated by the stable subvarieties $\{\overline{S}_p\}_{p \in \mathrm{Zero}(\varphi)}$. It follows that X is algebraic and that all cohomology theories on X (eg. algebraic cycles modulo rational equivalence, algebraic cycles modulo algebraic equivalence, singular cohomology) are naturally isomorphic. (See [5], [13], [15]).

When the fixed-point set has positive dimension, one can recover results of Carrell-Lieberman-Sommese for $\mathbf{C}^*$-actions ([9], [10]), which assert among other things that if $\dim(X^{\mathbf{C}^*}) = k$, then $H^{p,q}(X) = 0$ for $|p - q| > k$.

9. The local MacPherson formula

The ideas in this paper also have an interesting application to the study of curvature and singularities. Suppose

$$\alpha : E \longrightarrow F$$

is a map between smooth vector bundles with connection over a manifold X. Let $G = G_k(E \oplus F) \to X$ denote the Grassmann bundle of k-planes in $E \oplus F$ where $k = \mathrm{rank}(E)$. There is a flow φ_t on G induced by the flow $\psi_t : E \oplus F \to E \oplus F$ where $\psi_t(e, f) = (te, f)$. This is a very simple generalized Morse-Stokes flow on G. On the "affine chart"$\mathrm{Hom}(E, F)$, one has that $\varphi_t(A) = \frac{1}{t}A$.

Note that here it is more natural to consider the flow multiplicatively ($\varphi_{ts} = \varphi_t \circ \varphi_s$) than additively. Consider $\mathbf{R}^+ = \{[1 : t] \in \mathbf{P}^1(\mathbf{R}) \, : \, 0 < t < \infty\} \subset \mathbf{P}^1(\mathbf{R}) \subset \mathbf{P}^1(\mathbf{C})$. Note that this inclusion is compatible with the inclusion of the complex multiplicative group $\mathbf{C}^* \subset \mathbf{P}^1(\mathbf{C})$.

Definition 9.1. The section α is said to be **geometrically atomic** if the graph

$$\mathcal{T}(\alpha) \stackrel{\text{def}}{=} \left\{ (t, x, \frac{1}{t}\alpha(x)) \, : \, 0 < t \le 1 \quad \text{and} \quad x \in X \right\}$$

has finite volume in $\mathbf{P}^1(\mathbf{R}) \times X \times G$.

This hypothesis is sufficient to guarantee the existence of $\lim_{t \to 0} \alpha_t^* \Phi$ where $\alpha_t \equiv \frac{1}{t}\alpha$ and where Φ is any differential form on G. Choosing

$\Phi = \Phi_0(\Omega_U)$ where Φ_0 is an Ad-invariant polynomial on $\mathfrak{gl}_k(\mathbf{R})$ and Ω_U is the curvature of the tautological k-plane bundle over G, one can establish a local version of a basic formula of MacPherson [25], [26]. Details appear in [20].

10. Equivariant Morse theory

The ideas developed here carry over virtually intact to the setting of equivariant cohomology. The reason the method works directly is the simple but important fact that a closed, invariant submanifold is *equivariantly* closed (See Corollary 10.2 below). In this section we derive some consequences of the method which usefully apply to Morse functions arising in algebra and geometry (e.g., from moment map constructions). Deeper results have been obtained by J. Latschev [22]. We shall adopt the exposition of Cartan's equivariant de Rham theory found in [6] and [7].

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ acting on a compact n-manifold X. By an **equivariant differential form** on X we mean a G-equivariant polynomial map $\alpha : \mathfrak{g} \longrightarrow \mathcal{E}^*(X)$. The set of equivariant forms is denoted by

$$\mathcal{E}^*_G(X) \;=\; \{S^*(\mathfrak{g}^*) \otimes \mathcal{E}^*(X)\}^G$$

and is graded by declaring elements of $S^p(\mathfrak{g}^*) \otimes \mathcal{E}^q(X)$ to have total degree $2p+q$. The equivariant differential $d_G : \mathcal{E}^*_G(X) \longrightarrow \mathcal{E}^{*+1}_G(X)$ is defined by setting

$$(d_G\alpha)\,(V) \;=\; d\alpha(V) - i_{\widetilde{V}}\alpha(V)$$

for $V \in \mathfrak{g}$ where the vector field $\widetilde{V}$ is the image of V under the natural linear map $\mathfrak{g} \longrightarrow \Gamma(TX)$, and where $i_{\widetilde{V}}$ denotes contraction with $\widetilde{V}$.

The complex $\mathbb{E}^*_G(X)$ of **equivariant currents** is similarly defined by replacing smooth forms $\mathcal{E}^*(X)$ by forms $\mathbb{E}^*(X)$ with distribution coefficients.

Consider now a G-invariant function $f \in C^\infty(X)$ and suppose that X is provided with a G-invariant riemannian metric. Then $g_*\nabla f = \nabla f$ for all $g \in G$ and so the gradient flow of f commutes with the action of G. Suppose that the gradient flow has finite volume and let

$$\partial T \;=\; \Delta - P \tag{10.1}$$

denote the current equation derived in §1. Let G act on $X \times X$ by the diagonal action $g \cdot (x, y) = (gx, gy)$.

Lemma 10.1. *The current T satisfies:*

(i) $g_*T = T$ *for all $g \in G$, and*

(ii) $i_{\widetilde{V}}T = 0$ *for all $V \in \mathfrak{g}$.*

So also does the current P.

Proof. The current T corresponds to integration over the finite volume submanifold $\{(x,y) \in X \times X - \Delta \ : \ \exists t \in (0,\infty) \ \text{ s.t. } y = \varphi_t(x)\}$. Since $g\varphi_t(x) = \varphi_t(gx)$, assertion (i) is clear. The invariance of T implies the invariance of P by (10.1). For assertion (ii) note that for any n-form β on $X \times X$

$$\left(i_{\widetilde{V}}T\right)(\beta) \ = \ \int_T i_{\widetilde{V}}\beta \ = \ 0$$

since V is tangent to T. Similarly, $i_{\widetilde{V}}\Delta = 0$. Since $d \circ i_{\widetilde{V}} + i_{\widetilde{V}} \circ d = \mathcal{L}_{\widetilde{V}}$ (Lie derivative) and $\mathcal{L}_{\widetilde{V}}T = 0$, we conclude that $i_{\widetilde{V}}P = 0$. q.e.d.

Corollary 10.2. *Consider $T \equiv 1 \otimes T \in 1 \otimes \mathbb{E}^{n-1}(X \times X)^G$ as an equivariant current of total degree $(n-1)$ on $X \times X$. Consider Δ and P similarly as equivariant currents of degree n. Then*

$$\partial_G T \ = \Delta - P. \tag{10.2}$$

in the complex of equivariant currents on $X \times X$.

The correspondence between operators and kernels discussed in §1 carries over directly to the equivariant context. Currents in $\mathbb{E}_G^{n-\ell}(X \times X)$ yield G-equivariant operators $\mathcal{E}^*(X) \longrightarrow \mathbb{E}^{*+\ell}(X)$, and hence operators $\mathcal{E}_G^*(X) \longrightarrow \mathbb{E}_G^{*+\ell}(X)$. Equations of type (10.2) translate into operator equations

$$d_G \circ \mathbf{T} + \mathbf{T} \circ d_G \ = I - \mathbf{P}. \tag{10.3}$$

Applying the arguments of §3 proves the following.

Proposition 10.3. *Let φ_t be an invariant flow on a compact G-manifold X. If φ_t has finite volume, then the limit*

$$\mathbf{P}(\alpha) \ = \ \lim_{t\to\infty} \varphi_t^*\alpha \tag{10.4}$$

exists for all $\alpha \in \mathcal{E}_G^*(X)$ and defines a continuous linear operator $\mathbf{P} : \mathcal{E}_G^*(X) \longrightarrow \mathcal{D}'^*_G(X)$ of degree 0, which is equivariantly chain homotopic to the identity on $\mathcal{E}_G^*(X)$.

Applying the methods of §§2-3 and the fact that

$$S^*(\mathfrak{g}^*)^G \cong H^*(BG)$$

gives the following result.

Theorem 10.4. *Let $f \in C^\infty(X)$ be an invariant Morse function on a compact riemnannian G-manifold whose gradient flow φ_t is Morse-Stokes. Then the continuous linear operator (10.4) defines a map of equivariant complexes*

$$\mathbf{P} : \mathcal{E}^*_G(X) \longrightarrow S^*(\eth^*)^G \otimes \mathcal{S}_f$$

where $\mathcal{S}_f = \mathrm{span}\{[S_p]\}_{p\in Cr(f)}$ *as in (3.8) and where the differential on* $S^*(\eth^*)^G \otimes \mathcal{S}_f$ *is* $1 \otimes \partial$. *This map induces an isomorphism*

$$H^*_G(X) \stackrel{\cong}{\to} H^*(BG) \otimes H^*(X)$$

Examples of this phenomenon arise in moment map constructions. For a simple example consider $G = (S^1)^{n+1}/\Delta$ acting on $\mathbb{P}^n_{\mathbb{C}}$ via the standard action on homogeneous coordinates $[z_0, ..., z_n]$, and set $f([z]) = \sum k|z_k|^2/\|z\|^2$. One sees immediately the well-known fact that $H^*_G(\mathbb{P}^n_{\mathbb{C}})$ is a free $H^*(BG)$-module with one generator in each dimension $2k$ for $k = 0, ..., n$. This extends to all generalized flag manifolds and to products.

It has been pointed out by Janko Latschev that there exists an invariant Morse function for which no choice of invariant metric gives a Morse-Stokes flow.

On the other hand the method applies to much more general functions and yields results as in §§5-9. Suppose for example that f is an invariant function whose critical set consists of a finite number of non-degenerate critical orbits $\mathcal{O}_i = G/H_i$, $i = 1, ..., N$. Janko Latschev (cf. [22], [1]) has established a spectral sequence with (assuming for simplicity that everything is oriented)

$$E^1_{p,*} = \bigoplus_{\lambda_i = p} H^*_G(\mathcal{O}_i) = \bigoplus_{\lambda_i = p} H^*(BH_i)$$

and computable differentials such that

$$E^k_{*,*} \Rightarrow H^*_G(X).$$

11. Flat bundles and local coefficients

Our method applies immediately to forms with coefficients in a flat bundle $E \to X$. In this case the kernels of §2 are currents on $X \times X$

with coefficients in $\mathrm{Hom}(\pi_1^*E, \pi_2^*E)$. Given a Morse-Stokes flow φ_t on X we consider the kernel $T_E = h \otimes T$ where T is defined as in §2 and $h : E_{\varphi_t(x)} \to E_x$ is parallel translation along the flow line. One obtains the equation

$$\partial T_E = \Delta_E - P_E$$

where $\Delta_E = \mathrm{Id} \otimes \Delta$ and $P_E = \sum_p h_p \otimes ([U_p] \times [S_p])$ with $h_p : E_y \to E_x$ given by parallel translation along the broken flow line. Thus, h_p corresponds to $\mathrm{Id} : E_p \to E_p$ under the canonical trivializations $E\big|_{U_p} \cong U_p \times E_p$ and $E\big|_{S_p} \cong S_p \times E_p$. We obtain the operator equation

$$d \circ \mathbf{T}_E + \mathbf{T}_E \circ d \;=\; \mathbf{I} - \mathbf{P}_E \tag{11.1}$$

where $\mathbf{P}_E$ maps onto the finite complex

$$\mathcal{S}_E \overset{\text{def}}{=} \bigoplus_{p \in Cr(f)} E_p \otimes [S_p]$$

by integration of forms over the unstable manifolds. The restriction of d to $\mathcal{S}_E$ is given as in 4.3 by $d(e \otimes [S_p]) = \sum h_{p,q}(e)[S_q]$ where $h_{p,q} = (-1)^{\lambda_p} \sum_\gamma h_\gamma$ and $h_\gamma : E_p \to E_q$ is parallel translation along $\gamma \in \Gamma_{p,q}$. By (11.1) the complex $(\mathcal{S}_E, d)$ computes $H^*(X;\, E)$.

Reversing time in the flow shows that the complex $\mathcal{U}_E = \bigoplus_p E_p^* \otimes [U_p]$ with differential defined as above computes $H^*(X;\, E^*)$. As in §5 the obvious dual pairing of these complexes establishes the generalized Poincaré duality. Furthermore, one can extend all this to integral currents twisted by representations of $\pi_1(X)$ in $\mathrm{GL}_n(\mathbf{Z})$ or $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$ and obtain duality with local coefficient systems.

12. Products

Our method has a number of interesting extensions. For example, consider the triple diagonal $\Delta_3 \subset X \times X \times X$ as the kernel of the wedge-product operator

$$\wedge : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X).$$

Let f and f' be functions with Morse-Stokes flows φ_t and φ_t' respectively. Assume that for all $(p, p') \in Cr(f) \times Cr(f')$ the stable manifolds S_p and $S'_{p'}$ intersect transversely in a manifold of finite volume, and similarly for the unstable manifolds U_p and $U'_{p'}$. Degenerating Δ_3 gives a kernel

$$T \;\equiv\; \{(\varphi_t(x), \varphi_t'(x), x) \in X \times X \times X \;:\; x \in X \;\text{ and }\; 0 \le t < \infty\}$$

and a corresponding operator $\mathbf{T} : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ of degree -1. One calculates that $\partial T \;=\; \Delta_3 - M$ where

$$M = \sum_{(p,p')\in Cr(f)\times Cr(f')} [U_p] \times [U_{p'}] \times [S_p \cap S'_{p'}].$$

The corresponding operator $\mathbf{M} : \mathcal{E}^*(X) \otimes \mathcal{E}^*(X) \to \mathcal{E}^*(X)$ is given by

$$\text{(12.1)} \qquad \mathbf{M}(\alpha, \beta) \;=\; \sum_{(p,p')\in Cr(f)\times Cr(f')} \left(\int_{U_p} \alpha\right)\left(\int_{U_{p'}} \beta\right)[S_p \cap S'_{p'}].$$

The arguments of §§1-3 adapt to prove the following.

Theorem 12.1. *There is an equation of operators* $\wedge - \mathbf{M} = d \circ \mathbf{T} + \mathbf{T} \circ d$ *from* $\mathcal{E}^*(X \times X)$ *to* $\mathcal{E}^*(X)$ *(where* $\wedge$ *denotes restriction to the diagonal). In particular for* $\alpha, \beta \in \mathcal{E}^*(X)$ *we have the chain homotopy*

$$\text{(12.2)} \quad \alpha \wedge \beta - \mathbf{M}(\alpha, \beta) \;=\; d\mathbf{T}(\alpha, \beta) + \mathbf{T}(d\alpha, \beta) + (-1)^{deg\alpha}\mathbf{T}(\alpha, d\beta)$$

between the wedge product and the operator (12.1).

Note that the operator $\mathbf{M}$ has range in the finite dimensional vector space

$$\mathcal{M} \;\stackrel{\text{def}}{=}\; \text{span}_{\mathbf{R}}\{\,[S_p \cap S'_{p'}]\}_{\;(p,p')\in Cr(f)\times Cr(f')}.$$

It converts a pair of smooth forms α, β into a linear combination of the pairwise intersections of the stable manifolds $[S_p]$ and $[S'_{p'}]$. If $d\alpha = d\beta = 0$, then

$$\text{(12.3)} \qquad \mathbf{M}(\alpha, \beta) \;=\; \alpha \wedge \beta - d\mathbf{T}(\alpha, \beta),$$

and so $\mathbf{M}(\alpha, \beta)$ is a cycle homologous to the wedge product $\alpha \wedge \beta$. This operator also has the following properties.

Theorem 12.2. *The operator* $\mathbf{M}$ *maps onto the subspace* $\mathcal{M}$. *Furthermore, for forms* $\alpha, \beta \in \mathcal{E}^*(X)$, *it satisfies the equation*

$$\text{(12.4)} \qquad d\mathbf{M}(\alpha, \beta) \;=\; \mathbf{M}(d\alpha, \beta) + (-1)^{deg\alpha}\mathbf{M}(\alpha, d\beta)$$

Proof. To see that $\mathbf{M}$ is onto (as a linear map from $\mathcal{E}^*(X \times X)$) it suffices to see that for each non-empty intersection $S_p \cap S'_{p'}$ the current $[S_p \cap S'_{p'}]$ is in the range. However, by transversality we see that if $S_p \cap S'_{p'} \neq \emptyset$, then $\lambda_p + \lambda_{p'} \geq n$, and so $n \geq (n - \lambda_p) + (n - \lambda_{p'})$. Therefore, by transversality we have $\dim(U_q \cap U'_{q'}) \leq 0$ for all q, q'. It follows that we can find differential forms α and β such that $\int_{U_p}\alpha = \int_{U'_{p'}}\beta = 1$ and $\int_{U_q}\alpha = \int_{U'_{q'}}\beta = 0$ for all $q \neq p$ and $q' \neq p'$. Then by (12.1), $\mathbf{M}(\alpha,\beta) = [S_p \cap S'_{p'}]$ and the assertion is proved.

Equation (12.4) follows from the fact that $dM = 0$, which implies that $d\circ\mathbf{M}+\mathbf{M}\circ d = 0$, together with the standard formula for $d(\alpha\wedge\beta)$. q.e.d.

It follows immediately that $d(\mathcal{M}) \subset \mathcal{M}$. In fact one can see from the transversality assumptions that for $(p,p') \in Cr(f)\times Cr(f')$ one has

$$d\,[S_p \cap S'_{p'}] \;=\; \sum_{q\in Cr(f)} n_{pq}[S_q \cap U'_{p'}] + (-1)^{n-\lambda_p} \sum_{q'\in Cr(f')} n'_{p'q'}[S_p \cap U'_{q'}] \tag{12.5}$$

where the n_{pq} are defined as in §4. Thus we retrieve the cup product over the integers in the Morse complex.

Example 12.3. A fundamental example of a pair satisfying our hypotheses is given by $f, -f$ where the gradient flow is Morse-Stokes. In this case $U'_p = S_p$ and $S'_p = U_p$ for all $p \in Cr(f) = Cr(-f)$. Thus formula (12.1) becomes

$$\mathbf{M}(\alpha,\beta) \;=\; \sum_{p,p'\in Cr(f)} \left(\int_{U_p}\alpha\right)\left(\int_{S_{p'}}\beta\right)[S_p \cap U_{p'}].$$

In particular we have the following.

Proposition 12.4. *Suppose the gradient flow of f is Morse-Stokes. Then for any cycle Y in X which is transversal to all the U_p and S_p, $p \in Cr(f)$, we have the formula*

$$\int_Y \alpha\wedge\beta \;=\; \sum_{p,p'\in Cr(f)} \left(\int_{U_p}\alpha\right)\left(\int_{S_{p'}}\beta\right)[S_p \cap U_{p'} \cap Y]. \tag{12.6}$$

whenever $d\alpha = d\beta = 0$.

Example 12.5. Suppose $\deg\alpha = n - \deg\beta = k$. Then (cf. [23, Prop. 12])

$$\int_X \alpha\wedge\beta \;=\; \sum_{p\in Cr_k(f)} \left(\int_{U_p}\alpha\right)\left(\int_{S_{p'}}\beta\right)$$

13. A Lefschetz theorem for the Thom-Smale complex

Let X be a compact oriented riemannian manifold, and let U_p, S_p, $p \in Cr(f)$ be the unstable and stable manifolds of a Morse-Stokes flow on X, oriented as in §2. Recall the **Lefschetz number** of a smooth mapping $F : X \to X$ defined by

$$\mathrm{Lef}(F) \;=\; \sum_i (-1)^i \mathrm{trace}\,\{F_* : H_i(X;\mathbf{R}) \to H_i(X;\mathbf{R})\}$$

Theorem 13.1. *Suppose $F : X \to X$ is a smooth mapping such that F maps S_p transversally to U_q for all $q \succ p$ (i.e., $F\big|_{S_p}$ is transversal to U_p and $F(S_p) \cap U_q = \emptyset$ for all $q \succ p$ and $q \neq p$). Let $C_p = \{x \in S_p : F(x) \in U_p\}$. Then*

$$\mathrm{Lef}(F) \;=\; \sum_{p \in Cr(f)} (-1)^{\lambda_p} \sum_{x \in C_p} \sigma_x(F)$$

where

$$\sigma_x(F) \;\stackrel{\text{def}}{=}\; \begin{cases} 1 & \text{if } F_*T_x(S_p) \text{ agrees in orientation with the normal} \\ & \text{space to } U_p \text{ at } F(x) \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Our transversality assumption implies that the graph $\Gamma_F = \{(F(x), x) : x \in X\}$ in $X \times X$ meets the cycle P only in its regular points $\bigcup_p U_p \times S_p$ and it is transversal there. We recall that $\mathrm{Lef}(F) = [\Delta] \bullet [\Gamma_F]$ in $X \times X$. By (2.5) and the fact that Γ_F and P meet nicely, we conclude that $\mathrm{Lef}(F) = [P] \bullet [\Gamma_F]$ which is easily computed as claimed. q.e.d.

Note that when $F = \mathrm{Id}$, the hypotheses are satisfied and we get the standard computation of the Euler characteristic from the Morse complex.

14. The genericity theorem

Let $f \in C^\infty(X)$ be a Morse function on a compact manifold X. In this section we shall prove that there exists a riemannian metric on X for which the gradient flow satisfies the Morse-Stokes conditions of §2. The authors have subsequently discovered that some of the material in this section could be reduced by appealing to a paper of Laudenbach

[23]. For completeness we have kept the details. To begin we recall the following.

Definition 14.1. The gradient flow of f for a riemannian metric on X is called **Morse-Smale** if the stable and unstable manifolds S_p and U_q intersect transversely for all $p, q \in Cr(f)$.

To simplify arguments we shall demand a little more. Recall that at each $p \in Cr(f)$ there exist **canonical local coordinate systems** $(u, v) : \mathcal{O}_p \overset{\cong}{\to} \mathcal{V}_p$ where

$$(14.1) \qquad \mathcal{V}_p = \{(u, v) \in \mathbb{R}^{\lambda_p} \times \mathbb{R}^{n-\lambda_p} : |u|^2 \le r_p \text{ and } |v|^2 \le r_p\}$$

such that $u(p) = v(p) = 0$ and $f(u, v) = f(p) - |u|^2 + |v|^2$.

Definition 14.2. A riemannian metric ds^2 is said to be **canonically flat near** $Cr(f)$ if $ds^2 = |du|^2 + |dv|^2$ in some canonical linear coordinate system about each $p \in Cr(f)$.

We shall prove the following.

Theorem 14.3. *Let $f \in C^\infty(X)$ be a Morse function on a compact manifold X. Suppose X is given a riemannian metric which is canonically flat near $Cr(f)$ and for which the gradient flow φ_t is Morse-Smale. Then φ_t satisfies the Morse-Stokes conditions 2.1*

Theorem 14.4. *If $f \in C^\infty(X)$ is a Morse function and ds^2 is any riemannian metric on X, then ds^2 can be modified outside some neighborhood of $Cr(f)$ so that φ_t becomes Morse-Smale. In fact this modification can be made arbitrarily small in the C^∞-topology.*

Taken together these theorems prove the following.

Theorem 14.5. *Given any Morse function f on a compact manifold X, there exists a riemannian metric on X for which the gradient flow is Morse-Stokes.*

Furthermore, this metric can be chosen to be canonically flat near $Cr(f)$.

Proof of Theorem 14.3. We first observe that if the flow of f is Morse-Smale, then

$$(14.2) \qquad p \prec q \qquad \Rightarrow \qquad \lambda_p < \lambda_q$$

for all $p, q \in Cr(f)$. To see this suppose p and q are joined by an (unbroken) flow line ℓ. Then $U_p \cap S_q \supset \ell$ and so by the transversality condition, $\dim(U_p \cap S_q) = (n - \lambda_p) + \lambda_q - n \ge 1$.

Now let $a_1 < \cdots < a_m$ be the critical values of f. For each a_k let $Cr(f, a_k) \subset Cr(f)$ be the set of critical points of f with critical value a_k. Each $p \in Cr(f, a_k)$ has a canonical local coordinate system as in (14.1) where the metric is flat. We may assume that the radius r_p is the same for all $p \in Cr(f, a_k)$. Call this radius r_k. By shrinking the neighborhoods $\mathcal{O}_p$ we may assume that these canonical coordinate systems are pairwise disjoint and that $a_{k+1} - r_{k+1} > a_k + r_k$ for all k. Furthermore, by multiplying f by some scalar $a >> 1$ and further shrinking the $\mathcal{O}_p$ we can assume that $r_p = 2$ for all $p \in Cr(f)$.

Now our manifold decomposes into "blocks":

$$X \;=\; P_0 \cup Q_0 \cup P_1 \cup Q_1 \cup P_2 \cup Q_2 \cup \ldots P_m \tag{14.3}$$

where

$$P_k \;=\; f^{-1}[a_k - 1, a_k + 1] \qquad \text{and} \qquad Q_k \;=\; f^{-1}[a_k + 1, a_{k+1} - 1].$$

Note that P_k and Q_k are compact manifolds with boundary. The manifolds P_k can be further decomposed. Let $\mathcal{O}'_p \subset \mathcal{O}_p$ be the subset defined by the equations $|u||v| \leq 1$ and $-1 \leq |v|^2 - |u|^2 \leq 1$. Then

$$P_k \;=\; R_k \cup \bigcup_{p \in Cr(f, a_k)} \mathcal{O}'_p$$

where R_k is the closure of $P_k - \bigcup_p \mathcal{O}'_p$.

Let ψ_s be the (incomplete) flow on $X - Cr(f)$ generated by $W \equiv \operatorname{grad} f / \|\operatorname{grad} f\|^2$, so that

$$f(\psi_s(x)) = f(x) + s$$

whenever $\psi_s(x)$ is defined. Using this vector field in the obvious way (cf. [27]) we obtain smooth product structures

$$Q_k \;\cong\; (\partial^- Q_k) \times [0, 1] \tag{14.4}$$

$$R_k \;\cong\; (\partial^- R_k) \times [0, 1] \tag{14.5}$$

where $\partial^- Q_k = Q_k \cap f^{-1}(a_k + 1)$ and $\partial^- R_k = R_k \cap f^{-1}(a_k - 1)$. Note that $\partial^- R_k$ is a compact manifold with non-empty boundary.

Consider now the unstable manifold U_p for some $p \in Cr(f, a_k)$ and some k. We shall show that $\operatorname{vol}(U_p) < \infty$. To begin note that $U_p \cap$

$\mathcal{O}'_p \cong \{(0,v) \in \mathcal{V}_p \ : \ |v| \le 1\}$ is a smoothly embedded closed disk of dimension $\ell \stackrel{\text{def}}{=} n - \lambda_p$ and clearly has finite ℓ-volume. Its boundary $U_p \cap (\partial^- Q_k)$ is a smoothly embedded sphere, and via (14.4) we have $U_p \cap Q_k \cong (U_p \cap \partial^- Q_k) \times [0,1]$ which also has finite ℓ-volume.

We now show that $U_p \cap P_{k+1}$ has finite ℓ-volume. To begin note that via (14.5) we have a smooth product $U_p \cap R_{k+1} \cong (U_p \cap \partial^- R_{k+1}) \times [0,1]$ (which extends beyond the boundary of R_{k+1} so we needn't worry about how U_p meets this boundary). Since $U_p \cap \partial^- R_{k+1}$ is a subset of a compact $(\ell-1)$-manifold, we see that $U_p \cap R_{k+1}$ has finite ℓ-volume.

It remains to show that $U_p \cap \mathcal{O}'_q$ has finite volume for $q \in Cr(f, a_k)$. This is equivalent to showing that $U_p \cap \mathcal{O}''_q$ has finite volume, where $\mathcal{O}''_q$ is defined by $|u| \le 1$ and $|v| \le 1$. (To see this push inward along the flow.) For simplicity, from here on we shall denote $\mathcal{O}''_q$ by $\mathcal{O}_q$ and $\mathcal{V}''_q$ by $\mathcal{V}_q$.

Since $q \succ p$ we know from (14.2) that

$$\dim U_p \ > \ \dim U_q.$$

In our local coordinate box

$$\mathcal{V}_q = \{(u,v) \in \mathbb{R}^{\lambda_q} \times \mathbb{R}^{n-\lambda_q} \ : \ |u| \le 1, \ |v| \le 1\}$$

the flow is generated by

$$\varphi_t(u,v) \ = \ (e^{-t}u, e^t v) \tag{14.6}$$

We decompose $\partial \mathcal{V}_q$ into two pieces:

$$A = \{(u,v) \ : \ |u| = 1, \ |v| \le 1\} \qquad \text{and}$$
$$B = \{(u,v) \ : \ |u| \le 1, \ |v| = 1\}.$$

There are subsets

$$A_0 = A \cap S_q = \{(u,0) \ : \ |u| = 1\} \cong S^{\lambda_q - 1},$$

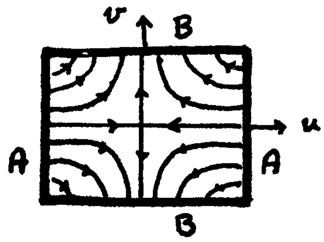
$$B_0 = B \cap U_q = \{(0,v) \ : \ |v| = 1\} \cong S^{n-\lambda_q-1}.$$

The flow determines a diffeomorphism

$$\Phi : A - A_0 \stackrel{\cong}{\to} B - B_0$$

given by

$$\Phi(u,v) \ = \ \left(|v|u, \frac{1}{|v|}v\right)$$



Note that $\Phi(u, v)$ is the unique point in B which lies on the flow line through (u, v).

Let $\widetilde{A} \overset{\pi_A}{\to} A$ be the "oriented blow-up" of A along A_0 where A_0 is replaced by the oriented normal lines to A_0. Let $\widetilde{B} \overset{\pi_B}{\to} B$ be defined similarly. $\widetilde{A}$ has coordinates $(\hat{u}, \hat{v}, t)$ where $|\hat{u}| = 1$, $|\hat{v}| = 1$ and $0 \leq t \leq 1$ and $\pi_A(\hat{u}, \hat{v}, t) = (\hat{u}, t\hat{v})$. $\widetilde{B}$ has the same coordinates with $\pi_B(\hat{u}, \hat{v}, t) = (t\hat{u}, \hat{v})$.

Φ lifts to a map $\widetilde{\Phi} : \widetilde{A} \longrightarrow \widetilde{B}$ which in these coordinates is the *identity map* (and so, in particular, a diffeomorphism).

Now since U_p is transversal to S_q we know that:

$$(?) \qquad (U_p \cap S_q) \cap \mathcal{V}_q \;\cong\; C(U_p \cap A_0)$$

where $U_p \cap A_0$ is a compact submanifold of codimension λ_p in A_0, and where for any subset $Y \subset A_0$, $C(Y)$ is the **cone** on Y defined by

$$C(Y) \;=\; \{(tu, 0) \;:\; u \in Y \text{ and } 0 \leq t \leq 1\}$$

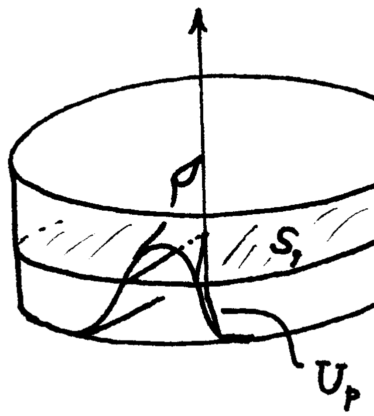
Now transversality also tells us that in a neighborhood of

$$Y \equiv U_p \cap A_0$$

the set $U_p \cap A$ is of the form $Y \times \mathbb{R}^{n-\lambda_q}$. Furthermore, $U_p \cap A$ has a "smooth proper transform" to $\widetilde{A}$, i.e., the lift of $U_p \cap (A - A_0)$ to $\widetilde{A}$ has closure which is a smooth manifold with boundary diffeomorphic to $Y \times S^{n-\lambda_q-1}$. Denote this closure by $\widetilde{U_p \cap A}$.

We can now describe the structure of $\overline{U}_p$ in our coordinate box. To begin we observe that

$$\overline{U_p \cap B} \;=\; \pi_B \widetilde{\Phi}(\widetilde{U_p \cap A})$$



and recall that $\widetilde{\Phi}$ is a diffeomorphism. It follows that $\overline{U_p \cap B}$ is a C^∞ stratified set with two strata. The top stratum is $U_p \cap B$. The singular stratum is exactly $B_0 \cong S^{n-\lambda_q-1}$. In a neighborhood of this singular stratum, $\overline{U_p \cap B}$ is diffeomorphic to $C(Y) \times S^{n-\lambda_q-1}$ where $Y = U_p \cap A_0$.

Since $\overline{U_p \cap B}$ is the image of a manifold of finite $(\ell - 1)$-volume (namely $\widetilde{U_p \cap A}$) under the smooth proper map $\pi_B \circ \widetilde{\Phi}$, it follows that $\overline{U_p \cap B}$ has finite $(\ell - 1)$-volume.

Conclusion 14.6 The closure of $U_p \cap \partial^- Q_{k+1}$ is a compact C^∞-stratified set of finite $(\ell - 1)$-volume, whose top stratum is $U_p \cap \partial^- Q_{k+1}$ and whose singular strata are exactly the spheres $U_q \cap \partial^- Q_{k+1}$ for $q \in Cr(f, a_{k+1})$.

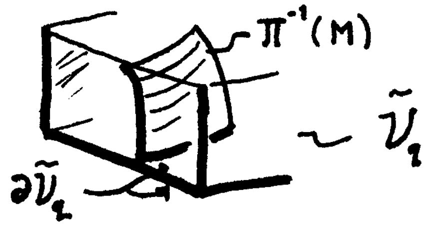
From the above analysis we can also conclude that U_p has finite volume in $\mathcal{V}_q$. Set

$$\widetilde{\mathcal{V}}_q \;=\; S^{\lambda_q - 1} \times S^{n-\lambda_q-1} \times [0,1] \times [0,1]$$

and consider the map $\Pi : \widetilde{\mathcal{V}}_q \to \mathcal{V}_q$ given by

$$\Pi(\hat{u}, \hat{v}, s, t) \;=\; (s\hat{u}, t\hat{v})$$

where $|\hat{u}| = |\hat{v}| = 1$. We identify $\widetilde{A}$ with the subset where $s = 1$ (and $\widetilde{B}$ with $t = 1$). Then Π restricts to be the projections π_A and π_B.



Consider the mapping

$$\Psi : \widetilde{\mathcal{V}}_q \longrightarrow \widetilde{A}$$

given by

$$\Psi(\hat{u}, \hat{v}, s, t) \;=\; (\hat{u}, \hat{v}, 1, st).$$

Then Ψ has the following two important properties. Let $M \subset \widetilde{A}$ be a closed submanifold with $\partial M \subset \partial\widetilde{A}$ which is transversal to $\partial_0\widetilde{A} = S^{\lambda_q-1} \times S^{n-\lambda_q-1} \times \{1\} \times \{0\}$. Then

$$\Psi \;\text{ is strongly transversal to }\; M \qquad \text{(See below)},$$

$$\Pi\left(\Psi^{-1}(M)\right) \;=\; \mathcal{V}_q \cap \bigcup_{t \geq 0} \varphi_t\left(\pi_A M\right)$$

By *strong transversality* we mean that if we extend M beyond $\partial_0\widetilde{A}$ by adding a collar, and if we extend Π to $S^{\lambda_q-1} \times S^{n-\lambda_q-1} \times (-\epsilon, 1] \times (-\epsilon, 1]$ using the same algebraic formula, then the extended Π is transversal to the extended M. Hence, $(\Pi^{-1}(M), \partial\Pi^{-1}(M))$ is a smooth submanifold with corners neatly embedded into $(\widetilde{\mathcal{V}}_q, \partial\widetilde{\mathcal{V}}_q)$.
In particular, if $\dim(M) = \ell - 1$, then$\Psi^{-1}(M)$ has finite ℓ-volume and so does $\Pi(\Psi^{-1}(M))$

Remark 14.7. Note that the above comments also apply if M is a compact C^∞ stratified set in $\widetilde{A}$ which in a collar neighborhood $\partial_0\widetilde{A}\times[0, \epsilon)$ of $\partial_0\widetilde{A}$ is of the form $M_0 \times [0, \epsilon)$.

We have now passed the first critical level beyond a_k and the closure of U_p has exited as a compact C^∞ stratified set, whose singularities are the submanifolds U_q for $q \succ p$ and $q \in Cr(f, a_{k+1})$.

We now compound the process. The product structure (14.4) shows that $U_p \cap Q_{k+1}$ has finite ℓ-volume (since it is a product of a smooth stratified set of finite $(\ell-1)$-volume with $[0,1]$). For similar reasons, $U_p \cap R_{k+2}$ has finite ℓ-volume, and it remains to examine what happens as we pass through the canonical coordinate boxes of critical points q at level $k+2$.

At points of $U_p \cap S_q$ for $q \in Cr(f, a_{k+2})$, the above analysis can be applied locally. However, to prove finite volume one must also consider points $x \in \overline{U}_p \cap S_q$ where $\overline{U}_p$ is singular. At such points x the singular set of $\overline{U}_p \cap S_q$ consists of all points of $U_{q'}$ in a neighborhood of x, for some $q' \prec q$. Now $U_{q'}$ is transversal to S_q, and $\overline{U}_p$ is locally of the form $C(Y) \times U_{q'}$ for some manifold Y. Consequently $\overline{U}_p \cap S_q$ is locally of the form $C(Y) \times \mathbb{R}^i$ and $\overline{U}_p$ is locally of the form $C(Y) \times \mathbb{R}^i \times \mathbb{R}^{n-\lambda_q}$ for some $i > 0$. In particular, $\overline{U}_p \cap A$ has a smooth proper transform to $\widetilde{A}$. This proper transform has a neat collar structure at the boundary as discussed in 14.7.

Applying the analysis above one concludes that $U_p \cap \partial^- Q_{k+2}$ is a compact C^∞ stratified set of finite ℓ-volume whose singular strata consist precisely of the sets $U_q \cap \partial^- Q_{k+2}$ for critical points $q \succ p$ of level $\leq k+2$. The same analysis also shows (using Remark 14.7) that U_q has finite ℓ-volume in a neighborhood of each critical point of level $k+2$.

One can now proceed inductively through the critical levels to prove that U_p has finite ℓ-volume.

Since S_p is the unstable manifold for -grad(f) = grad$(-f)$, we have also proved that each S_p has finite volume.

It remains to prove that the graph $\mathcal{T}$ has finite volume. For this we first observe that the arguments above apply directly to prove the following result. We say that the gradient flow of a smooth function F with non-degenerate critical manifolds is *tame* in a neighborhood of each critical point there are coordinates (x, u, v) such that $\nabla F = (0, -2u, 2v)$.

Proposition 14.8. *Let Φ_t be a tame gradient flow of a proper function $F : X \to \mathbb{R}$ with non-degenerate critical manifolds. Let c be a non-critical value of F and suppose that $\Sigma \subset \{F \leq c\}$ is defined as the backward time image of a compact manifold $\Sigma_0 \subset \{F = c\}$. Suppose that Σ is transversal to all the unstable manifolds of the flow. Then Σ has finite volume.*

We now consider the gradient flow

$$\Phi_t(s, x, y) \;=\; \big(e^t s, \varphi_{-t}(y), x\big)$$

of the function

$$F(s, x, y) \;=\; \frac{1}{2}s^2 - f(y)$$

on $\mathbb{R} \times X \times X$ (with metric ds^2 on $\mathbb{R}$). The critical set of F consists of the non-degenerate critical manifolds

$$P_p \;=\; \{0\} \times \{p\} \times X$$

for $p \in Cr(f)$. The stable and unstable manifolds at P_p are given by

$$\mathbf{S}_p = \{0\} \times U_p \times X \qquad \text{and} \qquad \mathbf{U}_p = \mathbb{R} \times S_p \times X.$$

We consider the invariant manifold

$$\widetilde{\mathcal{T}} \;\equiv\; \big\{(e^{-s}, \varphi_s(x), x) \;:\; -\infty < s < \infty \;\text{ and }\; x \in X\big\}$$

and the subdomain

$$\mathcal{T} \;\equiv\; \big\{(e^{-s}, \varphi_s(x), x) \;:\; 0 \leq s < \infty \;\text{ and }\; x \in X\big\}.$$

Note that $\widetilde{\mathcal{T}}$ is merely the union of orbits passing through $\Delta = \{(1, x, x) : x \in X\}$ and

$$\mathcal{T} \;=\; \bigcup_{s \leq 0} \Phi_s(\Delta)$$

is the union of the backward time orbits of Φ which begin at Δ. Recall (cf. Remark 1.6) that $\text{vol}(\mathcal{T}) < \infty \Rightarrow \text{vol}(T) < \infty$. Thus to complete the proof w shall prove $\text{vol}(\mathcal{T}) < \infty$ by applying Proposition 14.8.

To begin, a straightforward check verifies that

(14.7) $\qquad \widetilde{\mathcal{T}}$ is transversal to $\mathbf{S}_p$ and to $\mathbf{U}_p$ for all p.

Now the intersection

$$\widetilde{\mathcal{T}}(c) \;\stackrel{\text{def}}{=}\; \widetilde{\mathcal{T}} \cap \{F = c\},$$

is always a smooth submanifold since $\widetilde{\mathcal{T}}$ is Φ_t-invariant, i.e., $\text{grad}(F\big|_{\mathcal{T}}) = \text{grad}(F)\big|_{\mathcal{T}} \neq 0$ on $\widetilde{\mathcal{T}}$. Furthermore, if $c > \max|f|$, then $\widetilde{\mathcal{T}}(c)$ is compact. To see this note that $\widetilde{\mathcal{T}}(c)$ is not compact iff there exists a sequence $(s_j, x_j) \in \mathbf{R} \times X$ with

$$F(e^{s_j}, x_j, \varphi_{s_j}(x_j)) = \frac{1}{2}(e^{2s_j} - f(\varphi_{-s_j}(x)) = c$$

for all j and for which there is no convergent subsequence. Clear for such a sequence we have $s_j \to -\infty$ and so $e^{2s_j} \to 0$ implying $c \leq \max|f|$ as claimed. It therefore follows from Proposition 14.8 that for any $c > \max|f|$, the submanifold $\widetilde{\mathcal{T}}_{\leq c} \equiv \widetilde{\mathcal{T}} \cap \{F \leq c\}$ has finite volume in $\mathbf{R} \times X \times X$.

Since $\underset{\mathcal{T}}{\rightarrow} \max F = \frac{1}{2} + \max(f)$, we have $\mathcal{T} \subset \widetilde{\mathcal{T}}_{\leq c}$ for all $c > \max|f| + \frac{1}{2}$. Thus $\mathcal{T}$ has finite volume, and the proof is complete. q.e.d.

Proof of Theorem 14.4. The following proof is inspired by arguments of Milnor given for a similar result [28, §4]. Consider the block decomposition (14.3) and the product structure 14.4 given by the flow. Proceeding in order from $k = 1$ we shall modify the metric on each subset

$$\partial^- Q_k \times (\frac{1}{3}, \frac{2}{3}) \subset \partial^- Q_k \times [0,1]$$

so that under the new gradient flow (which agrees with the old one outside $\partial^- Q_k \times (\frac{1}{3}, \frac{2}{3})$), the unstable manifolds entering $\partial^- Q_k \times \{0\}$ become transversal to the stable manifolds at $\partial^- Q_k \times \{1\}$. By invariance under the flow this implies that each unstable manifold which meets Q_k is transversal everywhere on X to each stable manifold which meets Q_k. Modifying the metric at level k does not change the unstable manifolds below $\partial^- Q_k$. It also does not change the stable manifolds which originate below level k. From this one sees that after successively modifying the metric at level k for $k = 1, ..., m-1$, we have established the result.

We now show how to modify the metric. Fix a level k and consider the submanifolds

$$M_p = U_p \cap \partial^- Q_k \qquad \text{and} \qquad N_q = S_q \cap \partial^- Q_k$$

for $p, q \in Cr(f)$. We want to change the metric over $\partial^- Q_k \times (\frac{1}{3}, \frac{2}{3})$ so that after pushing each $M_p \times \{0\}$ forward by the new gradient flow, it becomes transversal to $N_q \times \{1\}$ for all q. We shall do this as follows. We shall construct a family of deformations of the metric, smoothly parameterized by an open subset $\mathcal{U} \subset \mathbb{R}^N$ for some N. This family will induce a smooth mapping

$$\mathcal{U} \times (\partial^- Q_k \times \{0\}) \xrightarrow{\Theta} \partial^- Q_k \times \{1\}$$

such that each $\Theta_u(\bullet) \equiv \Theta(u, \bullet)$ is the diffeomorphism induced by the new gradient flow (in the product structure of the old gradient flow).

By Sard's Theorem for Families it will suffice to prove that $\Theta\big|_{\mathcal{U} \times M_p}$ is transversal to N_q for all p, q (since then for almost all choices of $u \in \mathcal{U}$, we have that $\Theta_u\big|_{M_p}$ is transversal to N_q for each p, q). This condition will follow automatically if we show that

$$(14.8) \qquad \frac{\partial\Theta}{\partial u} \text{ is surjective at all points of } \mathcal{U} \times \partial^- Q_k$$

In fact it will suffice to show that there exists $u_0 \in \mathcal{U}$ such that

$$(14.9) \qquad \frac{\partial\Theta}{\partial u} \text{ is surjective at all points of } \{u_0\} \times \partial^- Q_k$$

for then, by the compactness of $\partial^- Q_k$, condition (14.11) will hold with $\mathcal{U}$ replaced by a small neighborhood of u_0 in $\mathcal{U}$.

We first construct such a map locally on $\partial^- Q_k$. Let $x = (x_1, ..., x_{n-1}) : \mathcal{O} \overset{\approx}{\to} \mathbb{R}^{n-1}$ be a local coordinate chart on Q_k (which maps *onto* $\mathbb{R}^{n-1}$). We then have coordinates $(x, t) \in \mathbb{R}^{n-1} \times [0, 1]$ for $\partial^- Q_k \times [0, 1]$ where $f(x, t) = t$. In these coordinates the given riemannian metric has the form

$$\langle\cdot,\cdot\rangle_{(x,t)} \ = \ (\cdot,\cdot)_{(x,t)} + dt^2$$

where $(\cdot,\cdot)$ is a family of inner products on $\mathbb{R}^{n-1}$.

Let $\mathbb{M}$ denote the space of symmetric $n \times n$-matrices. Fix a non-trivial smooth function $\psi : [0, 1] \to [0, 1]$ with support in $(\frac{1}{3}, \frac{2}{3})$. For each $A \in \mathbb{M}$ with eigenvalues of absolute value < 1, we define a new metric $\langle\cdot,\cdot\rangle_A$ by

$$(14.10) \qquad \langle\cdot,\cdot\rangle_A \ = \ \langle\cdot,\cdot\rangle + \psi\langle A(\cdot),\cdot\rangle.$$

Let $V = V(A)$ be the gradient of f in this new metric. Write $V = V' + V_0\partial/\partial t$ where V' is tangent to $\mathbb{R}^{n-1}$. Then for any vector field $W = W' + W_0\partial/\partial t$ we have that

$$W \cdot f = \langle V, W\rangle_A == \langle \partial/\partial t, W\rangle = \langle (I + \psi(t)A)V, W\rangle,$$

which implies that

$$\frac{\partial}{\partial t} \ = \ e_n \ = \ (I + \psi(t)A)V.$$

Our map Θ is given by taking the $\mathbb{R}^{n-1}$-component of the integral of the vector field $V = (I + \psi(t)A)^{-1}e_n$. We write the integal of V as $(\Theta_A(x, s), \tau_A(x, s))$ with respect to the decomposition $\mathbb{R}^{n-1} \times [0, 1] \subset \mathbb{R}^{n-1} \times \mathbb{R}$. Since V is translation invariant in the x-variables, we have

$$\tau_A(x, s) = \tau_A(s) \qquad \text{and} \qquad \Theta_A(x, s) = \Theta_A(s) + x$$

where $\Theta_A(s) = \Theta_A(0, s)$. Note that

$$\begin{aligned} \frac{d}{ds}\tau_A(s) &= n\text{-component of } (I + \psi(\tau_A(s))A)^{-1} \cdot e_n \\ &= (n,n)\text{-component of } \sum_{k=0}^{\infty} \psi(\tau_A(s))^k A^k. \end{aligned}$$

Similarly

$$\frac{d}{ds}\Theta_A(s) = \text{pr}\left\{(I + \psi(\tau_A(s))A)^{-1} \cdot e_n\right\}$$

where $\text{pr} : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is the linear projection. Hence for all A sufficiently small,

$$\begin{aligned} \Theta_A(1) &= \text{pr} \int_0^1 (I + \psi(\tau_A(s))A)^{-1} \cdot e_n \, ds \\ &= \text{pr} \int_0^1 \sum_{k=0}^{\infty} \psi(\tau_A(s))^k A^k \cdot e_n \, ds. \end{aligned}$$

Therefore, for each i, j with $1 \le i, j \le n$ we have

$$\begin{aligned} \left.\frac{\partial \Theta_A}{\partial A_{i,j}}\right|_{A=0} &= \text{pr} \int_0^1 \sum_{k=0}^{\infty} \frac{\partial}{\partial A_{i,j}} \psi(\tau_A(s))^k A^k \cdot e_n \, ds \Big|_{A=0} \\ &= \text{pr} \int_0^1 \{-\psi(\tau_0(s))E_{i,j}\} \cdot e_n \, ds \\ &= -\int_0^1 \psi(s) \, ds \begin{cases} E_{in} & \text{if } i > j = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

where E_{ij} is the elementary (i, j)-matrix. This proves the surjectivity of $\partial\Theta/\partial A$ in this coordinate system.

We now modify this family by replacing A with $\xi(x)A$ where $\xi \in C_0^\infty(\mathbb{R}^{n-1})$ satisfies $\xi(x) = 1$ for $|x| \le 2$. This family of deformations now extends trivially to all of $\partial^- Q_k$ and agrees with the one above in a neighborhood of $B_1 = \{|x| \le 1\}$.

We now choose a finite family of such local coordinate systems $x^k : \mathcal{O}_k \to \mathbb{R}^{n-1}$, $k = 1, ..., \nu$ such that the open sets $(x^k)^{-1}(B_1)$ cover $\partial^- Q_k$. Let $\alpha_k = \xi_k A_k$, for $A_k \in \mathbb{M}$, be the global section of $\text{End}(T(\partial^- Q_k))$ defined above, and consider the deformations of the metric given by

$$\langle \cdot, \cdot \rangle_{A_1,...,A_\nu} = \langle \cdot, \cdot \rangle + \psi(t) \sum_{j=1}^{\nu} \langle \xi_j A_j(\cdot), \cdot \rangle$$

for $\vec{A} = (A_1, ..., A_\nu) \in \mathbb{M}^\nu$. Then our calculation above shows that

$$\left.\frac{\partial \Theta_{\vec{A}}}{\partial \vec{A}}\right|_{\vec{A}=0} \quad \text{is surjective}$$

as desired. q.e.d.

Appendix: Currents and the kernel calculus

Let Z be a compact smooth n-dimensional manifold which is not necessarily orientable. In addition to the space $\mathcal{E}^k(Z)$, of smooth differential forms of degree k, one may consider the space $\widetilde{\mathcal{E}}^k(Z)$, of *twisted* smooth forms on Z of degree k. These are sections of the bundle $\Lambda^k T^* Z \otimes_{\mathbf{R}} \mathcal{O}_Z$ where $\mathcal{O}_Z$ is the orientation line bundle for Z. Since the transition functions for $\mathcal{O}_Z$ are $\mathbf{Z}_2 = \{-1, +1\} \subset \mathbf{R}$ valued, exterior differentiation d is naturally defined on any twisted form $\tilde{\alpha} \in \widetilde{\mathcal{E}}^k(Z)$. The resulting cohomology groups are the de Rham groups $H^k(Z; \mathcal{O}_Z)$ of Z with coefficients in the flat line bundle $\mathcal{O}_Z$.

A basic fact is that for any twisted n-form $\tilde{\alpha} \in \widetilde{\mathcal{E}}^n(Z)$ the integral $\int_Z \tilde{\alpha}$ is well defined. (This generalizes the usual definition since, if Z is oriented, then $\widetilde{\mathcal{E}}^k(Z) \cong \mathcal{E}^k(Z)$ are identified.) Following de Rham and Schwartz [31] we have the following.

Definition A.1. The space of **currents of degree** k **on** Z is the topological dual space

$$\mathcal{D}'^k(Z) \stackrel{\text{def}}{=} \widetilde{\mathcal{E}}^{n-k}(Z)'$$

of the space of twisted $(n-k)$-forms on Z.

Currents of degree k are a generalization of differential forms of degree k. In fact there is an embedding

$$\mathcal{E}^k(Z) \hookrightarrow \mathcal{D}'^k(Z) \tag{14.1}$$

which associates to $\alpha \in \mathcal{E}^k(Z)$, the current defined by

$$\alpha(\tilde{\beta}) \equiv \int_Z \alpha \wedge \tilde{\beta}, \qquad \text{for all } \tilde{\beta} \in \widetilde{\mathcal{E}}^{n-k}(Z). \tag{A.2}$$

Since $\int_Z d\alpha \wedge \tilde{\beta} = \int_Z d(\alpha \wedge \tilde{\beta}) - (-1)^k \int_Z \alpha \wedge d\tilde{\beta} = (-1)^{k-1} \int_Z \alpha \wedge d\tilde{\beta}$, it is natural to define the exterior derivative of a current $T \in \mathcal{D}'^*(Z)$ by:

(A.3) $$(dT)(\tilde{\beta}) = (-1)^{k-1}T(d\tilde{\beta}), \qquad \text{for all } \tilde{\beta} \in \widetilde{\mathcal{E}}^{n-k}(Z).$$

Thus the de Rham complex of smooth forms $(\mathcal{E}^*(Z), d)$ is a subcomplex of the de Rham complex of currents $(\mathcal{D}'^*(Z), d)$. (In fact $\mathcal{D}'^*(Z)$ are exactly the distributional sections of Λ^*T^*Z, and d the natural extension of exterior differentiation to these sections.)

Example A.2. Let S be a codimension-k submanifold of Z which has finite volume and *oriented normal bundle*. The identification $\mathcal{O}_Z\big|_S = \mathcal{O}_S$ enables us to pull back twisted forms via the immersion $i : S \to Z$, and so S **determines a current** $[S] \in \mathcal{D}'^*(\mathbf{Z})$ **by integration**:

(14.2) $$[S](\tilde{\beta}) = \int_S i^*(\tilde{\beta}), \qquad \text{for all } \tilde{\beta} \in \widetilde{\mathcal{E}}^*(Z).$$

It is sometiimes also useful to consider the **boundary operator** ∂ which is defined to be the dual of d on $\widetilde{\mathcal{E}}^*(Z)$, i.e.

$$\partial = (-1)^{k-1}d \qquad \text{on} \qquad \mathcal{D}'^k(Z).$$

Remark A.3. Note that in general an oriented compact submanifold does *not* define a current, but does define a **twisted current**, i.e., a linear functional on (untwisted) differential forms.

If $T \in \mathcal{D}'^p(Z)$ and $\alpha \in \mathcal{E}^q(Z)$ then the wedge product

(A.5) $$T \wedge \alpha \in \mathcal{D}'^{p+q}(Z) \text{ is defined by } (T \wedge \alpha)(\tilde{\beta}) = T(\alpha \wedge \tilde{\beta}).$$

More generally, for any flat bundle $E \to Z$ we have the spaces $\mathcal{E}^*(Z; E)$ of differential forms on Z with coefficients in E, and their extensions

(14.3) $$\mathcal{D}'^*(Z; E) \stackrel{\text{def}}{=} \mathcal{E}^{n-*}(Z; E^* \otimes \mathcal{O}_Z)'$$

to currents with coefficients in E.

Next we wish to represent operators on differential forms by currents on a product space. Let Y and X be manifolds with

$$\dim(Y) = n \qquad \text{and} \qquad \dim(X) = m$$

and note that $\mathcal{O}_{Y\times X} = \pi_Y^*\mathcal{O}_Y \otimes \pi_X^*\mathcal{O}_X$ where π_Y and π_X denote the projections to Y and X. Consider the space of differential forms $\mathcal{E}^*(Y \times \widetilde{X}) \overset{\text{def}}{=} \mathcal{E}^*(Y\times X; \pi_X^*\mathcal{O}_X)$ on $Y\times X$ twisted by the orientation line bundle $\pi_X^*\mathcal{O}_X$. For example $(\pi_Y^*\alpha) \wedge (\pi_X^*\tilde{\beta}) \in \mathcal{E}^*(Y \times \widetilde{X})$, if $\alpha \in \mathcal{E}^*(Y)$, $\tilde{\beta} \in \widetilde{\mathcal{E}}^*(X)$. The topological dual spaces

$$\mathcal{D}'^*(\widetilde{Y} \times X) \equiv \mathcal{D}'^*(Y \times X; \pi_Y^*\mathcal{O}_Y) \equiv \mathcal{E}^{n+m-*}(Y \times \widetilde{X})', \tag{14.4}$$

is called the space of **kernels** for operators from $\mathcal{E}^*(Y)$ to $\mathcal{D}'^*(X)$. Each kernel $K \in \mathcal{D}'^{n-r}(\widetilde{Y}\times X)$ determines an operator $\mathbf{K} : \mathcal{E}^*(Y) \to \mathcal{D}'^{*-r}(X)$ by setting

$$\mathbf{K}(\alpha)(\tilde{\beta}) = K(\pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta}). \tag{14.5}$$

Since

$$\pi_X^* : \widetilde{\mathcal{E}}^*(X) \to \mathcal{E}^*(Y \times \widetilde{X})$$

the dual map

$$(\pi_X)_* : \mathcal{D}'^*(\widetilde{Y} \times X) \to \mathcal{D}'^*(X)$$

pushes forward kernels on $Y \times X$ to currents on X. Now the right hand side of (A.7) can be rewritten as $((\pi_X)_*(K \wedge \pi_Y^*\alpha))(\tilde{\beta}))$, so that

$$(\text{A.8})'\mathbf{K}(\alpha) = (\pi_X)_*(K \wedge \pi_Y^*\alpha)$$

provides an alternate "pull-push" definition of **K**.

Proposition A.4. *Suppose $K \in \mathcal{D}'^{n-r}(\widetilde{Y} \times X)$ is a kernel whose operator $\mathbf{K}$ lowers degree by r. Then the kernel $\partial K \in \mathcal{D}'^{n-r+1}(\widetilde{Y} \times X)$, determines the operator*

$$\mathbf{K} \circ d + (-1)^{r-1} d \circ \mathbf{K}. \tag{14.10}$$

Proof. By definition

$$(\partial K)(\pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta}) = K(d(\pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta})$$

which equals

$$K(\pi_Y^*d\alpha \wedge \pi_X^*\tilde{\beta}) + (-1)^{\deg\alpha} K(\pi_Y^*\alpha \wedge \pi_X^*d\tilde{\beta}).$$

But

$$(\mathbf{K}(d\alpha))(\tilde{\beta}) = K(\pi_Y^* d\alpha \wedge \pi_X^* \tilde{\beta}),$$

and

$$\begin{aligned}(d(\mathbf{K}(\alpha)))(\tilde{\beta}) &\equiv (-1)^{\deg\alpha - r - 1}(\mathbf{K}(\alpha))(d\tilde{\beta}) \\ &= (-1)^{\deg\alpha - r - 1} K(\pi_Y^*\alpha \wedge \pi_X^* d\tilde{\beta}).\end{aligned}$$

q.e.d.

Now we list some examples of kernels and their corresponding operators.

Example A.5. Let $\Sigma \subset Y \times X$ be a finite volume submanifold of codimension-q with a given isomorphism $\pi_X^* \mathcal{O}_X\big|_\Sigma \cong \mathcal{O}_\Sigma$. Then integration of forms over Σ defines a current $[\Sigma] \in \mathcal{D}'^q(\tilde{Y} \times X)$.

Example A.6. (The Identity) The identity operator $\mathbf{I} : \mathcal{E}^*(X) \longrightarrow \mathcal{E}^*(X)$ on forms is represented by the kernel $I = [\Delta]$, corresponding to integration over the diagonal $\Delta \subset X \times X$, since

$$\alpha(\tilde{\beta}) \equiv \int_X \alpha \wedge \tilde{\beta} = \int_\Delta \pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta} = [\Delta](\pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta}).$$

Example A.7. (The Pull Back) Suppose $\varphi : X \longrightarrow Y$. Then the operator $\mathbf{P}_\varphi\, \alpha \equiv \varphi^*\alpha$ is represented by the kernel $P_\varphi = [\mathrm{graph}\,\varphi]$; that is

$$\mathbf{P}_\varphi \;=\; (\pi_X)_*([\mathrm{graph}\,\varphi] \wedge \pi_Y^*\alpha).$$

Example A.8. (Projection onto ψ along $\tilde{\varphi}$). Suppose $\psi \in \mathcal{E}^*(X)$ and $\tilde{\varphi} \in \widetilde{\mathcal{E}}^*(Y)$. Then $K = \pi_Y^*\tilde{\varphi} \wedge \pi_X^*\psi \in \mathcal{D}'^*(\tilde{Y} \times X)$ is a kernel inducing the operator

$$\mathbf{K}(\alpha) = \pm \left(\int_Y \tilde{\varphi} \wedge \alpha \right) \psi,$$

where $\pm$ equals $(-1)^{(\deg\psi)(\deg\alpha)}$. Note

$$K(\pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta}) = \int_{Y\times X} \pi_Y^*\tilde{\varphi} \wedge \pi_X^*\psi \wedge \pi_Y^*\alpha \wedge \pi_X^*\tilde{\beta}.$$

Example A.9. (Projection onto $[S]$ along $[U]$). Suppose S is a submanifold with oriented normal bundle in X and $[U]$ is an oiented submanifold of Y. Let $[S]$ denote the current in X and $[U]$ the twisted current in Y determined by integration. Then $K = [U] \times [S] \in \mathcal{D}'^*(\tilde{Y} \times X)$ is a kernel and the corresponding operator is $\mathbf{K}(\alpha) = \pm \left(\int_U \alpha\right)[S]$, where $\pm$ equals $(-1)^{(\deg S)(\deg\alpha)} = (-1)^{(\dim Y - \dim S)(\dim U)}$.

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