SINGULARITIES AND CHERN-WEIL THEORY, I

The Local MacPherson Formula

Dedicated to Kunihiko Kodaira, our teacher and friend

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Abstract.

Let $\alpha : E \to F$ be a smooth bundle map between vector bundles with connection on a manifold X, and let $\Phi(\Omega)$ be a Chern-Weil characteristic form of either E or F. A notion of "geometric atomicity" for α is introduced. For any such map α we establish a canonical cohomology

$$(*) \qquad \Phi(\Omega) - \sum_{k \geq 0} \mathrm{Res}_{\Phi,k}\,[\Sigma_k(\alpha)] \;=\; dT$$

where $\Sigma_k(\alpha) = \{x \in X \;:\; \dim\ker(\alpha) = k\}$, $\mathrm{Res}_{\Phi,k}$ is a smooth residue form along $\Sigma_k(\alpha)$, and T is a canonical L^1_{loc}-form on X. When rank E = rank F, (*) can be written

$$\Phi(\Omega^F) - \Phi(\Omega^E) \;=\; \sum_{k>0} \mathrm{Res}_{\Phi,k}\,[\Sigma_k(\alpha)] + dT.$$

Normal sections of $\mathrm{Hom}(E, F)$ (those by definition which are transversal to the universal singularity sets Σ_k) are always geometrically atomic, and for such maps equation (*) expresses a classical formula of R. MacPherson at the level of forms and currents. Every real analytic map α is geometrically atomic, no matter how misbehaved its singularities. For those where each $\Sigma_k(\alpha)$ has the expected dimension, analogous formulas are established. In all cases, each term in the sum in equation (*) is a d-closed current. Proofs entail a direct application of the methods of singular connections and of finite volume flows developed by the authors.

Geometrically atomic maps prove to be generic or "typical" in all structured situations such as: direct sum mappings, tensor product mappings, mappings given by Clifford multiplication, etc. In each case the methods yield new formulas. This will be done in Part II.

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§0. Introduction. Some of the most useful theorems in topology are those which relate singularities of maps to topological invariants, such as Hopf's Theorem on vector fields or the Lefschetz Fixed-point Theorem. One of the most general results of this type is the beautiful formula of R. MacPherson [Mac$_*$] which relates the topology of the primary singularities of a normal smooth bundle map $\alpha : E \to F$ to characteristic classes of E and F

In geometry the classical theory of Gauss-Chern-Weil relates topological invariants to local curvature data. Given two connections on a smooth bundle and a characteristic polynomial Φ, the theory produces a formula: $\Phi(\Omega_1) - \Phi(\Omega_2) = dT$, where Ω_i is the curvature of the i^{th} connection and T is a canonically defined smooth form. The gauge-invariant forms T are important in the study of the space of connections and they lead to well-known secondary invariants [CS], [ChS].

The aim here is to combine these results and derive MacPherson-type formulas locally on the manifold. Assume bundles E and F are equipped with metrics and connections, and let $\alpha : E \to F$ be a smooth bundle map. We shall derive formulas which explicitly express each Chern-Weil form $\Phi(\Omega)$ of E or F as a sum

$$\Phi(\Omega) \;=\; \sum_k \mathrm{Res}_{\Phi,k}[\Sigma_k(\alpha)] + dT \tag{0.1}$$

where $\Sigma_k(\alpha) = \{x : \dim\ker(\alpha_x) = k\}$, $\mathrm{Res}_{\Phi,k}$ is a smooth *residue form* defined along $\Sigma_k(\alpha)$, and T is a canonical *transgression form* with L^1_{loc}-coefficients. The sum on the right in (0.1) is a *characteristic current*. It is the Chern-Weil representative of the class Φ for a certain singular connection in the sense of [HL$_1$].

When rank E = rank F, equation (0.1) has the form

$$\Phi(\Omega^F) - \Phi(\Omega^E) \;=\; \sum_{k>0} \mathrm{Res}_{\Phi,k}\,[\Sigma_k(\alpha)] + dT,$$

expressing the difference of the Φ-characteristic classes of E and F in terms of the singularities of α.

If α is a *normal* bundle map (cf. Definition 9.3) on a compact manifold, then passing to cohomology in (0.1) yields MacPherson's formula.

However, there are many important types of bundle mappings which are far from normal, such as direct sum mappings

$$\alpha_1 \oplus \cdots \oplus \alpha_\ell : E_1 \oplus \cdots \oplus E_\ell \longrightarrow F_1 \oplus \cdots \oplus F_\ell,$$

tensor product mappings

$$\alpha_1 \otimes \cdots \otimes \alpha_\ell : E_1 \otimes \cdots \otimes E_\ell \longrightarrow F_1 \otimes \cdots \otimes F_\ell,$$

and mappings given by Clifford multiplication. We shall establish MacPherson-type formulas in all of these cases. In fact we shall present a method for deriving such formulas in any case of interest. The method is based on a "finite-volume" property of bundle maps called **geometric atomicity** – one of the key ideas of the paper. This property guarantees the existence of formulas for *every* characteristic polynomial Φ. It holds for normal bundle maps and for *all real analytic bundle maps.* Furthermore, it cuts robustly across the cases mentioned above. Within each special case the geometrically atomic maps are generic.

The concept of geometric atomicity strictly generalizes the notion of *atomicity* introduced in [HS], that is, any section $\alpha : \underline{\mathbf{R}} \to F$ which is atomic is geometrically atomic. Furthermore, there is an analytic criterion analogous to that in [HS], which implies geometric atomicity. This will be discussed in part II.

A basic feature of geometric atomicity is that it enables the construction of canonical homologies between universal singularity sets. (See §4.) The main ideas involved here carry over to dynamical systems and have yielded a new approach to Morse Theory [HL_3].

Geometric atomicity guarantees the existence of the limit of characteristic forms for the families of approximate push-forward connections constructed in [HL_1]. Here in Part I we examine the resulting formulas (0.1) in detail for normal maps and for real analytic maps whose singularity sets have the expected dimension. For each k, it is proved that

$$d([\Sigma_k(\alpha)]) \;=\; d(\mathrm{Res}_{\Phi,k}[\Sigma_k(\alpha)]) \;=\; 0.$$

We explicitly compute the residue forms in many cases. We also show they are completely canonical in the following sense. Along $\Sigma_k(\alpha)$ there are orthogonal splittings:

$$E \;=\; \ker\alpha \oplus \mathrm{Im}\,\alpha \qquad \text{and} \qquad F \;=\; \mathrm{coker}\;\alpha \oplus \mathrm{Im}\alpha$$

with respect to which $\alpha = 0 \oplus I$. The given connections induce direct sum connections with respect to these splittings. This in turn induces a connection on the bundle $\mathrm{Hom}(\ker\alpha, \mathrm{coker}\,\alpha)$ which is equivalent to the normal bundle of $\Sigma_k(\alpha)$. The residue form $\mathrm{Res}_{\Phi,k}$ is expressed directly, in the spirit of Chern-Weil, from these bundles and connections.

It is a philosophically significant point that all the formulas here drop out directly from the methods of singular connections introduced in [HL_1]. The idea is this. Given any bundle map $\alpha : E \to F$ between bundles with connection, one can construct canonical families $\overrightarrow{D}_s$, $0 \le s < \infty$, of smooth "push-forward" connections on F (and "pull-back" connections $\overleftarrow{D}_s$ on E) which begin with the given connection at infinity and limit to a "singular push-forward connection" (or "pull-back connection") at 0. Applying standard Chern-Weil theory to this family essentially yields the results. MacPherson's special blow-ups, the canonical residue forms, and (therefore) the topological formula all fall out.

It is possible that versions of the these local formulas over $\mathbf{Z}/2$ can be established using ideas and results in [HZ].

§1. Characteristic currents. Let E and F be smooth vector bundles of rank m and n respectively over a manifold X, and let

$$\alpha : E \to F$$

be a smooth vector bundle map. We suppose that E and F are provided with metrics and with connections D^E and D^F (which need not respect the metrics). From this data the authors have constructed in [HL_1] certain smooth 1-parameter families of connections $\overleftarrow{D}_t$ on E and $\overrightarrow{D}_t$ on F, for $0 < t \le \infty$, which connect the background connections

$$\overleftarrow{D}_\infty = D^E \qquad \text{and} \qquad \overrightarrow{D}_\infty = D^F$$

at time $t = \infty$ to certain "singular" *pullback* and *pushforward* connections at time $t = 0$ on E and F respectively. These limiting connections are well defined only outside the singularities of the map α. However, for Ad-invariant polynomials Φ and Ψ on the Lie algebras of the structure groups of E and F, it is possible that the limits

$$(1.1) \qquad \Phi(\!(\overleftarrow{D})\!) \equiv \lim_{t \to 0} \Phi(\overleftarrow{D}_t) \qquad \text{and} \qquad \Psi(\!(\overrightarrow{D})\!) \equiv \lim_{t \to 0} \Psi(\overrightarrow{D}_t)$$

exist in the space of generalized forms (i.e., currents) on X. Here $\Phi(\overleftarrow{D}_t) \equiv \Phi(\overleftarrow{\Omega}_t)$ denotes the smooth characteristic form obtained by applying Φ to the curvature 2-form $\overleftarrow{\Omega}_t$ of $\overleftarrow{D}_t$ in the standard way. In [$\mathrm{HL}_{1,2}$] it is shown that for certain classes of bundle mappings these limits, called **characteristic currents**, do exist and give rise to formulas of the sort

$$\Phi(\Omega^E) - \Phi(\!(\overleftarrow{D})\!) = dT \qquad \text{and} \qquad \Psi(\Omega^F) - \Psi(\!(\overrightarrow{D})\!) = dT'$$

where Ω^E, Ω^F denote the curvature 2-forms of E and F, and where T, T' are forms with L^1_{loc}-coefficients on X. Such formulas give a direct relationship between the singularities of the bundle map a and characteristic forms of E and F. They generalize classical results of Poincaré and Hopf and lead to a wide variety of interesting geometric residue theorems. (See [HL_2].)

For example in the category of real oriented bundles, suppose that $E = \underline{\mathbf{R}}$ is the trivial line bundle and Ψ is the normalized Pfaffian, so that $\Psi(\Omega^F) = \chi(\Omega^F)$ is the Euler-Chern form of F. Then for cross-sections $\alpha : \underline{\mathbf{R}} \longrightarrow F$ which are transversal to 0, one obtains

$$\chi(\Omega^F) \; - \; \mathrm{Div}(\alpha) \;=\; dT$$

where $\mathrm{Div}(\alpha)$ is the oriented submanifold of zeros of α. This result extends to quite general cross-sections of F, referred to as *atomic sections*. (See [HS].)

More generally one can consider the singularity sets

$$\Sigma_k(\alpha) \equiv \{x \in X \; : \; \dim\,(\ker \alpha) = k\} \tag{1.2}$$

for general k. There are similar results relating these singularities to Shur polynomials in the Chern classes (or Pontrjagin classes) of E and F (See [HL_2]). One also gets local versions of the differentiable Riemann-Roch Theorem for embeddings.

In the general case one expects to find a formula of the sort

$$\Psi(\Omega^F) \;=\; \sum_{k \geq 0} \mathrm{Res}_{\Psi,k}[\Sigma_k] + dT$$

where $\mathrm{Res}_{\Psi,k}$ is a smooth form defined on Σ_k universally in terms of Ψ. When $\mathrm{rank}(E) = \mathrm{rank}(F)$, it would have the form

$$\Psi(\Omega^F) - \Psi(\Omega^E) \;=\; \sum_{k > 0} \mathrm{Res}_{\Psi,k}[\Sigma_k] + dT$$

The point of this paper is to derive these general formulas and to establish their existence under fairly weak hypotheses on α. For normal maps we recover the formula of R. MacPherson [$\mathrm{Mac}_{1,2,3}$] concerning characteristic classes and singularities of bundle maps. Our formula is "local" on X, in the spirit of modern versions of the Atiyah-Singer Index Theorem. It is an equation of forms and currents with an explicit transgression term T. The MacPherson formula is obtained by passing to cohomology. The class of bundle maps α for which our local formula holds is broad and includes arbitrary real analytic maps whose singularity sets have the expected dimension.

Note 1. To simplify exposition we shall assume that E and F are complex bundles. Modifications required for the real case will be discussed in the last section on real vector bundles.

Note 2. The results in [HL_1] allow a choice of approximation mode. Here we shall always work with the *algebraic* approximation mode, where $\overrightarrow{D}_t$ has a particularly nice form. For example if $m \leq n$

$$\overrightarrow{D}_t \;=\; (t^2 D^F + \alpha D^E \alpha^*)(\alpha\alpha^* + t^2)^{-1}$$

§2. The universal case.

A bundle morphism

$$\begin{array}{ccc} E & \xrightarrow{\;\alpha\;} & F \\ & \searrow \quad \swarrow & \\ & X & \end{array} \tag{2.1}$$

as above can be considered to be a cross-section of the vector bundle

$$\mathrm{Hom}(E,F) \xrightarrow{\pi} X.$$

Now over the total space of Hom(E,F) there is a tautological bundle morphism

$$\begin{array}{ccc} \pi^*E & \xrightarrow{\boldsymbol{\alpha}} & \pi^*F \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}(E,F) & \end{array} \tag{2.2}$$

which at $A \in \mathrm{Hom}(E,F)$ is given by A itself. Everything is induced by pullback from this universal case. In particular $\alpha^*(\boldsymbol{\alpha}) = \alpha$, and the set-up in (2.1), metrics and connections included, is the pullback of that in (2.2). Our methods proceed as follows. We first analyse the problems posed in §1 for the universal case. We then examine normal maps, which are transversal to the universal singularities, and show that the universal formula can essentially be pulled back to X. Finally, using the notion of *geometric atomicity*, we establish results for quite general maps α.

We now focus our attention on the universal case (2.2). We begin by observing that there is a natural compactification

$$\mathrm{Hom}(E,F) \subset G \tag{2.3}$$

of Hom(E,F) given by

$$G \equiv G_m(E \oplus F) \xrightarrow{\pi} X,$$

the Grassmann bundle of complex m-planes in $E \oplus F$. The embedding (2.3) assigns to a linear map $A : E_x \to F_x$ at $x \in X$ its graph P_A in $E_x \oplus F_x$. Over G there is a natural orthogonal decomposition

$$\pi^*(E \oplus F) = U \oplus U^\perp \tag{2.4}$$

where U is the tautological m-plane bundle over G.

The multiplicative flow $\varphi_t : E \oplus F \longrightarrow E \oplus F$ defined by $\varphi_t(e,f) = (te, f)$ naturally induces a flow

$$\varphi_t : G \longrightarrow G \qquad \text{for } \ t \in \mathbf{C}^*$$

which restricts to the linear flow

$$\varphi_t(A) = \tfrac{1}{t}A \tag{2.5}$$

on Hom(E,F). The importance of this flow comes from the following fact proved in [HL$_1$; Section I.8].

Proposition 2.1. *Let $\overrightarrow{D}_t$ and $\overleftarrow{D}_t$ be the families of connections and Φ, Ψ the Ad-invariant polynomials discussed in §1. Then for all $t > 0$,*

$$\Phi(\overleftarrow{D}_t) = \varphi_t^* \, \Phi(\Omega^U) \qquad \text{and} \qquad \Psi(\overrightarrow{D}_t) = \varphi_t^* \, \Psi(\Omega^{U^\perp}).$$

Note 2.2. In the case where $\mathrm{rank}(E) = 1$ John Zweck [Z] uses 2.1 to calculate the characteristic currents associated to a section of $\mathbf{P}(E \oplus F)$ over X.

§3. Morse-Stokes kernels. Proposition 2.1 brings us to study the limits of differential forms under the flow $\varphi_t : G \to G$ for $0 < t < \infty$. (Examination of this question led to the new approach to Morse Theory in [HL$_3$].)

Denote by $G^{\oplus 2}$ the fibre product of G with itself over X, and consider the standard embedding $\mathbf{R} \subset \mathbf{P}^1(\mathbf{R}) = \mathbf{R} \cup \{\infty\}$ as an affine algebraic chart. We consider the submanifold

$$\mathcal{T} \stackrel{\text{def}}{=} \left\{(t, \varphi_t(P), P) \in \mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2} \ : \ 0 < t < \infty \ \text{ and } \ P \in G\right\},$$

called the *total graph of the flow*, and orient $\mathcal{T}$ by some choice of orientation on G. (We are essentially working locally on X, so its orientability is not a question.) Let $[\mathcal{T}]$ denote the current given by integration over $\mathcal{T}$ and define

$$T \ = \ \mathrm{pr}_*[\mathcal{T}] \tag{3.1}$$

where $\mathrm{pr} : \mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2} \longrightarrow G^{\oplus 2}$ is the projection. Closely related to this is the family

$$\mathcal{T}_{s,s'} \stackrel{\text{def}}{=} \{(t, \varphi_t(P), P) \in \mathcal{T} \ : \ s < t < s'\}$$

and its pushforward

$$T_{s,s'} \ = \ \mathrm{pr}_*[\mathcal{T}_{s,s'}] \tag{3.2}$$

for $0 < s < s' < \infty$. Note that $\mathcal{T}_{s,s'}$ is a compact manifold with boundary

$$\partial \mathcal{T}_{s,s'} \ = \ \{s'\} \times \Gamma_{s'} - \{s\} \times \Gamma_s$$

where

$$\Gamma_s \stackrel{\text{def}}{=} \{(\varphi_s(P), P) \in G^{\oplus 2} \ : \ P \in G\}. \tag{3.3}$$

It follows that

$$dT_{s,s'} \ = \ \Gamma_{s'} - \Gamma_s \tag{3.4}$$

in $G^{\oplus 2}$. This brings us to our main observation.

Proposition 3.1. *$\mathcal{T}$ is a submanifold of finite volume in $\mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2}$ over each compact subset of X.*

Proof. This follows from real analyticity. In local trivializations of E and F and local coordinates on X we have that $\mathrm{Hom}(E, F) \cong \mathbf{R}^p \times \mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n)$ and $G \cong \mathbf{R}^p \times G_m(\mathbf{C}^{m+n})$

where $G_m(\mathbf{C}^{m+n})$ denotes the Grassmannian of complex m-planes in $\mathbf{C}^{m+n}$ and $p = \dim(X)$. Now from (2.5) we deduce that in this presentation $\overline{\mathcal{T}}$ has the form $\mathbf{R}^p \times A$ where A is a semi-algebraic subset of $\mathbf{P}^1(\mathbf{R}) \times G_m(\mathbf{C}^{m+n}) \times G_m(\mathbf{C}^{m+n})$. Because it is semi-algebraic, A has finite volume. (See, for example [F].) It follows that $\mathcal{T}$ does also. ∎

Corollary 3.2. *The limit*

$$T = \lim_{\substack{s\to 0\\ s'\to\infty}} T_{s,s'} \tag{3.5}$$

exists in the mass topology on currents on $G^{\oplus 2}$.

Proof. By Proposition 3.1 the analogous limit of $\mathcal{T}_{s,s'}$ exists and equals $\mathcal{T}$ on $\mathbf{P}^1(\mathbf{R})\times G^{\oplus 2}$. Now apply the projection pr which decreases mass. ∎

Corollary 3.3. *The limits*

$$\Gamma_0 = \lim_{s\to 0} \Gamma_s \qquad \text{and} \qquad \Gamma_\infty = \lim_{s\to\infty} \Gamma_s$$

exist in integrally flat currents on $G^{\oplus 2}$ *and*

$$dT \;=\; \Gamma_\infty - \Gamma_0. \tag{3.6}$$

§4. **Morse-Stokes operators.** Each of the results of the previous section can be reinterpreted from the point of view of operators – operating on forms on G (cf. [HP], [HL$_3$]). As noted above, we want to understand the limit of the pull-back of differential forms under the flow φ_s on G. Consider

$$\begin{array}{ccc} G^{\oplus 2} & \xrightarrow{\ \mathrm{pr}_2\ } & G \\ {\scriptstyle \mathrm{pr}_1}\big\downarrow & & \\ G & & \end{array}$$

where pr_1 and pr_2 are the projections in the fibre product, and note that the submanifold $\Gamma_s = [\mathrm{graph}\varphi_s]$ determines the pullback operator φ_s^* via the equation

$$\varphi_s^*(\omega) \;=\; (\mathrm{pr}_2)_* \left\{(\mathrm{pr}_1^*\,\omega) \wedge \Gamma_s\right\}. \tag{4.1}$$

This leads us to consider, for each smooth p-form ω on G, the smooth $(p-1)$-form defined by the expression

$$\mathbf{T}_{s,s'}(\omega) \;\equiv\; (-1)^{\deg\omega}(\mathrm{pr}_2)_* \left\{(\mathrm{pr}_1^*\,\omega) \wedge T_{s,s'}\right\}. \tag{4.2}$$

Note that $\mathbf{T}_{s,s'}$ defines a continuous linear operator of degree -1

$$\mathbf{T}_{s,s'} : \mathcal{E}^*(G) \longrightarrow \mathcal{E}^*(G).$$

with "kernel" $T_{s,s'}$, on the space of smooth forms on G. The current equation (3.4) gives rise to the following operator equation.

Proposition 4.1.

$$\{d \circ \mathbf{T}_{s,s'} + \mathbf{T}_{s,s'} \circ d\}(\omega) \;=\; \varphi_{s'}^*\omega \;-\; \varphi_s^*\omega$$

for all differential forms $\omega \in \mathcal{E}^*(G)$.

Proof. By (3.4) and (4.1),

$$\begin{aligned} d\mathbf{T}_{s,s'}(\omega) &= (\mathrm{pr}_2)_* \left\{(-1)^{\deg\omega} d\left\{(\mathrm{pr}_1^*\,\omega) \wedge T_{s,s'}\right\}\right\} \\ &= (\mathrm{pr}_2)_* \left\{(-1)^{\deg\omega} (\mathrm{pr}_1)^* d\omega \wedge T_{s,s'} + (\mathrm{pr}_1^*\,\omega) \wedge dT_{s,s'}\right\} \\ &= (\mathrm{pr}_2)_* \left\{(-1)^{\deg\omega} (\mathrm{pr}_1)^* d\omega \wedge T_{s,s'} + (\mathrm{pr}_1^*\,\omega) \wedge (\Gamma_{s'} - \Gamma_s)\right\} \\ &= -\mathbf{T}_{s,s'}(d\omega) \;+\; \varphi_{s'}^*\omega \;-\; \varphi_s^*\omega \qquad \blacksquare \end{aligned}$$

As in (4.2) above the current or "kernel" T can be used to define the operator

(4.3) $$\mathbf{T}(\omega) \;\equiv\; (-1)^{\deg\omega}(\mathrm{pr}_2)_* \left\{(\mathrm{pr}_1^*\,\omega) \wedge T\right\}.$$

Theorem 4.2. *For any smooth k-form ω on G, the limits*

$$\mathbf{\Gamma}_0(\omega) = \lim_{s \to 0} \varphi_s^*\omega \qquad \textit{and} \qquad \mathbf{\Gamma}_\infty(\omega) = \lim_{s \to \infty} \varphi_s^*\omega$$

exist in the space of flat currents on G, and are given by the formulas

(4.4) $$\mathbf{\Gamma}_0(\omega) = (-1)^k (\mathrm{pr}_2)_* \left\{(\mathrm{pr}_1^*\,\omega) \wedge \Gamma_0\right\} \quad \textit{and} \quad \mathbf{\Gamma}_\infty(\omega) = (-1)^k (\mathrm{pr}_2)_* \left\{(\mathrm{pr}_1^*\,\omega) \wedge \Gamma_\infty\right\},$$

Furthermore, these limits satisfy the equation

(4.5) $$d\mathbf{T}(\omega) + \mathbf{T}(d\omega) = \mathbf{\Gamma}_\infty(\omega) - \mathbf{\Gamma}_0(\omega).$$

Proof. By Corollary 3.2 and equation (4.2) we see that

$$\mathbf{T}(\omega) = \lim_{\substack{s \to 0 \\ s' \to \infty}} \mathbf{T}_{s,s'}(\omega)$$

for all ω. The result now follows from Corollary 3.3, equation (4.1) and Proposition 4.1. $\blacksquare$

§5. Analysis of the currents Γ_0 and Γ_∞ – the decomposition. To understand the operators $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_\infty$ we must analyse the currents Γ_0 and Γ_∞ that define them in (4.4). Since these currents are interchanged by time reversal $t \mapsto \frac{1}{t}$, it will suffice to study Γ_0. Note that each of the graphs $\Gamma_t \subset G^{\oplus 2}$ is independent of base parameters. That is, if $G\big|_U \cong U \times G_m$ is a local trivialization of G, then over U, Γ_t has the form $\{(x, \varphi_t(P), P) : x \in U \text{ and } P \in G_m\}$, which is invariant under changes of the trivialization of E and F because the flow commutes with such changes. We conclude that the limit is similarly independent of base parameters. Consequently we shall drop all mention of X and simply analyse the multiplicative flow φ_t on $G \equiv G_m(\mathbf{C}^m \oplus \mathbf{C}^n)$ induced by the map $(z, w) \mapsto (tz, w)$ on $\mathbf{C}^m \oplus \mathbf{C}^n$.

To simplify the formulas we assume that $m \leq n$. The results hold in all cases as the reader will easily see.

Our first observation is that the fixed point set of the flow in a disjoint union of submanifolds

$$F = \coprod_{k \geq 0} F_k$$

where

$$(5.1) \qquad \begin{aligned} F_k &= \{P \in G_m(\mathbf{C}^m \oplus \mathbf{C}^n) : \dim(P \cap \mathbf{C}^m) = k \text{ and } \dim(P \cap \mathbf{C}^n) = m - k\} \\ &\cong G_k(\mathbf{C}^m) \times G_{m-k}(\mathbf{C}^n) \end{aligned}$$

Consider the subsets

$$\Sigma_k \equiv \{P : \dim(P \cap \mathbf{C}^m) = k\} \qquad \text{and} \qquad \Upsilon_k \equiv \{P : \dim(P \cap \mathbf{C}^n) = m - k\}$$

and note that

$$\Sigma_k \cap \mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n) = \{A : \dim(\ker A) = k\} \qquad \text{and}$$

$$\{0\} = \Sigma_m \subset \overline{\Sigma}_{m-1} \subset \overline{\Sigma}_{m-2} \subset \overline{\Sigma}_{m-3} \subset \cdots \subset \overline{\Sigma}_0 = G.$$

Furthermore, we observe that

$$\Upsilon_k \cap \mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n) = \emptyset \text{ if } k < m \qquad \text{and}$$

$$G = \overline{\Upsilon}_m \supset \overline{\Upsilon}_{m-1} \supset \overline{\Upsilon}_{m-2} \supset \overline{\Upsilon}_{m-3} \supset \ldots \Upsilon_0 = G_m(\mathbf{C}^n)$$

and furthermore

$$(5.2) \qquad \mathrm{Hom}(E, F) = G - \overline{\Upsilon}_{m-1}.$$

Note 5.1. Σ_k and Υ_k are the stable and unstable manifolds of F_k for φ_t^{-1}, that is,

$$\Sigma_k = \{P \in G : \lim_{t \to 0} \varphi_t(P) \in F_k\} \qquad \text{and} \qquad \Upsilon_k = \{P \in G : \lim_{t \to \infty} \varphi_t(P) \in F_k\}.$$

This follows immediately from the next Lemma whose proof is easy.

Lemma 5.2. *For any $P \in G$,*

$$\lim_{t\to 0} \varphi_t(P) = (P \cap \mathbf{C}^m) \oplus \mathrm{pr}_{\mathbf{C}^n}(P) \stackrel{\mathrm{def}}{=} \pi_1(P)$$

$$\lim_{t\to \infty} \varphi_t(P) = \mathrm{pr}_{\mathbf{C}^m}(P) \oplus (P \cap \mathbf{C}^n) \stackrel{\mathrm{def}}{=} \pi_2(P)$$

where $\mathrm{pr}_{\mathbf{C}^m}$ and $\mathrm{pr}_{\mathbf{C}^n}$ are the projections of $\mathbf{C}^m \oplus \mathbf{C}^n$ onto the factors.

The right hand side of the formulas in Lemma 5.2 give us projections

$$\begin{array}{ccccc} \Upsilon_k & \xrightarrow{\ \pi_2\ } & G_k(\mathbf{C}^m) \times G_{m-k}(\mathbf{C}^n) & \xleftarrow{\ \pi_1\ } & \Sigma_k \\ & & \| & & \\ & & F_k & & \end{array}$$

Proposition 5.3.

$$\Gamma_\infty = \sum_{k=0}^{m} [\Sigma_k \times_{F_k} \Upsilon_k] \qquad \textit{and} \qquad \Gamma_0 = \sum_{k=0}^{m} [\Upsilon_k \times_{F_k} \Sigma_k]$$

where

$$\Upsilon_k \times_{F_k} \Sigma_k = \{(P, Q) \in \Upsilon_k \times \Sigma_k \subset G \times G \ : \ \pi_2(P) = \pi_1(Q)\}$$

and $\Sigma_k \times_{F_k} \Upsilon_k$ is defined similarly.

Proof. We shall only sketch the argument since a similar, more general assertion is proved in [HL$_3$]. From its definition (cf. (3.3) and Corollary 3.3) it is straightforward to show that $\mathrm{supp}\Gamma_0 \subseteq \Upsilon_k \times_{F_k} \Sigma_k$. Now each of the submanifolds $\Upsilon_k \times_{F_k} \Sigma_k$ is a Zariski dense subset of an algebraic subvariety; in particular it has finite volume in $G \times G$. The Federer Flat Support Lemma [F; 4.1.15] now implies that $\Gamma_0 = \sum_k n_k [\Upsilon_k \times_{F_k} \Sigma_k]$. Analysis of the limit at points of F_k shows that $n_k = 1$ (cf. (10.2)). ∎

Note that $\Upsilon_k \times_{F_k} \Sigma_k$ is a fibre product over F_k embedded diagonally in $G \times G$. For $x \in F_k$ the fibre $\pi_2^{-1}(x, x)$ lies in $G \times \{x\}$ and the fibre $\pi_1^{-1}(x, x)$ lies in $\{x\} \times G$.

Combining Proposition 5.3 with (4.4) above gives the following.

Corollary 5.4. *The operators $\mathbf{\Gamma}_0$ and $\mathbf{\Gamma}_\infty$ can be written as*

$$\mathbf{\Gamma}_0 \ = \ \sum_{k=0}^{m} \mathbf{P}_k \qquad \textit{and} \qquad \mathbf{\Gamma}_\infty \ = \ \sum_{k=0}^{m} \widetilde{\mathbf{P}}_k$$

where

$$\mathbf{P}_k(\omega) \ = \ (\mathrm{pr}_2)_* \left\{ (\mathrm{pr}_1^* \omega) \wedge [\Upsilon_k \times_{F_k} \Sigma_k] \right\} \quad \textit{and} \quad \widetilde{\mathbf{P}}_k(\omega) \ = \ (\mathrm{pr}_2)_* \left\{ (\mathrm{pr}_1^* \omega) \wedge [\Sigma_k \times_{F_k} \Upsilon_k] \right\}$$

for any smooth form ω on the Grassmann bundle G.

§6. Analysis of the currents Γ_0 and Γ_∞ – the residues. We now show that the operator $\mathbf{P}_k$ in Corollary 5.4 has the form $\mathbf{P}_k(\omega) = \mathrm{Res}_k(\omega)[\Sigma_k]$ where $\mathrm{Res}_k(\omega)$ is an explicitly computable residue form defined on Σ_k in terms of ω, and analogously $\widetilde{\mathbf{P}}_k(\omega) = \mathrm{Res'}_k(\omega)[\Upsilon_k]$ where $\mathrm{Res'}_k(\omega)$ is a smooth residue form on Υ_k.

We begin with the following observation.

Lemma 6.1. *Each of the submanifolds Υ_k and Σ_k has finite volume in G. So also do the fibre products $\Upsilon_k \times_{F_k} \Sigma_k$.*

Proof. This is evident from the fact that their closures are algebraic subvarieties. ∎

Definition 6.2. Given a smooth differential form ω defined in a neighborhood of $\overline{\Upsilon}_k$, and a smooth differential form ω' defined in a neighborhood of $\overline{\Sigma}_k$, set

$$\mathrm{Res}_k(\omega) \;=\; (\pi_1)^*\left\{(\pi_2)_*\omega\right\} \qquad \text{and} \qquad \mathrm{Res'}_k(\omega') \;=\; (\pi_2)^*\left\{(\pi_1)_*\omega'\right\}$$

using the projections

$$\Upsilon_k \xrightarrow{\pi_2} F_k \xleftarrow{\pi_1} \Sigma_k.$$

We shall show that the maps π_1 and π_2 have the natural structure of algebraic vector bundles over F_k.

Proposition 6.3. *Each operator $\mathbf{P}_k$ can be expressed by the formula*

$$\mathbf{P}_k(\omega) \;=\; Res_k(\omega)[\Sigma_k].$$

Furthermore, for any smooth form ω on G, $\mathrm{Res}_k(\omega)$ is a smooth form on Σ_k which has finite L^1-norm (i.e., finite mass), so that $Res_k(\omega)[\Sigma_k]$ is a well defined current on G. Similarly, $\widetilde{\mathbf{P}}_k$ can be written as

$$\widetilde{\mathbf{P}}_k(\omega) \;=\; Res'_k(\omega)[\Upsilon_k].$$

where $Res'_k(\omega)$ is a smooth L^1 form on Υ_k.

Proof. To begin we note that there is a commutative diagram

$$\begin{array}{ccc} \Upsilon_k \times_{F_k} \Sigma_k & \xrightarrow{p_2} & \Sigma_k \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ \Upsilon_k & \xrightarrow[\pi_2]{} & F_k \end{array} \tag{6.1}$$

where p_1 and p_2 are induced from the two projections $G \times G$ onto G, i.e., there is a commutative diagram

$$\begin{array}{ccccc} \Upsilon_k & \xleftarrow{p_1} & \Upsilon_k \times_{F_k} \Sigma_k & \xrightarrow{p_2} & \Sigma_k \\ \downarrow & & \downarrow & & \downarrow \\ G & \xleftarrow{\mathrm{pr}_1} & G \times G & \xrightarrow{\mathrm{pr}_2} & G \end{array}$$

Each of the maps π_1 and π_2 in (6.1) has the natural structure of an algebraic vector bundle over F_k. We show this explicitly as follows. Let

$$\xi \longrightarrow G_k(\mathbf{C}^m) \qquad \text{and} \qquad \eta \longrightarrow G_{m-k}(\mathbf{C}^n)$$

be the tautological bundles of rank k and $m-k$ respectively, and extend them by pullback to $F_k = G_k(\mathbf{C}^m) \times G_{m-k}(\mathbf{C}^n)$. Then there are commutative diagrams

$$\begin{array}{ccc} \mathrm{Hom}(\xi, \eta^\perp) & \xrightarrow{j_2} & \Upsilon_k \\ {\scriptstyle \pi_2'}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ G_k(\mathbf{C}^m) \times G_{m-k}(\mathbf{C}^n) & \xrightarrow{\cong} & F_k \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{Hom}(\eta, \xi^\perp) & \xrightarrow{j_1} & \Sigma_k \\ {\scriptstyle \pi_1'}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ G_k(\mathbf{C}^m) \times G_{m-k}(\mathbf{C}^n) & \xrightarrow{\cong} & F_k \end{array}$$

where π_1' and π_2' are bundle projections and where at $(\xi, \eta) \in G_k \times G_{m-k}$

$$j_2(a) \ = \ \eta \oplus \mathrm{graph}(a) \qquad \text{and} \qquad j_1(b) \ = \ \xi \oplus \mathrm{graph}(b).$$

The maps j_1 and j_2 are biholomorphisms and give Υ_k and Σ_k the structure of vector bundles as claimed. Consider the Grassmann compactificatons

$$\begin{array}{ccccc} \mathrm{Hom}(\xi, \eta^\perp) & & \subset & & G(\xi \oplus \eta^\perp) \\ \pi_2' & \searrow & & \swarrow & \widetilde{\pi}_2 \\ & & F_k & & \end{array} \qquad\qquad \begin{array}{ccccc} \mathrm{Hom}(\eta, \xi^\perp) & & \subset & & G(\eta \oplus \xi^\perp) \\ \pi_1' & \searrow & & \swarrow & \widetilde{\pi}_1 \\ & & F_k & & \end{array}$$

where $G(\xi \oplus \eta^\perp)$ is the bundle of k-planes in $\xi \oplus \eta^\perp$ and $G(\eta \oplus \xi^\perp)$ is the bundle of $(m-k)$-planes in $\eta \oplus \xi^\perp$. The maps j_i extend to surjective algebraic maps

$$G(\xi \oplus \eta^\perp) \xrightarrow{\bar{j}_2} \overline{\Upsilon}_k \qquad\qquad G(\eta \oplus \xi^\perp) \xrightarrow{\bar{j}_1} \overline{\Sigma}_k$$

given on the fibres above $(\xi, \eta) \in G_k \times G_{m-k}$ by

(6.2) $$\bar{j}_2(\ell_2) = \eta \oplus \ell_2 \qquad \text{and} \qquad \bar{j}_1(\ell_1) = \xi \oplus \ell_1$$

Note 6.4. The normal bundle to Σ_k is equivalent to the pullback of the vector bundle Υ_k via the map π_2. Similarly the normal bundle to Υ_k is the π_1-pullback of Σ_k.

We now observe that by the commutativity of (6.1) we have

$$\begin{aligned} (\mathrm{pr}_2)_* \left\{ (\mathrm{pr}_1^* \omega) \wedge [\Upsilon_k \times_{F_k} \Sigma_k] \right\} \ &= \ (p_2)_* \left\{ (p_1^* \omega) \right\} \\ &= \ (\pi_1)^* \left\{ (\pi_2)_* \omega \right\} \\ &= \ \mathrm{Res}_k(\omega)[\Sigma_k] \end{aligned}$$

This proves the formula asserted in 6.3. The integrability of $\mathrm{Res}_k(\omega)$ on Σ_k is equivalent to the fact that the current $\mathrm{Res}_k(\omega)[\Sigma_k]$ has finite mass. This finiteness of mass is a consequence of Lemma 6.1, which implies that $(\mathrm{pr}_1^*\,\omega)\wedge[\Upsilon_k\times_{F_k}\Sigma_k]$ has finite mass, and the fact that pushforward of currents is mass non-increasing.

This completes the proof of 6.3 for $\mathbf{P}_k$. The argument for $\widetilde{\mathbf{P}}_k$ is completely analogous. ∎

The proof above used the "Grassmann desingularization" of $\overline{\Upsilon}_k$ and $\overline{\Sigma}_k$ by the maps $\overline{j}_2$ and $\overline{j}_1$. This gives us another way to look at the residues which will be useful to us when we consider characteristic forms in §9 and onward.

Proposition 6.5. *The form* $\mathrm{Res}_k(\omega)$ *can be expressed as*

$$\mathrm{Res}_k(\omega) \;=\; (\pi_1)^*\left\{(\widetilde{\pi}_2)_*\widetilde{\omega}\right\}$$

where $\widetilde{\pi}_2 : G(\xi\oplus\eta^\perp)\longrightarrow F_k$ *is the Grassmann compactification above, and where* $\widetilde{\omega} = \overline{j}_2^*\omega$. *The analogous statements hold for* $\mathrm{Res}'_k(\omega')$.

Proof. We have seen that $(\pi_2)_*(\omega)=(\pi'_2)_*(j_2^*\omega)$. Since the fibres of π'_2 are Zariski dense, and in particular of full measure, in the fibres of $\widetilde{\pi}_2$, we see that integration of a smooth form on $G(\xi\oplus\eta^\perp)$ over the fibres of $\widetilde{\pi}_2$ and over the fibres of π'_2 are equal. ∎

§**7. The first main theorem.** Combining 4.2, 5.4 and 6.3 immediately yields our first main result.

Theorem 7.1. *Let* $G=G_m(E\oplus F)$ *be the Grassmann bundle of* m*-planes in the smooth vector bundle* $E\oplus F\to X$, *and let* $\varphi_t,\ \ 0<t<\infty$ *be the multiplicative flow on* G *engendered by* $(e,f)\mapsto(te,f)$ *in* $E\oplus F$. *Then there are continuous linear operators* $\mathbf{\Gamma}_0$, $\mathbf{\Gamma}_\infty$, $\mathbf{T}:\mathcal{E}^*(G)\longrightarrow\mathcal{E}'^*(G)$ *from smooth differential forms to generalized differential forms (in fact, flat currents) on* G *with the following properties. For all* $\omega\in\mathcal{E}^*(G)$,

$$\mathbf{\Gamma}_0(\omega) \;=\; \lim_{t\to 0}\varphi_t^*\omega \;=\; \sum_{k=0}^{m}\mathrm{Res}_k(\omega)[\Sigma_k] \qquad \textit{and} \qquad \mathbf{\Gamma}_\infty(\omega) \;=\; \lim_{t\to\infty}\varphi_t^*\omega \;=\; \sum_{k=0}^{m}\mathrm{Res}'_k(\omega)[\Upsilon_k]$$

where

$$\mathrm{Res}_k(\omega)=\pi_1^*\{(\pi_2)_*\omega\} \qquad \textit{and} \qquad \mathrm{Res}'_k(\omega)=\pi_2^*\{(\pi_1)_*\omega\}.$$

Furthermore, $\mathbf{T}$, *defined by (4.3), is an operator of degree -1 which satisfies the equation*

$$d\circ\mathbf{T}+\mathbf{T}\circ d \;=\; \mathbf{\Gamma}_\infty-\mathbf{\Gamma}_0 \tag{7.1}$$

§**8. The formula in the universal case.** Let $U\longrightarrow G$ be the tautological bundle and let Φ and Ψ be Ad-invariant polynomials on the Lie algebras of the structure groups of E and F respectively. Because of Proposition 2.1 we want to apply the Theorem above to the situation where

$$\omega=\Phi(\Omega^U) \qquad \text{or} \qquad \omega=\Psi(\Omega^{U^\perp}).$$

The main point is to compute the residues. For simplicity we will treat the first case. According to Proposition 6.3 $\mathrm{Res}_k(\omega)$ on Σ_k is computed by restricting ω to Υ_k, integrating over the projection π_2 and then pulling back to Σ_k via π_1. Note that by the naturality of the Chern construction

$$\omega\big|_{\Upsilon_k} = \Phi\left(\Omega^{U\big|_{\Upsilon_k}}\right).$$

From the proof of Proposition 6.3 we see that on Υ_k the bundle U splits as

$$U\big|_{\Upsilon_k} \cong \eta \oplus U_k$$

where U_k is the restriction to $\mathrm{Hom}(\xi, \eta^\perp) \subset G(\xi \oplus \eta^\perp)$ of the **tautological k-plane bundle**

$$U_k \longrightarrow G(\xi \oplus \eta^\perp).$$

Consequently we have from 7.1 that

$$\mathrm{Res}_k(\omega) = \pi_1^*(\pi_2)_*\Phi\left(\Omega^{\eta\oplus U_k}\right) \tag{8.0}$$

Restricting to the coordinate chart $\mathrm{Hom}(E, F)$ we get

$$\mathrm{Res}_k(\omega) = (\pi_2)_*\Phi\left(\Omega^{\mathrm{Im}\boldsymbol{\alpha}\oplus U_k}\right) \tag{8.1}$$

where U_k is the tautological bundle over the Grassmann compactification $G_k(\ker\boldsymbol{\alpha} \oplus \mathrm{coker}\boldsymbol{\alpha})$ of the normal bundle $\mathrm{Hom}(\ker\boldsymbol{\alpha}, \mathrm{coker}\boldsymbol{\alpha})$ to Σ_k.

On the other hand by (5.2) one sees directly that on the chart $\mathrm{Hom}(E, F)$

$$\mathrm{Res}'_k(\omega) = 0 \qquad \text{for} \quad k < m \tag{8.2}$$

and

$$\mathrm{Res}'_m(\omega) = \pi_1^*\left(\omega\big|_{\Sigma_m}\right) = \Phi(\Omega^E) \tag{8.3}$$

Theorem 8.1. *Let $E \to X$ and $F \to X$ be smooth complex vector bundles with* $\mathrm{rank}(E) \le \mathrm{rank}(F)$*, and let Φ be an invariant polynomial on the Lie algebra of the structure group of E. Then for any choice of connections on E and F there exists an L^1_{loc}-form T on* $\mathrm{Hom}(E, F)$ *so that*

$$\Phi(\Omega^E) = \sum_{k=0}^{m} \mathrm{Res}_{\Phi,k}[\Sigma_k] + dT \tag{8.4}$$

$$\mathrm{Res}_{\Phi,k} = (\pi_2)_*\Phi\left(\Omega^{\mathrm{Im}\,\boldsymbol{\alpha}\oplus U_k}\right)$$

where $\boldsymbol{\alpha} : \pi^*E \to \pi^*F$ denotes the tautological bundle map on $\mathrm{Hom}(E, F)$ *and where $(\pi_2)_*$ denotes integration over the fibres of the Grassmann compactification $G_k = G_k(\ker\boldsymbol{\alpha} \oplus \mathrm{coker}\,\boldsymbol{\alpha})$ of the normal bundle $N_{\Sigma_k} \cong \mathrm{Hom}(\ker\boldsymbol{\alpha}, \mathrm{coker}\,\boldsymbol{\alpha})$, and where U_k is the tautological k-plane bundle over G_k.*

If $\operatorname{rank}(E) = \operatorname{rank}(F)$*, then formula (8.1) becomes*

$$\Phi(\Omega^E) - \Phi(\Omega^F) = \sum_{k=1}^{m} \operatorname{Res}_{\Phi,k}[\Sigma_k] + dT$$

If Ψ *is an invariant polynomial on the Lie algebra of the structure group of* F*, then there exists an* L^1_{loc}*-form* T' *on* $\operatorname{Hom}(E, F)$ *so that*

$$\Psi(\Omega^F) = \sum_{k=0}^{m} \operatorname{Res}_{\Psi,k}[\Sigma_k] + dT' \tag{8.5}$$

$$\operatorname{Res}_{\Psi,k} = (\pi_2)_* \Psi\left(\Omega^{(\ker\boldsymbol{\alpha})^\perp \oplus U_k^\perp}\right)$$

Note 8.2. The case where $\operatorname{rank}(E) > \operatorname{rank}(F)$ follows by applying Theorem 8.1 to the adjoint of $\boldsymbol{\alpha}$.

Note 8.3. The bundle $\operatorname{Hom}(\operatorname{coker}\boldsymbol{\alpha}, \ker\boldsymbol{\alpha})$ is the dual of the normal bundle $\operatorname{Hom}(\ker\boldsymbol{\alpha}, \operatorname{coker}\boldsymbol{\alpha})$ of Σ_k.

Proof. If $\omega = \Phi(\Omega^U)$, then $d\omega = 0$. We apply Theorem 7.1 to ω and apply (8.1) to calculate the term $\mathbf{\Gamma}_0(\omega)$. We then restrict to $\operatorname{Hom}(E, F) \subset G$ and apply (8.2) and (8.3) to calculate $\mathbf{\Gamma}_\infty(\omega)$. This proves the first part of the theorem. The calculations for $\Psi(\Omega^{U^\perp})$ are completely analogous. ∎

Remark 8.4. The bundle $\operatorname{Im}(\boldsymbol{\alpha})$ is a pull-back to G_k of a bundle defined on Σ_k via the fibration $G_k \to \Sigma_k$. The tautological bundle U_k carries a natural connection which along the fibres of $G_k \to \Sigma_k$ is the *standard* connection. In §12 we shall see that the connection yielding the curvature form in formula (8.2) for the residue can be assumed to be the *direct sum* of the pull-back connection on $\operatorname{Im}(\boldsymbol{\alpha})$ with the projected connection on U_k. This has particularly nice consequences when Φ is a multiplicative series of characteristic polynomials.

§**9. Existence for normal bundle maps — geometric atomicity.** Let E and F be smooth complex vector bundles over a manifold X of dimension ν, and suppose that

$$\alpha : E \longrightarrow F$$

is a smooth bundle map. Given Ad-invariant polynomials Φ and Ψ as above, one can ask when the limits (1.1) exist. We shall now answer this question in some generality, and also establish the local MacPherson formula for α.

The following concept is crucial here. To begin we recall that a Borel measurable subset A of a locally compact topological space Z is said to have **locally finite** μ-measure if each point $z \in Z$ has a compact neighborhood U such that $\mu(A \cap U) < \infty$.

Definition 9.1. The section α is called **geometrically atomic** if the subset

$$(9.1) \qquad T_\alpha \;\equiv\; \left\{ (\tfrac{1}{t}\alpha_x, \alpha_x) \in G^{\oplus 2} \;:\; x \in X \;\text{ and }\; 0 < t < \infty \right\}$$

has locally finite $(\nu+1)$-dimensional measure in $G^{\oplus 2}$.

Note. Above the open set $X - \text{Zero}(\alpha)$ where $\alpha \neq 0$, T_α is a submanifold. In fact, it is a line bundle over this set. The remaining points of T_α consist of the zeros of α and therefore have locally finite ν-dimensional measure. Hence they can be ignored, and the condition in 9.1 can be replaced by requiring that the remaining submanifold have locally finite volume in $G^{\oplus 2}$. (Thus, the zero-section $\alpha = 0$ is always geometrically atomic.)

Note. The condition in Definition 9.1 is equivalent to the requirement that for each compact $K \subset X$, the subset

$$T_{K,\alpha} \;\equiv\; \left\{ (\tfrac{1}{t}\alpha_x, \alpha_x) \in G^{\oplus 2} \;:\; x \in K \;\text{ and }\; 0 < t < \infty \right\}$$

has finite $(\nu+1)$-dimensional measure in $G^{\oplus 2}$.

The generality of Definition 9.1 is clear from the following result.

Proposition 9.2. *If α is real analytic, then it is geometrically atomic.*

Proof. The closure of the submanifold

$$(9.2) \qquad \mathcal{T}_\alpha = \left\{ (t, \tfrac{1}{t}\alpha_x, \alpha_x) \in \mathbf{R} \times G^{\oplus 2} \;:\; x \in X \;\text{ and }\; 0 < t < \infty \right\} \subset \mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2}$$

is an analytic subvariety of dimension $(\nu+1)$ in $\mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2}$ and hence has locally finite $(\nu+1)$-measure. It follows that its image $T_\alpha = \text{pr}_* \mathcal{T}_\alpha$, where $\text{pr} : \mathbf{P}^1(\mathbf{R}) \times G^{\oplus 2} \to G^{\oplus 2}$ is the projection, also has locally finite $(\nu+1)$-measure. ∎

Note The singularities of a real analytic map can be monstrous. In particular, the sets $\Sigma_k(\alpha)$, defined in (1.2), need not have the expected dimension.

Definition 9.3. A bundle map $\alpha : X \longrightarrow \text{Hom}(E, F)$ is called **normal** if it is transversal to the submanifolds Σ_k for all k.

Proposition 9.4. *Any normal bundle map is geometrically atomic.*

The proof is postponed to section 10. Our first main result is the following.

Theorem 9.5. *If α is geometrically atomic, then the limits*

$$(9.3) \qquad \Phi(\overleftarrow{\mathbb{D}}) \equiv \lim_{t \to 0} \Phi(\overleftarrow{D}_t) \qquad \text{and} \qquad \Psi(\overrightarrow{\mathbb{D}}) \equiv \lim_{t \to 0} \Psi(\overrightarrow{D}_t)$$

exist on X for all Ad-invariant polynomials Φ and Ψ on the Lie algebras of the structure groups of E and F. Furthermore, for all Φ, Ψ there exist L^1_{loc}-forms T_Φ, T_Ψ on X such that

$$(9.4) \qquad \Phi(\Omega^E) - \Phi(\overleftarrow{\mathbb{D}}) = dT_\Phi \qquad \text{and} \qquad \Psi(\Omega^F) - \Psi(\overrightarrow{\mathbb{D}}) = dT_\Psi$$

where Ω^E, Ω^F denote the curvature 2-forms E and F respectively.

Proof. Fix $s' > s > 0$ and consider the subset

$$T_{\alpha,s,s'} \ \equiv\ \left\{ \left(\tfrac{1}{t}\alpha_x, \alpha_x\right) \in G^{\oplus 2} \ :\ x \in X \quad \text{and} \quad s < t < s' \right\}.$$

(Note that $T_{\alpha,s,s'} = \mathrm{pr}_* \mathcal{T}_{\alpha,s,s'}$ where $\mathcal{T}_{\alpha,s,s'}$ is the compact submanifold with boundary defined as in 9.2 with $s \le t \le s'$.) As in §3 the assumption of locally finite volume implies that

$$\lim_{\substack{s \to 0 \\ s' \to \infty}} T_{\alpha,s,s'} \ =\ T_\alpha$$

in locally integral currents on $G^{\oplus 2}$, and that

$$\lim_{\substack{s \to 0 \\ s' \to \infty}} dT_{\alpha,s,s'} \ =\ \lim_{s' \to \infty} \Gamma_{\alpha,s'} - \lim_{s \to 0} \Gamma_{\alpha,s} \ \equiv\ \Gamma_{\alpha,\infty} - \Gamma_{\alpha,0} \tag{9.5}$$

where

$$\Gamma_{\alpha,s} = \left\{ \left(\tfrac{1}{s}\alpha_x, \alpha_x\right) \in G^{\oplus 2} \ :\ x \in X \right\}$$

We now reinterpret these equations as operator equations and apply them to the forms $\Phi(\Omega^U)$ and $\Psi(\Omega^{U^\perp})$ as in §4. Specifically, to each integral current S of dimension $n+\ell$ on $G^{\oplus 2}$ we associate the operator $\mathbf{S} : \mathcal{E}^*(G) \to \mathfrak{E}^*(X)$ of degree $-\ell$ from forms on G to currents on X by setting

$$\mathbf{S}(\omega) \ =\ p_* \left\{ (\mathrm{pr}_1^* \,\omega) \wedge S \right\} \tag{9.6}$$

where $p : G \to X$ is the bundle projection. Note that $\mathbf{\Gamma}_{\alpha,s}(\omega) = \alpha^* \varphi_s^*(\omega)$ where φ_s is the flow on G defined in (2.5). In particular, if $\omega = \Phi(\Omega^U)$, then by Proposition 2.1 and the universality of the construction of $\overleftarrow{D}_s$ we have that

$$\mathbf{\Gamma}_{\alpha,s}(\omega) \ = \alpha^* \varphi_s^* \Phi(\Omega^U) \ =\ \alpha^* \Phi(\overleftarrow{\boldsymbol{\Omega}}_s) \ =\ \Phi(\overleftarrow{\Omega}_s). \tag{9.7}$$

where $\overleftarrow{\boldsymbol{\Omega}}_s$ is the curvature of the universal pushforward connection on $\mathrm{Hom}(E,F) \subset G$. Consequently (9.5) and (9.7) imply that

$$\lim_{s \to \infty} \Phi(\overleftarrow{\Omega}_s) \ -\ \lim_{s \to 0} \Phi(\overleftarrow{\Omega}_s) \ =\ dT_\Phi$$

where $T_\Phi = \mathbf{T}_\alpha(\Phi(\Omega^U))$. By the continuity of $\overleftarrow{D}_s$ at infinity we have $\lim_{s\to\infty} \Phi(\overleftarrow{\Omega}_s) = \Phi(\Omega^E)$. This proves the result for Φ. The result for Ψ is similar. ∎

A determination of the limits in (9.3) for all Φ and Ψ will follow from understanding the limiting current $\Gamma_{\alpha,0}$. When α is normal we shall see that this current is modeled on the universal case.

§**10. The local MacPherson formula.** In this section we analyse the current $\Gamma_{\alpha,0}$. Our discussion is local on X, so we shall assume that E and F are trivialized bundles. Our section α is then just a map from X to $\mathrm{Hom}(\mathbf{C}^m, \mathbf{C}^n) \subset G_m(\mathbf{C}^{m+n}) = G$.

To begin the analysis we give a simple presentation of the flow φ_s in a neighborhood of $F_k \cong G_k(E) \times G_{m-k}(F)$ in G. We return to the notation of §6 and consider the vector bundle

$$\begin{array}{c} H_k \equiv \mathrm{Hom}(\xi, \eta^\perp) \oplus \mathrm{Hom}(\eta, \xi^\perp) \\ \Big\downarrow {\scriptstyle \pi_2' \oplus \pi_1'} \\ F_k . \end{array}$$

There is a map $j : H_k \to G$ defined by

$$j(a, b) \;=\; \mathrm{gr}(a) \oplus \mathrm{gr}(b) \tag{10.1}$$

where $\mathrm{gr}(a)$ denotes the graph of a. This map gives a diffeomorphism from a neighborhood of the zero-section to a neighborhood of F_k in G. We introduce fibre metrics and identify such a neighborhood of F_k with

$$\mathcal{U} \;=\; \{(a, b) \in H_k \;:\; |a| \leq 1 \;\text{ and }\; |b| \leq 1\}$$

In this presentation the flow φ_s has the form

$$\varphi_s(a, b) \;=\; \left(\tfrac{1}{s}a, sb\right).$$

Now in this neighborhood our set T can be written as

$$\begin{aligned} T \cap (\mathcal{U} \times \mathcal{U}) \;&=\; \left\{(\tfrac{1}{s}a, sb, a, b) \;:\; |a| \leq 1, |b| \leq 1, |\tfrac{1}{s}a| \leq 1, |sb| \leq 1, \text{ and } 0 < s \leq 1\right\} \\ &=\; \{(a, sb, sa, b) \;:\; |a| \leq 1, |b| \leq 1, \text{ and } 0 < s \leq 1\} \end{aligned}$$

The boundary of this set is clearly given by

$$\partial\{T \cap (\mathcal{U} \times \mathcal{U})\} \;=\; \{(a, 0, 0, b) \;:\; |a| \leq 1, |b| \leq 1\} \;\cong\; \Upsilon_k \times_{F_k} \Sigma_k. \tag{10.2}$$

This is in fact a manifold with boundary in $\mathcal{U} \times \mathcal{U} - F_k$. We can resolve the singularity at F_k by considering

$$\{(s, a, sb, sa, b) \;:\; |a| \leq 1, |b| \leq 1, \text{ and } 0 \leq s \leq 1\} \;\subset\; \mathbf{R} \times G \times G.$$

This is a manifold with boundary whose projection is the set above.

Suppose now that α is a normal bundle map and fix $x_0 \in X$ with $\alpha_{x_0} \in \Sigma_k$. Let $z_0 = \pi_1(\alpha_{x_0}) \in F_k$ and fix a neighborhood V of z_0 in F_k with local trivializations

$$\xi\big|_V \cong\; V \times \mathbf{C}^k \qquad \text{and} \qquad \eta\big|_V \cong\; V \times \mathbf{C}^{m-k}.$$

Thus over V we have

$$(10.3) \qquad \mathrm{Hom}(\xi,\eta^{\perp}) \oplus \mathrm{Hom}(\eta,\xi^{\perp})\big|_V \cong V \times \mathrm{Hom}_{k,n-m+k} \times \mathrm{Hom}_{m-k,m-k}.$$

where $\mathrm{Hom}_{r,s} \equiv \mathrm{Hom}(\mathbf{C}^r,\mathbf{C}^s)$. In this picture $\alpha_{x_0} \cong (z_0, 0, b_0)$ for some point $b_0 \in \mathrm{Hom}_{m-k,m-k}$, which we may assume (by homothety) to satisfy $|b_0| < 1$. Restricting to triples (v, a, b) with $|a| \le 1$ and $|b| \le 1$ parameterizes a neighborhood in G containing z_0 and α_{x_0}. In this neighborhood

$$\Sigma_k \cong V \times \{0\} \times \mathrm{Hom}_{m-k,m-k}.$$

Now the transversality of α to Σ_k implies the following.

Lemma 10.1. *Suppose α is normal and $x_0 \in \Sigma_k(\alpha)$. Then there exist local coordinates on a neighborhood U of x_0 in X of the form*

$$(y, a) \in \mathbf{R}^N \times \mathrm{Hom}_{k,n-m+k}$$

where $N = \nu - 2k(n-m+k)$, such that in the coordinates (10.3) above

$$(10.4) \qquad \alpha(y, a) = (v(y), a, b(y)).$$

Note that in U, $\Sigma_k(\alpha)$ is the submanifold corresponding to $a = 0$. Furthermore, Lemma 10.1 shows that in U

$$(10.5) \qquad \Sigma_\ell(\alpha) = \mathbf{R}^N \times \{a \in \mathrm{Hom}_{k,n-m+k} : \dim\ker(a) = \ell\}$$

for all $\ell \le k$. We conclude the following.

Corollary 10.2. *If α is normal , then each $\Sigma_\ell(\alpha)$ has locally finite volume in X.*

Proof of Proposition 9.4 Fix $x_0 \in \Sigma_k(\alpha)$ and choose coordinates on G and X as above. Note that the map j defined in (10.1) extends smoothly to the compactification $G(\xi \oplus \eta^{\perp}) \oplus G(\eta \oplus \xi^{\perp})$. In particular, via (10.3) this gives a map

$$V \times G_k(\mathbf{C}^k \oplus \mathbf{C}^{n-m+k}) \times \mathrm{Hom}_{m-k,m-k} \longrightarrow G$$

which we compose with our coordinate representation of α above. We then consider the map

$$(0,1] \times U \longrightarrow G \times G$$

given by

$$(s, y, a) \mapsto (v(y), \tfrac{1}{s}a, sb(y);\, v(y), a, b(y)).$$

The volume element induced by this map is dominated by the volume element induced by the product mapping

$$(s, y, a) \mapsto (v(y), \tfrac{1}{s}a, b(y);\, v(y), a, b(y)).$$

Now the map $(s,a) \mapsto (\frac{1}{s}a, a)$ is algebraic and its image is a submanifold of finite volume in $G_k(\mathbf{C}^{n-m+2k}) \times G_k(\mathbf{C}^{n-m+2k})$. This proves that $T_{\alpha,K}$ has finite volume for compact subsets $K \subset U$. ∎

We now consider the local MacPherson formula for a normal bundle map α. We have seen that for each k, $\Sigma_k(\alpha)$ is a smooth submanifold of locally finite volume and of (real) codimension $2k(n-m+k)$ in X. Along each $\Sigma_k(\alpha)$ it is clear that $\ker\alpha \subset E$ and $\operatorname{Im}\alpha \subset F$ are smooth vector bundles. Furthermore, the bundle $\operatorname{Hom}(\ker\alpha, \operatorname{coker}\alpha)$ is naturally equivalent to the normal bundle of $\Sigma_k(\alpha)$ in X. (See [Mac$_1$] for example.) Now fix $x_0 \in \Sigma_k(\alpha)$ and choose coordinates as in Lemma 10.1. Then from (10.5) we see that for each $\ell \le k$ we have a splitting in U:

$$\Sigma_\ell(\alpha) \;=\; \mathbf{R}^N \times \Sigma_\ell$$

where

$$\Sigma_\ell \subset \operatorname{Hom}_{k,n-m+k} \subset G_k(\mathbf{C}^{n-m+2k})$$

is the universal degeneracy locus (where $\dim\ker a = \ell$). Our section of $G^{\oplus 2}$ can now be written

$$(\varphi_s\alpha(y,a),\, \alpha(y,a)) \;=\; (v(y), \tfrac{1}{s}a, sb(y);\, v(y), a, b(y)).$$

From here it is straightforward to see that in these coordinates on $G^{\oplus 2}\big|_U$

$$\Gamma_{\alpha,0} \;=\; \mathbf{R}^N \times \Gamma_0 \tag{10.6}$$

where Γ_0 is the current in $G_k(\mathbf{C}^{n-m+2k}) \times G_k(\mathbf{C}^{n-m+2k})$ defined universally in §3 as the limit of the sets $\Gamma_s = \operatorname{Cl}\{(\frac{1}{s}a, a) : a \in \operatorname{Hom}_{k,n-m+k}\}$ as $s \to 0$. Thus the analysis of §§5-6 applies directly and we conclude the following.

Theorem 10.3. *Let $E \to X$ and $F \to X$ be smooth complex vector bundles with $\operatorname{rank}(E) \le \operatorname{rank}(F)$, and let $\alpha : E \to F$ be a normal bundle map. Suppose Φ is an invariant polynomial on the Lie algebra of the structure group of E. Then for any choice of connections on E and F there exists an $L^1_{\rm loc}$-form T on X so that*

$$\Phi(\Omega^E) \;=\; \sum_{k=0}^{m} \operatorname{Res}_{\Phi,k}[\Sigma_k(\alpha)] \;+\; dT \tag{10.7}$$

$$\operatorname{Res}_{\Phi,k} \;=\; (\pi_2)_*\Phi\left(\Omega^{\operatorname{Im}\alpha \oplus U_k}\right)$$

where $(\pi_2)_*$ denotes integration over the fibres of the Grassmann compactification $G_k = G_k(\ker\alpha \oplus \operatorname{coker}\alpha)$ of the normal bundle $N_{\Sigma_k} \cong \operatorname{Hom}(\ker\alpha, \operatorname{coker}\alpha)$ to $\Sigma_k(\alpha)$, and where $U_k \to G_k$ is the tautological k-plane bundle.

If $\operatorname{rank}(E) = \operatorname{rank}(F)$, then formula (10.7) becomes

$$\Phi(\Omega^E) \;-\; \Phi(\Omega^F) \;=\; \sum_{k=1}^{m} \operatorname{Res}_{\Phi,k}[\Sigma_k(\alpha)] \;+\; dT$$

If Ψ is an invariant polynomial on the Lie algebra of the structure group of F, then there exists an $L^1_{\rm loc}$-form T' on X so that

$$\Psi(\Omega^F) \ = \ \sum_{k=0}^{m} \mathrm{Res}_{\Psi,k}[\Sigma_k(\alpha)] \ + \ dT' \tag{10.8}$$

$$\mathrm{Res}_{\Psi,k} \ = \ (\pi_2)_* \Psi\left(\Omega^{(\ker\alpha)^\perp \oplus U_k^\perp}\right)$$

§**11. Real analytic bundle maps.** In this section we derive a Local MacPherson Formula for real analytic bundle maps under the assumption that the degeneracy loci have the expected dimension. We begin with the following general result.

Lemma 11.1. *Let $\alpha : E \to F$ be a geometrically atomic bundle map over a smooth manifold X, and let $\Gamma_{\alpha,0}$ be the current from (9.5). Then*

$$\mathrm{supp}\ \Gamma_{\alpha,0} \ \subset \ \bigcup_k \Upsilon_k \times_{F_k} \Sigma_k(\alpha) \tag{11.1}$$

Proof. Fix $x \in \Sigma_k(\alpha)$ and choose local trivializations of E and F in a neighborhood U of x so that $G^{\oplus 2}\big|_U \cong U \times G_m \times G_m$ where $G_m = G_m(\mathbf{C}^{m+n})$. Suppose $(x, P', P) \in \mathrm{supp}\ \Gamma_{\alpha,0}$. Then there exist sequences $x_j \to x$ and $t_j \to 0$ such that

$$\gamma_j \equiv \mathrm{gr}(\alpha_{x_j}) \longrightarrow P \qquad \text{and} \qquad \gamma_j' \equiv \mathrm{gr}(\tfrac{1}{t_j}\alpha_{x_j}) = \varphi_{t_j}\gamma_j \longrightarrow P'$$

where φ_t is the flow from §2. It is now an elementary argument (as in [HL$_3$, Lemma 2.10]) to see that P must be joined to P' by a piecewise flow line in G_m which passes through F_k. In particular, $P' \in \Upsilon_k$ and $\pi_2(P') = \pi_1(P)$, where π_1 and π_2 are the projections from §6. Thus $(P', P) \in \Upsilon_k \times_{F_k} \Sigma_k(\alpha)$ as claimed. ∎

Suppose now that $\alpha : E \to F$ is a real analytic bundle map between complex vector bundles over a ν-dimensional manifold X, and that

$$\mathrm{rank}(E) = m \leq n = \mathrm{rank}(F).$$

Then for each k the degeneracy locus $\Sigma_k(\alpha)$ is an analytic subset of X of some dimension, say $\nu_{\alpha,k}$. Therefore, $\Sigma_k(\alpha)$ has locally finite $\nu_{\alpha,k}$-measure, and integration over the regular points of $\Sigma_k(\alpha)$ defines an integral current $[\Sigma_k(\alpha)]$ of dimension $\nu_{\alpha,k}$. Recall that the "expected" dimension of $\Sigma_k(\alpha)$ in X is $\nu_k \equiv \nu - 2k(n-m+k)$.

Theorem 11.2. *Let $\alpha : E \to F$ be as above and suppose that*

$$\dim \Sigma_k(\alpha) \leq \nu_k \tag{11.2}$$

for each k. Then there exist integer-valued functions n_k on $\Sigma_k(\alpha)$, constant on each irreducible component, such that for all Φ, Ψ as in 10.3 and all connections on E and F, there exist L^1_{loc}-forms T, T' on X such that

$$\Phi(\Omega^E) \;=\; \sum_{k=0}^{m} n_k \operatorname{Res}_{\Phi,k}[\Sigma_k(\alpha)] \;+\; dT \qquad \text{and}$$

$$\Psi(\Omega^F) \;=\; \sum_{k=0}^{m} n_k \operatorname{Res}_{\Psi,k}[\Sigma_k(\alpha)] \;+\; dT'$$

where $\operatorname{Res}_{\Phi,k}$ and $\operatorname{Res}_{\Psi,k}$ are smooth forms defined on the regular set of $\Sigma_k(\alpha)$ exactly as in Theorem 10.3.

Proof. By Theorem 9.5 and its proof (in particular the discussion from (9.5) to (9.7)) we need only to compute

$$\mathbf{\Gamma}_{\alpha,0}(\omega) \;=\; p_*\{(\mathrm{pr}_1^*\,\omega)\wedge \Gamma_{\alpha,0}\} \tag{11.3}$$

where $\omega = \Phi(\Omega^U)$ or $\Psi(\Omega^{U^\perp})$. Now it follows from assumption (11.2) that

$$\dim\{\Upsilon_k \times_{F_k} \Sigma_k(\alpha)\} \;\le\; \nu = \dim \Gamma_{\alpha,0}$$

for all k. Hence, from (11.1), the fact that $d\Gamma_{\alpha,0} = 0$, and the Federer Flat Support Lemma [F;4.1.15] it follows that

$$\Gamma_{\alpha,0} = \sum_k n_k[\Upsilon_k \times_{F_k} \Sigma_k(\alpha)].$$

where $n_k : \Sigma_k(\alpha) \to \mathbf{Z}$ is locally constant on the regular set and 0 on any component of dimension $< \nu_k$. Computing (11.3) at regular points of $\Sigma_k(\alpha)$ gives the residue forms as in §8. ∎

Note. The function n_k represents the *order of k-degeneracy* of α.

§12. Analysis of the currents $\operatorname{Res}_{\Phi,k}[\Sigma_k]$ and residue calculations. In this section we shall study the singular currents which appear in our formulas. They have a surprizingly regular structure and the residues are explicitly computable in many cases. Our first result is that for regular bundle maps, each of the terms $\operatorname{Res}_{\Phi,k}[\Sigma_k]$ occuring in the main formula is a d-closed current of finite mass. We begin with the following.

Proposition 12.1. *Let $\alpha : E \to F$ be as in Theorem 10.3 or Theorem 11.2. Then for each k, integration over the regular points of $\Sigma_k(\alpha)$ defines a locally rectifiable current $[\Sigma_k(\alpha)]$ in X with*

$$d[\Sigma_k(\alpha)] \;=\; 0 \tag{12.1}$$

Proof. Corollary 10.2 and the discussion prior to 11.2 show that $\Sigma_k(\alpha)$ has locally finite volume in X and therefore defines a locally rectifiable current. Note that

$$\operatorname{supp}\{d[\Sigma_k(\alpha)]\} \subseteq \bigcup_{\ell>k} \Sigma_\ell(\alpha),$$

and that $\operatorname{codim}_{\mathbf{R}} \Sigma_\ell = 2\ell(n-m+\ell)$. Since $d[\Sigma_k(\alpha)]$ is a flat current of codimension $2k(n-m+k)-1$, it follows from [F, 4.1.15] that $d[\Sigma_k(\alpha)] = 0$. ∎

Proposition 12.2. *Let $\alpha : E \to F$, Φ and Ψ be as in 10.3 or 11.2. Then for each k, the currents*

$$R_k \equiv \operatorname{Res}_{\Phi,k}[\Sigma_k(\alpha)] \qquad \text{and} \qquad R'_k \equiv \operatorname{Res}_{\Psi,k}[\Sigma_k(\alpha)]$$

have locally finite mass in X and satisfy

$$dR_k = dR'_k = 0.$$

Proof. We begin with the universal case. In Propositions 6.3 and 6.5 it is proved that the currents R_k and R'_k have locally finite mass. We recall that this is done as follows. Consider the desingularization

$$\overline{j}_1 : G(\eta \oplus \xi^\perp) \longrightarrow \overline{\Sigma}_k$$

of the closure of Σ_k given by the Grassmann compactification of $\Sigma_k \cong \operatorname{Hom}(\eta, \xi^\perp)$ (cf. (6.2)). The projection $\pi_1 : \Sigma_k \to F_k$ extends to a smooth map

$$\widetilde{\pi}_1 : G(\eta \oplus \xi^\perp) \longrightarrow F_k.$$

We pull the bundle $\Upsilon_k \cong \operatorname{Hom}(\xi, \eta^\perp)$ back via $\overline{j}_1$ and take its Grassmann compactification $G(\xi \oplus \eta^\perp)$. Let $U_k \to G(\xi \oplus \eta^\perp)$ be the tautological bundle. Then we have the identity $U = U_k \oplus \eta$ (cf. §8), and

$$Res_{\Phi,k} = (\overline{j}_1)_* \Phi\left(\Omega^{U_k \oplus \eta}\right).$$

Since $\Phi\left(\Omega^{U_k\oplus\eta}\right)$ is a smooth form on the manifold $G(\eta \oplus \xi^\perp)$, its push-forward by $\overline{j}_1$ has finite mass. Furthermore, since $\Phi\left(\Omega^{U_k\oplus\eta}\right)$ is d-closed on $G(\eta \oplus \xi^\perp)$, its push-forward is d-closed on X. A similar argument applies for R'_k. This gives the result in the universal case.

The normal case is proved in parallel fashion by using the regular singular structure of $\Sigma_k(\alpha)$ established in §10. Namely, from Lemma 10.1 we see that $\Sigma_k(\alpha)$ has the same singular structure as Σ_k in the universal case, and the arguments above apply straightforwardly. In the analytic case one replaces the desingularization $\overline{j}_1$ by resolution of singularities. ∎

We now address the question of the residue forms themselves. From forumla (8.0) we see that it would be particularly nice if the connection on $U\big|_{\Upsilon_k} = \eta \oplus U_k$ were a direct-sum connection $D^\eta \oplus D^{U_k}$. Explicit calculation shows that this is not the case. However, one could hope that for an invariant form Φ, there is an equality

$$\pi_{2*}\Phi\left(\Omega^{\eta\oplus U_k}\right) \;=\; \pi_{2*}\Phi\left(\Omega^\eta \oplus \Omega^{U_k}\right) \tag{12.2}$$

where

$$\Omega^\eta \oplus \Omega^{U_k} \;=\; \begin{pmatrix} \Omega^\eta & 0 \\ 0 & \Omega^{U_k} \end{pmatrix}$$

is the curvature of the direct-sum connection obtained from the given connection by taking orthogonal projection of the covariant derivative $D^{\eta\oplus U_k}$ back onto the factors η and U_k. This is often the case (cf. [HL_1], [Z]). However, the work of John Zweck [Z, Thm. 4.17] on meromorphic sections of vector bundles shows that (12.2) does not always hold. Nevertheless, we do have the following.

Lemma 12.3. *The general residue form $\mathrm{Res}_{\Phi,k}$ on Σ_k appearing in Theorems 10.3 and 11.2 can be written as*

$$\mathrm{Res}_{\Phi,k} \;=\; \pi_{2*}\Phi\left(\Omega^{\mathrm{Im}\,\alpha} \oplus \Omega^{U_k}\right) + dS_k \tag{12.3}$$

where S_k is a smooth form written universally in terms of the curvature and connection of E and F.

Proof. Consider the linear family $D^t = (1-t)D^U + t(D^{\mathrm{Im}\,\alpha} \oplus D^{U_k})$ joining the given connection on $U = \mathrm{Im}\,\alpha \oplus U_k$ and the direct-sum connection, and set S_k be the standard Chern transgression form (cf. [HL_1]). ∎

Remark 12.4. The universal expression (12.3) can be regarded in another way. To derive it, it suffices to consider the universal case. In fact via [NS] it suffices to consider the case where $X = G_m(\mathbf{C}^M)\times G_n(\mathbf{C}^N)$ and E and F are the pull-backs of the tautological bundles $\mathbf{E} \to G_m(\mathbf{C}^M)$ and $\mathbf{F} \to G_n(\mathbf{C}^N)$ respectively. Here $\pi_{2*}\Phi\left(\Omega^{\eta\oplus U_k}\right)$ is a $U_M \times U_N$-invariant form on the universal k^{th} fixed-point set $G_k(E) \times G_N(F)$. Now $F_k = G_k(E) \times G_N(F)$ is a product of two-stage flag manifolds, and the invariant forms in a given cohomology class are not unique. However, any two cohomologous invariant forms differ by the exterior derivative of an invariant form. To derive equation (12.3) explicitly in any given case, it suffices to do it on this particular manifold X.

Proposition 12.5. *Let $\alpha : E \to F$, Φ and Ψ be as in 10.3 and 11.2. For each k let*

$$\mathrm{Res}_{\Phi,k} \;=\; \widetilde{\mathrm{Res}}_{\Phi,k} + dS_k$$

be the canonical decomposition of the residue form given in (12.3). Then each of the currents

$$\widetilde{R}_k \;\equiv\; \widetilde{\mathrm{Res}}_{\Phi,k}[\Sigma_k(\alpha)], \qquad S_k[\Sigma_k(\alpha)], \qquad \text{and} \qquad (dS_k)[\Sigma_k(\alpha)]$$

has locally finite mass in X, and furthermore the following equation holds on X:

$$d\left(S_k[\Sigma_k(\alpha)]\right) \;=\; d(S_k)[\Sigma_k(\alpha)]. \tag{12.4}$$

The parallel results hold for $\mathrm{Res}_{\Psi,k}$.

Proof. The proof that these currents have finite mass follows exactly the lines of the proof of 12.2. Assertion (12.4) then follows from (12.1). ∎

Corollary 12.6. *The cycles* $R_k \equiv \mathrm{Res}_{\Phi,k}[\Sigma_k(\alpha)]$ *and* $\widetilde{R}_k \equiv \widetilde{\mathrm{Res}}_{\Phi,k}[\Sigma_k(\alpha)]$ *are cohomologous in the complex of locally flat currents on* X. *In particular they represent the same class in* $H^*(X;\mathbf{R})$. *(The analogous assertion holds for* $\mathrm{Res}_{\Psi,k}[\Sigma_k(\alpha)]$.*)*

Combining the above gives the following.

Theorem 12.7. *Let* $\alpha : E \to F$, Φ *and* Ψ *be as in Theorem 10.3 or Theorem 11.2. Then the following equation holds on* X*:*

$$\Phi(\Omega^E) \;=\; \sum_k \widetilde{\mathrm{Res}}_{\Phi,k}[\Sigma_k(\alpha)] + d\widetilde{T}$$

where

$$\widetilde{\mathrm{Res}}_{\Phi,k} \;=\; \int_{\pi_k} \Phi\left(\Omega^{\pi_k^*\mathrm{Im}\,\alpha} \oplus \Omega^{U_k}\right)$$

and where $\pi_k : G_k(\ker\alpha \oplus \mathrm{coker}\alpha) \longrightarrow \Sigma_k(\alpha)$ *is the Grassmann compactification of the normal bundle to* $\Sigma_k(\alpha)$, $\mathrm{Im}\ \alpha \subset F$ *carries the induced connection,* $\Omega^{\pi_k^*\mathrm{Im}\,\alpha} \oplus \Omega^{U_k}$ *denotes the curvature of the direct sum connection, and* $\widetilde{T}$ *is a flat current on* X. *In particular, if* Φ *is a multiplicative series, then*

$$\widetilde{\mathrm{Res}}_{\Phi,k} \;=\; \left\{\int_{\pi_k} \Phi(\Omega^{U_k})\right\} \Phi(\Omega^{\mathrm{Im}\ \alpha}).$$

The analogous result holds for $\Psi(\Omega^F)$.

It is interesting to examine some basic examples. For convenience we shall drop the tilde from our notation.

Example 12.8. Let

$$\Phi(\Omega) = c(\Omega) \stackrel{\mathrm{def}}{=} \det\left(I + \tfrac{i}{2\pi}\Omega\right)$$

be the **total Chern class** and suppose that $m = n$. Then

$$\begin{aligned}\mathrm{Res}_{c,1} &= c(\Omega^{\mathrm{Im}\ \alpha})\\ \mathrm{Res}_{c,k} &= 0 \quad \text{for all} \quad k > 1.\end{aligned}$$

Example 12.9. Let

$$\Phi(\Omega) = \mathrm{ch}(\Omega) \stackrel{\mathrm{def}}{=} \exp\{\tfrac{i}{2\pi}\Omega)\}$$

be the **Chern character** and suppose that $m = n$. Then

$$\mathrm{Res}_{\mathrm{ch},1} \;=\; -\frac{\mathrm{ch}(\Omega^{\ker\alpha}) - \mathrm{ch}(\Omega^{\mathrm{coker}\,\alpha})}{c_1(\Omega^{\ker\alpha}) - c_1(\Omega^{\mathrm{coker}\,\alpha})}$$

$$\mathrm{Res}_{\mathrm{ch},k} \;=\; 0 \quad \text{for all} \quad k > 1.$$

Example 12.10. Let

$$\Phi(\Omega) = c^{\perp}(\Omega) \stackrel{\text{def}}{=} \det\left\{(I + \tfrac{i}{2\pi}\Omega)^{-1}\right\}$$

be **the total dual Chern class** and suppose that $m = n$. Then a calculation shows that for any $\ell \geq 1$

$$\begin{aligned}
\mathrm{Res}_{(c^{\perp})^{\ell},1} &= -c^{\perp}(\Omega^{\mathrm{Im}\ \alpha})^{\ell} \left\{ \frac{c^{\perp}(\Omega^{\ker\alpha})^{\ell} - c^{\perp}(\Omega^{\mathrm{coker}\,\alpha})^{\ell}}{c_1(\Omega^{\ker\alpha}) - c_(\Omega^{\mathrm{coker}\,\alpha})} \right\} \\
\mathrm{Res}_{(c^{\perp})^{\ell},\ell} &= c^{\perp}(\Omega^{\mathrm{Im}\ \alpha})^{\ell} c^{\perp}(\Omega^{\ker\alpha})^{\ell} c^{\perp}(\Omega^{\mathrm{coker}\,\alpha})^{\ell} \\
\mathrm{Res}_{(c^{\perp})^{\ell},k} &= 0 \quad \text{for all} \quad k > \ell.
\end{aligned}$$

From this example we get the following pretty formulas for a normal bundle map $\alpha : E \to F$ over a manifold X where $\mathrm{rank}(E) = \mathrm{rank}(F)$. Fix any integer $\ell \geq 1$. Then there exists a flat current S_ℓ on X such that

$$\boxed{\begin{aligned}
c^{\perp}(\Omega^{E})^{\ell} - c^{\perp}(\Omega^{F})^{\ell} &= -c^{\perp}(\Omega^{\mathrm{Im}\,\alpha})^{\ell} \left\{ \frac{c^{\perp}(\Omega^{\ker\alpha})^{\ell} - c^{\perp}(\Omega^{\mathrm{coker}\,\alpha})^{\ell}}{c_1(\Omega^{\ker\alpha}) - c_(\Omega^{\mathrm{coker}\,\alpha})} \right\} [\Sigma_1] \\
&\quad + \dots \\
&\quad + c^{\perp}(\Omega^{\mathrm{Im}\,\alpha})^{\ell} c^{\perp}(\Omega^{\ker\alpha})^{\ell} c^{\perp}(\Omega^{\mathrm{coker}\,\alpha})^{\ell} [\Sigma_\ell] \ + \ dS_\ell
\end{aligned}}$$

§**13. Results for real vector bundles.** Up to this point the bundles E and F have been assumed to be complex. We now re-examine our results under the assumption that E and F are real vector bundles. One verifies directly that in this case the fundamental constructions presented in §§2—12 carry through with virtually no change provided that the fibre diagonal $\Delta_G \subset G^{\oplus 2}$ and its isotopic deformations Γ_s, (cf. (3.3)) define currents on $G^{\oplus 2}$, and provided that the projection pr_2 induces a map $(\mathrm{pr}_2)_*$on currents (cf. (4.1) and (4.2)).

We recall that **currents of dimension** p on a manifold Y are defined as the topological dual space of the space $\widetilde{\mathcal{E}}^p(Y)$ of compactly supported, smooth p-forms twisted by the orientation bundle Or_Y of Y (cf. [deR], [S]). Any p-dimensional submanifold with *oriented* normal bundle defines such a current. So also does any smooth form of degree $n-p$ on Y, and in fact every current can be considered to be an $(n-p)$-form on Y with generalized coefficients.

Note that a smooth mapping $f : Y \to Y'$ between smooth manifolds induces a continuous linear map f_* on currents if and only if $f^*Or_{Y'} = Or_Y$. When f is a submersion, this condition is guaranteed if the fibres of f are orientable.

Now the normal bundle N to the fibre diagonal $G \cong \Delta_G \subset G^{\oplus 2}$ is isomorphic to the bundle of tangent vectors to the fibres of the fibration $\pi : G \to X$. That is, there is a bundle equivalence

$$N \;\cong\; \mathrm{Hom}(U, U^{\perp}) \tag{13.1}$$

We recall an important elementary fact. (See [HL_2, A.13], for example.)

Lemma 13.1. *Let V, V' be real vector bundles over a space Y. If* $\mathrm{rank}(V) - \mathrm{rank}(V')$ *is even, then $Hom(V, V')$ is orientable.*

Recall that $\mathrm{rank}(U) = \mathrm{rank}(E)$ and $\mathrm{rank}(U^{\perp}) = \mathrm{rank}(F)$, and note that the fibres of the projections $\mathrm{pr}_i : G^{\oplus 2} \to G$ are diffeomorphic to the Grassmannian $G_m(\mathbf{R}^{m+n})$ whose tangent bundle is $\mathrm{Hom}(U, U^{\perp})$. Observe also that for a normal bundle map $\alpha : E \to F$ the normal bundle to each $\Sigma_k(\alpha)$ is isomorphic to $\mathrm{Hom}(\ker\alpha, \mathrm{coker}\,\alpha)$. Thus from (13.1) and the Lemma one deduces

Lemma 13.2. *Suppose that* $\mathrm{rank}(F) - \mathrm{rank}(E)$ *is even. Then the submanifolds $\Gamma_s \subset G^{\oplus 2}$ define currents on $G^{\oplus 2}$, and the projections pr_i, $i = 1, 2$, induce continuous maps on currents. Furthermore, if $\alpha : E \to F$ is a normal bundle map, then each submanifold $\Sigma_k(\alpha)$ defines a current $[\Sigma_k(\alpha)]$ on X.*

Note that these results are independent of all considerations of orientability for E and F on X.

This brings us to the main result of this section.

Theorem 13.3. *Let $E \to X$ and $F \to X$ be smooth real vector bundles where* $\mathrm{rank}(F) - \mathrm{rank}(E)$ *is even. Then the analogues of all results in §§2—12 hold for these bundles. Furthermore, if E and F are given orthogonal connections, then in formulas (6.3), (8.4-5), (10.7-8) and in Theorems 11.2 and 12.6, one has that*

$$\mathrm{Res}_k \;= 0 \qquad \textit{for all } k \textit{ odd.} \tag{13.2}$$

Proof. Once one knows that the submanifolds Γ_s define currents in $G^{\oplus 2}$, that pr_2 induces a continuous map on currents, and that the submanifolds Σ_k and Υ_k define currents in X, the discussion given in §§2—12 carries through without change in the real case. This gives the first part of the theorem.

To prove (13.2) we use Theorem 12.6. Observe first that every O_m-invariant polynomial Φ is a polynomial in the Pontrjagin forms and that for the total Pontrjagin form we have

$$p\left(\widetilde{\Omega}^U\right) \;=\; p\left(\Omega^{U_k}\right)\pi^* p\left(\Omega^{\mathrm{Im}\,\alpha}\right).$$

Hence any polynomial in the Pontrjagin classes $p_j\left(\widetilde{\Omega}^U\right)$ can be expressed as a polynomial in the Pontrjagin classes $p_j\left(\Omega^{U_k}\right)$ with coefficients which are pull-backs over π of forms on $\Sigma_k(\alpha)$. Thus to prove (13.2) it will suffice to prove the following.

Lemma 13.4. *Let $V \to Y$ be a smooth riemannian vector bundle of rank M with orthogonal connection, and let $G = G_\ell(V) \to Y$ be the Grassmann bundle of unoriented ℓ-planes in V. Let $\pi : U \to G$ be the tautological ℓ-plane bundle, and write $\pi^*V = U \oplus U^\perp$. Give U the connection obtained by projection of the pull-back connection on π^*V. Then if rank(V) is odd, one has*

$$\int_\pi \Phi\left(\Omega^U\right) \;=\; 0$$

for all O_M-invariant polynomials Φ on the Lie algebra $\mathfrak{O}_M$.

Proof. To begin we observe that the result holds in the special case where $V = V_0$ is the tautological bundle over the real Grassmannian $G_M(\mathbf{R}^N) \equiv Y_0$. This follows because the closed form

$$\int_\pi \Phi\left(\Omega^{U_0}\right)$$

(where U_0 is the tautological bundle over Y_0) is O_N-invariant and hence harmonic, but also of odd degree and therefore zero since $H^{\mathrm{odd}}(Y_0;\,\mathbf{R}) = 0$.

The general case follows from the special one because by [NS] there exists an embedding $j : Y \hookrightarrow Y_0$ such that $j^*V_0 \cong V$ as bundles with connection. It follows from the naturality of the constructions that there exists a bundle map $\widetilde{j} : G_\ell(V) \to G_\ell(V_0)$ covering j such that $\widetilde{j}^*U_0 \cong U$ as bundles with connection. Thus there is a commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{\;\widetilde{\widetilde{j}}\;} & U_0 \\
\downarrow & & \downarrow \\
G_\ell(V) & \xrightarrow{\;\widetilde{j}\;} & G_\ell(V_0) \\
\pi\downarrow & & \downarrow\pi_0 \\
Y & \xrightarrow{\;j\;} & Y_0
\end{array}$$

where $\widetilde{j}$ and $\widetilde{\widetilde{j}}$ are bundle maps. One concludes that

$$0 \;=\; j^*(\pi_0)_*\Phi\left(\Omega^{U_0}\right) \;=\; (\pi)_*\left(\widetilde{j}^*\Phi\left(\Omega^{U_0}\right)\right) \;=\; \pi_*\left(\Phi\left(\Omega^U\right)\right). \;\blacksquare\blacksquare$$

Note 13.5 . When F (or E) is orientable, these results extend to invariant polynomials on the Lie Algebra of SO_n (or SO_m respectively).

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