POTENTIAL THEORY FOR NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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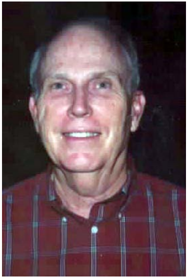
with Reese Harvey



The Outline

1. GENERALIZED POTENTIAL THEORY

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2. EXAMPLES

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3. **THE RIESZ CHARATERISTIC**

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3. **THE RIESZ CHARATERISTIC**

4. **TANGENTS AND DENSITIES**

GENERALIZED POTENTIAL THEORY

The Idea

To every differential equation on $\mathbb{R}^n$ of the form:

$$f(D^2 u) \ = \ 0$$

there is an associated "Potential Theory"

based on the functions which satisfy the condition:

$$f(D^2 u) \ \geq \ 0$$

in a generalized sense.

Classical Examples

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The Theory of Subharmonic Functions

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"$\Delta u \;\geq\; 0$".

The upper semi-continuous functions which are
"sub the harmonics":

$$u \;\leq\; h \quad \text{on} \quad \partial K \qquad \Rightarrow \qquad u \;\leq\; h \quad \text{on} \quad K$$

Classical Examples

2. **The Real Monge-Ampère Equation:**

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The Theory of Convex Functions

$$\text{“}D^2 u \geq 0\text{”}.$$

Classical Examples

3. **The Complex Monge-Ampère Equation:**

$$\mathrm{det}_{\mathbb{C}}\left\{(D^2u)_{\mathbb{C}}\right\} = 0 \quad \text{and} \quad (D^2u)_{\mathbb{C}} \geq 0.$$

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The Associated Potential Theory:

The Theory of Plurisubharmonic Functions

"$(D^2u)_{\mathbb{C}} \geq 0$".

The upper semi-continuous functions on $\mathbb{C}^n$ whose restriction to every affine complex line is subharmonic

Poincaré, Oka, Grauert, Lelong, Hörmander, etc.

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$$\arctan\left(D^2 u\right) = c$$ **Special Lagrangian Potential Equation**

The Usual Set-up

One fixes a continuous function

$$f : \mathrm{Sym}^2(\mathbb{R}^n) \to \mathbb{R},$$

and associates to it the nonlinear **differential equation**

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$$A_{\mathbb{C}} \;=\; \tfrac{1}{2}(A - JAJ)$$

A General Geometric Approach

Start with a general closed **constraint set**

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on second derivatives:

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for $u \in C^2(X)$.

Obective: To extend this to a larger class of functions u,

called the ***F*-subharmonic functions** $F(X)$,

and to develop an accompanying potential theory for this class.

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Take $u \in \mathrm{USC}(X)$ and require that for any $\varphi \in C^2(X)$,

the condition

$$u \leq \varphi \qquad \text{and} \qquad u(x_0) = \varphi(x_0)$$

must imply

$$D^2_{x_0}\varphi \in F.$$

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Note:

If φ satisfies the conditions above for u at a point x_0, then so does

$$\widetilde{\varphi}(x) \;=\; \varphi(x) + \tfrac{1}{2}\langle A(x - x_0), x - x_0\rangle$$

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Conclusion: We must require $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ to satisfy the condition

$$\mathbf{F} + \mathcal{P} \;\subset\; \mathbf{F}$$

where $\mathcal{P} \equiv \;\{A : A \geq 0\}$.

Definitions

A closed subset $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is called a **subequation**

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We want to **extend these notions to upper semi-continuous functions.**

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- If $u \in C^2(X)$, then

$$u \in F(X) \quad \iff \quad D_x^2 u \in F \quad \forall\, x \in X.$$

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- Note that
$$F \cap -\widetilde{F} = \partial F$$

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The homogeneous real Monge-Ampère Equation

$D^2 u \geq 0$ **and** $\det(D^2 u) = 0$.

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(1) $u\big|_{\Omega}$ *is F-harmonic, and*

(2) $u\big|_{\partial\Omega} = \varphi.$

EXAMPLES

Standing Assumption from now on:

The subequation $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is a *cone* which is invariant under a subgroup

$$G \subset \mathrm{O}(n)$$

which acts transitively on the sphere

$$S^{n-1} \subset \mathbb{R}^n.$$

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If $F \subset \mathrm{Sym}^2(\mathbb{R}^n)$ is O(n)-invariant,

then it is completely determined by a condition

on the **eigenvalues** of $A \in \mathrm{Sym}^2(\mathbb{R}^n)$.

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These subequations are U(n) and Sp(n) invariant.

$$\mathbb{C}^n = (\mathbb{R}^{2n}, J)$$

The **complex hermitian symmetric part** of $A \in \mathrm{Sym}^2(\mathbb{R}^{2n})$ is

$$A_{\mathbb{C}} \ = \ \frac{1}{2}(A - JAJ).$$

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The eigenspaces of $A_{\mathbb{H}}$ are quaternionic lines with eigenvalues $\lambda_1, ..., \lambda_n$.

Example: Monge-Ampère Equations

$$\mathcal{P} \qquad \Rightarrow \qquad \mathcal{P}^{\mathbb{C}}, \ \ \mathcal{P}^{\mathbb{H}}$$

The complex and quaternionic

Monge-Ampère Equations

Examples: Other Elementary Symmetric Functions Hessian Equations – Trudinger-Wang-Labutin

$$\Sigma_k \equiv \{A : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\}$$

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This is the **principal branch** of the equation

$$\sigma_k(D^2 u) = 0.$$

The equation has $(k-1)$ other branches.

It also has **complex and quaternionic counterparts.**

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This is always a subequation.

Geometric Subequations – Examples

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$$MA_p(A) = \prod_{i_1 < \cdots < i_p} \left(\lambda_{i_1}(A) + \cdots + \lambda_{i_p}(A) \right) = 0.$$

THE RIESZ CHARACTERISTIC

The Riesz Kernel

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$$K_p(x) = r^{p-2} K(rx) \qquad \text{for all} \quad r > 0 \quad \text{when } p \neq 2$$

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The Pucci equation: For $0 < \lambda < \Lambda$

$$\mathcal{P}_{\lambda,\Lambda} \equiv \{A \in \mathrm{Sym}^2(\mathbb{R}^n) : \lambda \mathrm{tr} A^+ + \Lambda \mathrm{tr} A^- \geq 0\}, \qquad p = \frac{\lambda}{\Lambda}(n-1) + 1.$$

SOME INITIAL RESULTS

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If u is continuous on Ω and F-harmonic on $\Omega - E$,

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When $F = \mathcal{P}^{\mathbb{C}}$

$$u(z) \;=\; \log|f(z)| \quad \text{with } f \text{ holomorphic}$$

is plurisubharmonic.

TANGENTS AND DENSITIES

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\end{aligned}$$

Note: One has $u_r \in F(B_{R/r})$ and $B_{R/r}$ expands to $\mathbb{R}^n$ as $r \downarrow 0$.

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2. Uniqueness fails utterly for $\mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{H}}$.

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THEOREM (Strong Uniqueness II). *Fix $p \geq 2$ and $n \geq 3$. Then strong uniqueness of tangents to $F(\mathbb{G})$-subharmonic functions holds for:*

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(b) Every compact $Sp(n) \cdot \mathrm{Sp}(1)$-invariant subset $\mathbb{G} \subset G^{\mathbb{R}}(p, \mathbb{H}^n)$ with three exceptions, namely the sets of real p-planes which lie in a quaternion line for $p = 2, 3, 4$ (when $p = 4$ this is $G^{\mathbb{H}}(1, \mathbb{H}^n)$),

(c) For $p \geq 5$, every compact $Sp(n)$-invariant subset $\mathbb{G} \subset G^{\mathbb{R}}(p, \mathbb{H}^n)$.

Tangents to Subsolutions – Strong Uniqueness

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THEOREM (Strong Uniqueness III). *If $\mathbb{G}$ has the transitivity property, then*

$$\text{strong uniqueness holds for } F(\mathbb{G}).$$

Special Cases

This proves the strong uniqueness of tangent cones when:

(a) $\mathbf{G} = \text{ASSOC}$ (Associative subharmonic functions in $\mathbb{R}^7$) ($p = 3$).

(b) $\mathbf{G} = \text{COASSOC}$ (Coassociative subharmonic functions in $\mathbb{R}^7$) ($p = 4$).

(c) $\mathbf{G} = \text{CAYLEY}$ (Cayley subharmonic functions in $\mathbb{R}^8$) ($p = 4$).

(d) $\mathbf{G} = \text{LAG}$ (Lagrangian subharmonic functions in $\mathbb{C}^n$) ($p = n$).

Strong Uniqueness Fails

Recall that in the three cases:

$$G(1, \mathbb{R}^n) \quad \text{(i.e., } F = \mathcal{P}\text{)}$$

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In the complex case, uniqueness fails. A complete characterization of the possible $T_0 u$ is due to **KISELMAN.**

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is an increasing radial F-subharmonic function iff $\psi(t)$ satisfies the one-variable subequation

$$\psi''(t) + \frac{(p-1)}{t}\psi'(t) \;\geq\; 0 \quad \text{and} \quad \psi'(t) \;\geq\; 0.$$

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where $C \geq 0$, $k \in \mathbb{R}$, and $K_p(t)$ is the p^{th} Riesz function defined on $0 < t < \infty$ by

$$K_p(t) = \begin{cases} t^{2-p} & \text{if } 1 \leq p < 2 \\ \log t & \text{if } p = 2 \\ -\frac{1}{t^{p-2}} & \text{if } 2 < p < \infty. \end{cases}$$

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exists and defines the **Ψ-density**, $0 \le \Theta^{\Psi} < \infty$, *of u at 0.*

$\Psi = M, S$ **or** V

Densities

In fact,

$$\Theta^{\Psi}(u, 0) \;=\; \lim_{r \downarrow 0} \frac{\Psi(r)}{K(r)}.$$

(When $1 \leq p < 2$, we must normalize so that $\Psi(0) = 0$.)

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For each $c > 0$ *the set*

$$E_c \equiv \{x : \Theta^\Psi(u, x) \geq c\} \text{ is closed.}$$

The Hörmander- Bombieri-Siu Theorem

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Question: Are there analogous results for other subequations?

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This result is essentially sharp.

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(1) *H is F-harmonic on $\Omega - \bigcup_j \{x_j\}$,*

(2) $H\big|_{\partial B} = \varphi$,

(3) *There exists constants c, C. so that for each j,*

$$\Theta_j K_p(|x - x_j|) + c \ \leq\ H(x)\ \leq\ \Theta_j K_p(|x - x_j|) + C$$

HAPPY BIRTHDAY OUSSAMA !!

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We look forward to many more years
of your
Wisdom, Leadership and Guidance.

OUR GREAT APPRECIATION TO THE ORGANIZERS:

Nicolas GINOUX

Emmanuel HUMBERT

Marie-Amélie LAWN

Andrei MOROIANU