Lagrangian potential theory and a Lagrangian equation of Monge-Ampère type

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Abstract. The purpose of this paper is to establish a Lagrangian potential theory, analogous to the classical pluripotential theory, and to define and study a Lagrangian differential operator of Monge-Ampère type. This developement is new even in $\mathbf{C}^n$. However, it applies quite generally – perhaps most importantly to symplectic manifolds equipped with a Gromov metric.

The Lagrange operator is an explicit polynomial on $\mathrm{Sym}^2(TX)$ whose principle branch defines the space of Lag-harmonics. Interestingly the operator depends only on the Laplacian and the SKEW-Hermitian part of the Hessian. The Dirichlet problem for this operator is solved in both the homogeneous and inhomogeneous cases. It is also solved for each of the other branches.

This paper also introduces and systematically studies the notions of Lagrangian plurisubharmonic and harmonic functions, and Lagrangian convexity. An analogue of the Levi Problem is proved. In $\mathbf{C}^n$ there is another concept, Lag-pluriharmonics, which relate in several ways to the harmonics on any domain. Parallels of this Lagrangian potential theory with standard (complex) pluripotential theory are constantly emphasized.

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1. Introduction

The aim of this paper is to establish a Lagrangian potential theory, analogous to the classical pluripotential theory, and to define and study a Lagrangian differential operator of Monge-Ampère type. This developement is new even in $\mathbf{C}^n$. However, it applies quite generally – perhaps most importantly to symplectic manifolds equipped with a Gromov compatible metric, which we shall call *Gromov manifolds*.

On $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ the Lagrangian operator $\mathrm{M}_{\mathrm{Lag}}(D^2u)$ is a homogeneous polynomial in $A = D^2u \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{R}^{2n})$ of degree $N = 2^n$. Its principal branch – the closure of the component of $\{\mathrm{M}_{\mathrm{Lag}}(A) \neq 0\}$ containing the identity matrix – has a fundamental geometric interpretation as the set of A's satisfying

$$\text{(1.1)} \qquad \mathrm{tr}\left\{A\big|_W\right\} \ \geq\ 0 \qquad \text{for all Lagrangian } \ n \text{ planes } \ W.$$

Interestingly, this operator only depends on $\mathrm{tr}(A)$ and the skew-Hermitian part $A^{\mathrm{skew}} \equiv \frac{1}{2}(A+JAJ)$ of A. The value is unchanged if one adds to A any Hermitian symmetric matrix $B = \frac{1}{2}(B - JBJ)$ of trace 0. There are several constructions of this operator, two of which have intrinsic interpretations. Either of these allows one to define the operator on any Gromov manifold X. One construction uses the basic Spin^c-bundle $\Lambda^{0,*}(X)$. The other comes from a certain derivation on the bundle $\Lambda^n_{\mathbf{R}}\mathbf{R}^{2n}$.

One of the basic results here establishes the existence and uniqueness of the solution to the Dirichlet problem for both the homogeneous and the inhomogeneous Lagrangian Monge-Ampère operator. In $\mathbf{C}^n$, the existence holds for any smoothly bounded domain $\Omega \subset\subset \mathbf{C}^n$ such that $\partial\Omega$ satisfies a strict *Lagrangian convexity condition* at each point. One very geometric version of this condition is that the trace of the second fundamental form on any tangential Lagrangian n-plane is strictly inward-pointing.

The operator $\mathrm{M}_{\mathrm{Lag}}(A)$ is actually Gårding hyperbolic with respect to the identity $I \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{R}^{2n})$, which implies that it has $N \equiv 2^n$ nested subequations corresponding to the increasing Gårding eigenvalues $\Lambda_1(A) \leq \Lambda_2(A) \leq \cdots \leq \Lambda_N(A)$. The condition $\Lambda_k \geq 0$ is the k^{th}-*branch* of the Lagrangian Monge-Ampère operator. (The homogeneous equation for the operator above is the first branch.) For each $k = 1, ..., N$, it is shown that solutions to the homogenous Dirichlet problem for the equation $\Lambda_k(D^2u) = 0$

are unique on all bounded domains $\Omega \subset\subset \mathbf{C}^n$ as above. The appropriate boundary convexity becomes less stringent as k decreases from N to $N/2$.

Furthermore, all of these results concerning the Dirichlet problem carry over to any Gromov manifold X. The statements are essentially the same. For $\mathrm{M}_{\mathrm{Lag}}$ one considers domains $\Omega \subset\subset X$ with smooth, strictly Lagrangian-convex boundary. The only additional hypothesis, which is global and always true for $X = \mathbf{C}^n$, is that there exist some smooth strictly Lagrangian plurisubharmonic function defined on a neighborhood of $\overline{\Omega}$. (See the following paragraph for the terminology here.)

The original motivation for this study was to develop Lagrangian potential theory as another tool in complex and symplectic analysis. For this one can start with the notion of a *smooth Lagrangian plurisubharmonic* function defined on $\Omega \subset \mathbf{C}^n$. This is one whose second derivative (or riemannian Hessian in the general case) satisfies (1.1), i.e., that its trace on every Lagrangian n-plane at every point is ≥ 0. This notion of Lagrangian plurisubharmonic (or Lag-psh) can be carried over to the class of upper semicontinuous functions by using viscosity test functions. The result enjoys all the basic properties of classical plurisubharmonic functions.

One can show that Lag-psh functions are subharmonic, and so, in particular they are in L^1_{loc}. There is also a Restriction Theorem which says that the restriction of a (viscosity) Lag-psh function to any minimal Lagrangian submanifold M is subharmonic on M (or $\equiv -\infty$). We note, by the way, the converse in $\mathbf{C}^n$, that if its restriction to every affine Lagrangian plane is subharmonic, then it is (viscosity) Lag-psh. (This result, as in complex analysis, justifies the word "plurisubharmonic".)

We now return to smooth functions and assume that X is non-compact and connected. Then given a compact subset $K \subset X$ we define the *Lagrangian hull* $\widehat{K}$ *of* K to be those points $x \in X$ such that $u(x) \leq \sup_K u$ for all Lag-psh $u \in C^\infty(X)$. Then X is defined to be *Lagrangian convex* if $K \subset\subset X \Rightarrow \widehat{K} \subset\subset X$. This is meaningful on any non-compact Gromov manifold. We prove that X is Lagrangian convex iff X admits a smooth proper exhaustion function which is Lag-psh. Moreover, if X carries some smooth strictly Lag-psh function (for example, when $X = \mathbf{C}^n$), then this exhaustion can be made strict. In this case X has the homotopy-type of a complex of dimension $\leq 2n - 2$.

A submanifold $M \subset X$ is defined to be *Lag-free* (or free of lagrangian tangents) if it has no tangential Lagrangian planes. (Thus, any manifold of dimension $< n$ is automatically free.) If $M \subset X$ is closed and free, then the domains $M_\epsilon \equiv \{x : \mathrm{dist}(x, M) \leq \epsilon\}$ admit strictly Lag-psh exhaustions for all $\epsilon > 0$ sufficiently small.

In Section 10 we discuss the notion of strict Lagrangian boundary convexity (referred to above). The notion has several equivalent definitions. A local-to-global theorem is proved – namely, if a domain $\Omega \subset X$ has a strictly Lagrangian convex boundary, then there exists a smooth exhaustion function which is strictly Lag-psh at infinity. (If, in addition, X carries some strictly Lag-psh function, the exhaustion can be taken to be strictly Lag-psh everywhere.) This all can be viewed as a (weaken) form of the Levi-problem in complex analysis.

Let us return now to the Lagrangian Monge-Ampère equation. Let

$$\mathcal{P}(\mathrm{LAG}) \equiv \left\{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \mathrm{tr}\left(A\big|_W\right) \geq 0 \ \ \forall \,\mathrm{Lagrangian} \ \ W\right\},$$

and note that a C^2-function u is Lag-psh if $D^2u(x) \in \mathcal{P}(\mathrm{LAG})$ for all x (and it is strictly Lag-psh if $D^2u(x) \in \mathrm{Int}\mathcal{P}(\mathrm{LAG})$ for all x). Now for a C^2-function u it is natural to define u to be *Lagrangian harmonic* if equality holds in the sense that $D^2u(x) \in \partial\mathcal{P}(\mathrm{LAG})$ for all x. Using the dual subequation $\{\Lambda_N \geq 0\}$ to $\mathcal{P}(\mathrm{LAG}) = \{\Lambda_1 \geq 0\}$, this notion can be carried over to continuous functions, and all of this makes sense on a Gromov manifold by replacing D^2u with $\mathrm{Hess}\, u$, the riemannian Hessian of u. These Lagrangian harmonic functions are exactly the solutions to the principle branch of the Lagrangian Monge-Ampère equation. We also see that these solutions, or Lag-harmonic functions, are the *maximal functions* in our Lagrangian pluripotential theory.

By the way, there is another notion, distinct from those of Lag-psh, dually Lag-psh and Lag-harmonic, which is the analogue of pluriharmonic in complex analysis. Namely, a function u is *Lagrangian pluriharmonic* if its restriction to each affine Lagrangian plane is classically harmonic. (There is a notion of the Lagrangian Hessian, denoted D^2_{LAG}, and u is Lagrangian pluriharmonic if and only if $D^2_{\mathrm{LAG}}u = 0$.) In Section 4 the Lagrangian pluriharmonics are characterized and then put to use by obtaining a classical (non-viscosity) interpretation of the notion of dually Lag-psh (see Theorem 4.6). There is also an interesting result about their contact sets with Lag-harmonics (see Theorem 4.8).

Section 11 is concerning with the Dirichlet Problem for the Lagrangian Monge-Ampère operator for domains in a Gromov manifold. The results are valid for the full inhomogenous case $\mathrm{M}_{\mathrm{Lag}}(u) = \psi$ where $\psi \in C(\overline{\Omega})$ satisfies $\psi \geq 0$. This result also carries over to certain operators associated to the branches (see Note 11.5) $\mathrm{M}_{\mathrm{Lag},k}(u) = \psi$, again with $\psi \geq 0$.

In Section 12 we give a simple proof that the linearization of the operator at a smooth strictly LAG-psh function on a compact set, is uniformly elliptic.

2. Four fundamental concepts

We first consider complex euclidean space $V \equiv \mathbf{C}^n$ with all of its standard structures: the complex structure J, the euclidean inner product $\langle\cdot,\cdot\rangle$, and the hermitian inner product $(\cdot,\cdot) \equiv \langle\cdot,\cdot\rangle - i\omega(\cdot,\cdot)$ where $\omega(v,w) \equiv \langle Jv, w\rangle$ is the standard symplectic form. Given a smooth function u, its hessian, or second derivative $A \equiv D^2_z u \in \mathrm{Sym}^2_{\mathbf{R}}(V)$ is a real symmetric bilinear form on V, or equivalently (by polarization) a real quadratic form on V. It can also be considered as a linear map $A \equiv D^2_z u \in \mathrm{End}_{\mathbf{R}}(V)$ which is symmetric. All three of these natural isomorphisms are used without mention throughout the paper, and when an orthonormal basis is present, $A \equiv D_z^2 u$ can also be considered a symmetric $2n \times 2n$-matrix.

We emphasize the parallels with complex potential theory where the focus is on $\mathbf{C}$-plurisubharmonic functions, that is, real-valued functions u with the property that the restriction of u to complex lines is Δ-subharmonic. Equivalently, the restriction of the quadratic form $A \equiv D_z^2 u$ to each complex line W has non-negative trace, i.e., $\operatorname{tr}\left(A\big|_W\right) \geq 0$ for all $W \in G_{\mathbf{C}}(1, \mathbf{C}^n) = \mathbf{P}_{\mathbf{C}}^{n-1}$, the Grassmannian of complex lines in $\mathbf{C}^n$.

Now our Lagrangian case can be presented in a manner analogous to the complex case. The subset $\mathrm{LAG} \subset G_{\mathbf{R}}(n, \mathbf{C}^n)$ of Lagrangian n-planes in $\mathbf{C}^n$ is defined by

$$W \in \mathrm{LAG} \quad \Longleftrightarrow \quad JW = W^{\perp} \quad \Longleftrightarrow \quad \omega\big|_W = 0. \tag{2.1}$$

Smooth functions. Throughout this subsection u is a smooth function defined on an open subset of $\mathbf{C}^n$. Note that for any real affine subspace W of $\mathbf{C}^n$ we have that $\operatorname{tr}\left(D^2 u\big|_W\right) = \Delta\left(u\big|_W\right)$.

Definition 2.1. We say that u is **Lagrangian plurisubharmonic** if at each point x in its domain and for each affine Lagrangian n-plane W containing x, one has

$$\operatorname{tr}\left(D^2 u\big|_W\right) = \Delta\left(u\big|_W\right) \geq 0. \tag{2.2}$$

The constraint set on the second derivative $A = D_z^2 u$ will be denoted by

$$\mathcal{P}(\mathrm{LAG}) \equiv \left\{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \operatorname{tr}\left(A\big|_W\right) \geq 0 \ \ \forall\, W \in \mathrm{LAG}\right\}. \tag{2.3}$$

Note that this defines a convex cone in $\mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, and that the definition can be reformulated by requiring

$$D_z^2 u \in \mathcal{P}(\mathrm{LAG}) \quad \text{for all} \ \ z \ \ \text{in the domain of} \ \ u. \tag{2.2$'$}$$

The next concept is central to the Lagrangian Dirichlet Problem (and some possible notions of Lagrangian capacity, which we do not discuss). It is analogous to the notion of a *maximal function* or a solution of the complex Monge-Ampère equation in complex potential theory (cf. [**BT**$_1$, **BT**$_2$]).

Definition 2.2. We say that u is **Lagrangian harmonic** if it is Lagrangian plurisubharmonic and if at each point z in its domain there exists an affine Lagrangian n-plane through z such that

$$\operatorname{tr}\left(D^2 u\big|_W\right) = \Delta\left(u\big|_W\right) = 0. \tag{2.4}$$

In terms of the constraint set $\mathcal{P}(\mathrm{LAG})$ this definition is equivalent to the requirement that

$$D_z^2 u \in \partial\mathcal{P}(\mathrm{LAG}) \quad \text{for all} \ \ z \ \ \text{in the domain of} \ \ u. \tag{2.4$'$}$$

In complex potential theory a function u is said to be pluriharmonic if its restriction to each complex line is harmonic. Analogously we have the following concept.

Definition 2.3. We say that u is **Lagrangian pluriharmonic** if

$$\text{tr}\left(D^2u\big|_W\right) \;=\; \Delta\left(u\big|_W\right) \;=\; 0 \quad \text{for all affine Lagrangian planes } W. \tag{2.5}$$

In terms of the constraint set $\mathcal{P}(\text{LAG})$ this definition is equivalent to the requirement that

$$D^2_z u \in \mathcal{P}(\text{LAG}) \cap (-\mathcal{P}(\text{LAG})) \;\equiv\; E \qquad \text{for all } \; z \; \text{ in the domain of } \; u.$$

We note that E is a vector subspace of the convex cone $\mathcal{P}(\text{LAG})$, and it contains all other vector subspaces of $\mathcal{P}(\text{LAG})$. This set E is called the **edge of the convex cone** $\mathcal{P}(\text{LAG})$.

Upper semi-continuous functions. Each of these three concepts can be extended from smooth function to the general class of upper semi-continuous functions by using viscosity test functions. We begin with some generalities. Given $X \subset \mathbf{R}^N$, let $\text{USC}(X)$ denote the set of upper semi-continuous functions $u : X \to \mathbf{R} \cup \{-\infty\}$.

Definition 2.4. (a) For $u \in \text{USC}(X^{\text{open}})$ and $x \in X$, a smooth function φ, defined on a neighborhood of x, is called a **(viscosity) test function** for u at x if

$$u \;\le\; \varphi \quad \text{near } \; x \; \text{ with equality at } \; x. \tag{2.6}$$

(b) A closed subset $F \subset \text{Sym}^2(\mathbf{R}^N)$ is called a **subequation** if $F + \mathcal{P} \subset F$.

(c) Given u and F as above, we say that u is **F-subharmonic** if

$$D^2_x\varphi \in F, \tag{2.7}$$

for each $x \in X$ and each test function φ for u at x.

(d) Given two subequations F and G, the set

$$E \;\equiv\; F \cap (-G) \quad \text{is called a } \textbf{generalized equation}. \tag{2.8}$$

(e) A continuous function u is a **solution to** E if

$$U \;\text{ is } F \text{ subharmonic and } \; -u \; \text{ is } G \text{ subharmonic} \tag{2.9}$$

(f) Given a subequation F, the **dual subequation** is defined to be

$$\widetilde{F} \;\equiv\; -(\sim \text{Int}F) \;=\; \sim(-\text{Int}F) \tag{2.10}$$

(g) A generalized equation of the form $\partial F = F \cap (-\widetilde{F})$ is called an **equation**, and its solutions are called **F-harmonic**. Note that u is a solution if and only if u is F-subharmonic and $-u$ is $\widetilde{F}$-subharmonic.

Note that, in particular, we have the dual subequation $\widetilde{\mathcal{P}}(\text{LAG})$ of $\mathcal{P}(\text{LAG})$. It is given by

$$\widetilde{\mathcal{P}}(\text{LAG}) = \left\{A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \exists\, W \in \text{LAG} \ni \text{tr}\left(A\big|_W\right) \ge 0\right\} \tag{2.11}$$

Now we can define our fourth fundamental concept. First, a smooth function u is **dually Lagrangian plurisubharmonic** if $D^2_z u \in \widetilde{\mathcal{P}}(\mathrm{LAG})$ at each point z in its domain, that is, at each z there exists $W \in \mathrm{LAG}$ such that $\mathrm{tr}\{D^2_z u\big|_W \geq 0\}$

It is now easy to extend all four concepts for smooth functions to upper semi-continuous functions on an open set $X \subset \mathbf{C}^n$.

DEFINITION 2.5. Given $u \in \mathrm{USC}(X)$, we say that:

(1) u is **Lagrangian plurisubharmonic** if u satisfies the subequation $\mathcal{P}(\mathrm{LAG})$.

(2) u is **dually Lagrangian plurisubharmonic** if u satisfies the subequation $\widetilde{\mathcal{P}}(\mathrm{LAG})$.

(3) u is **Lagrangian harmonic** if u is a solution to the equation $\partial\mathcal{P}(\mathrm{LAG})$. Note that this holds if and only if u is Lag-psh and $-u$ is dually Lag-psh.

(4) u is **Lagrangian pluriharmonic** if u is a solution to the generalized equation $E \equiv \mathcal{P}(\mathrm{LAG}) \cap (-\mathcal{P}(\mathrm{LAG}))$, i.e., u is Lag-psh and $-u$ is also Lag-psh.

The more succinct expressions are respectively that u is

(1) **Lag-psh** (2) **dually Lag-psh**
(3) **Lag-harmonic** (4) **Lag-pluriharmonic.**

The terminology *Lagrangian plurisubharmonic* is justified by the Restriction Theorem (see [**HL**$_9$]).

THEOREM 2.6. *An upper semi-continuous function u is Lag-psh if and only if its restriction to each affine Lagrangian n-plane is Δ-subharmonic (or $\equiv -\infty$).*

This is directly analogous to the subequation $\mathcal{P}$ for convex functions (restrict to affine lines) and to the subequation $\mathcal{P}(G_{\mathbf{C}}(1, \mathbf{C}^n))$ for plurisubharmonic functions (restrict to affine complex lines).

Theorem 4.6 gives a classically formulated (i.e., non-viscosity) characterization of dually Lag-psh functions, which complements Theorem 2.6

The following is a useful fact.

PROPOSITION 2.7.

$$\mathcal{P}(\mathrm{LAG}) \subset \Delta \equiv \{\mathrm{tr}A \geq 0\}, \tag{2.12a}$$

or equivalently,

(2.12b) Each Lag–psh function is classically subharmonic.

PROOF. The x-axis $\mathbf{R}^n$ and the y-axis $i\mathbf{R}^n$ in $\mathbf{C}^n$ are both Lagrangian n-planes. They define the "partial" Laplacians

$$\Delta^x \equiv \left\{A : \mathrm{tr}\left(A\big|_{\mathbf{R}^n}\right) \geq 0\right\} \qquad \text{and} \qquad \Delta^y \equiv \left\{A : \mathrm{tr}\left(A\big|_{i\mathbf{R}^n}\right) \geq 0\right\}.$$

Since $\mathcal{P}(\mathrm{LAG}) \subset \Delta^x \cap \Delta^y \subset \Delta$, (2.12a) follows easily. $\square$

We conclude this section by pointing out that for any domain $\Omega \subset \mathbf{C}^n$ the set, denoted $\mathcal{P}_{\text{Lag}}(\Omega)$, of all (u.s.c.) Lag-psh functions on Ω, has all the classical properties, namely:

(1) $u \in \mathcal{P}_{\text{Lag}}(\Omega) \ \Rightarrow u \in L^1_{\text{loc}}$ (by (2.12b)),

(2) $u, v \in \mathcal{P}_{\text{Lag}}(\Omega) \ \Rightarrow \max\{u, v\} \in \mathcal{P}_{\text{Lag}}(\Omega)$,

(3) $\mathcal{P}_{\text{Lag}}(\Omega)$ is closed under uniform limits and decreasing limits,

(4) if $\mathcal{F} \subset \mathcal{P}_{\text{Lag}}(\Omega)$ is any family locally bounded above, and if $U(x) \equiv \sup\{u(x) : u \in \mathcal{F}\}$ for $x \in \Omega$, then the u.s.c. regularization $U^* \in \mathcal{P}_{\text{Lag}}(\Omega)$, and

(5) a function $u \in C^2(\Omega) \cap \mathcal{P}_{\text{Lag}}(\Omega))$ is Lag-psh in the sense of Def. 2.1.

The properties (2) – (5) have been well known for a long time for any subequation. Proofs can be found, for example, in Appendix B in [**HL**$_4$]. Finally note the *convex composition property*:

(6) If $u \in \mathcal{P}_{\text{Lag}}(\Omega)$ and if χ is a convex, increasing $\mathbf{R}$-valued function on Image(u), then $\chi \circ u \in \mathcal{P}_{\text{Lag}}(\Omega)$.

This follows since $\mathcal{P}(\text{LAG})$ is a convex cone (see Fact (2) in §6 of [**HL**$_3$]).

3. The Lagrangian subequation and Lagrangian Hessian

A more detailed algebraic (and self-contained) discussion of this subequation is presented in Appendix A. Here we again emphasize the parallels with complex potential theory, where the pertinent subequation is defined by the subset $\mathbf{P}^{n-1}_{\mathbf{C}} \subset G_{\mathbf{R}}(2, \mathbf{C}^n)$ of complex lines in $\mathbf{C}^n$ – namely, the subequation:

$$\mathcal{P}(\mathbf{P}^{n-1}_{\mathbf{C}}) \ = \ \left\{A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \text{tr}\left(A\big|_W\right) \geq 0 \ \ \forall\, W \in \mathbf{P}^{n-1}_{\mathbf{C}}\right\}.$$

Note that for any unit vector $e \in W \in \mathbf{P}^{n-1}_{\mathbf{C}}$,

$$\text{tr}\left(A\big|_W\right) \ = \ \langle Ae, e\rangle + \langle AJe, Je\rangle \ = \ \langle (A - JAJ)e, e\rangle. \tag{3.1}$$

Consequently, the subspace $\text{Herm}^{\text{skew}}(\mathbf{C}^n) \subset \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ of real symmetric maps A which anti-commute with J is unimportant here since, given $A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$,

$$\text{tr}\left(A\big|_W\right) \ = \ 0 \quad \forall\, W \in \mathbf{P}^{n-1}_{\mathbf{C}} \qquad \iff \qquad A \in \text{Herm}^{\text{skew}}(\mathbf{C}^n) \tag{3.2}$$

So in classical pluripotential theory it is best to focus on its orthogonal complement in $\text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, namely the space $\text{Herm}^{\text{sym}}(\mathbf{C}^n)$ of real symmetric maps on $\mathbf{C}^n$ which commute with J. The opposite is true for $\mathcal{P}(\text{LAG})$, defined by (2.3), where $\mathbf{P}^{n-1}_{\mathbf{C}}$ is replaced by LAG.

Since this algebra will be used here, we summarize as follows. The space $\text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ decomposes as an orthogonal direct sum (with respect to $\langle A, B\rangle = \text{tr}(AB)$) as

$$\text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) \ = \ \text{Herm}^{\text{sym}}(\mathbf{C}^n) \oplus \text{Herm}^{\text{skew}}(\mathbf{C}^n). \tag{3.3}$$

Note that the dimensions are $n(2n+1) = n^2 + (n^2+n)$. The orthogonal projections are defined by

$$A^{\rm sym} \equiv \tfrac{1}{2}(A - JAJ) \qquad \text{and} \qquad A^{\rm skew} \equiv \tfrac{1}{2}(A + JAJ) \tag{3.4}$$

for all $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. The projection of the hessian onto the hermitian symmetric part:

$$\left(D^2u\right)^{\rm sym} \in \mathrm{Herm}^{\rm sym}(\mathbf{C}^n) \tag{3.5}$$

is called the *complex part* of the hessian of u, or just the *complex hessian* in pluripotential theory. This is the important part of the second derivative for complex analysis, and the subequation $\mathcal{P}(\mathbf{P}^{n-1}_{\mathbf{C}})$ can be defined by the constraint $(D^2u)^{\rm sym} \geq 0$. Also note that $\mathrm{Herm}^{\rm skew}(\mathbf{C}^n) = \mathcal{P}(\mathbf{P}^{n-1}_{\mathbf{C}}) \cap (-\mathcal{P}(\mathbf{P}^{n-1}_{\mathbf{C}}))$ is the edge of the convex cone $\mathcal{P}(\mathbf{P}^{n-1}_{\mathbf{C}})$.

Note (Complex notation). Besides the four ways of looking at an element $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ (discussed at the beginning of Section 2), there are additional identifications that come into play because of the complex structure. First, $\mathrm{Herm}^{\rm sym}(\mathbf{C}^n)$ can be identified with the space of complex linear maps $A \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n)$ which are self-adjoint, as well as the space of complex matrices $A \subset \mathrm{M}_n(\mathbf{C})$ with $A = \overline{A}^t$. In coordinates

$$(D^2u)^{\rm sym} \cong \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right) \qquad \text{is the complex hessian.} \tag{3.6}$$

The second space $\mathrm{Herm}^{\rm skew}(\mathbf{C}^n)$ is isomorphic as a real vector space to $\mathrm{Sym}^2_{\mathbf{C}}(\mathbf{C}^n)$, the space of symmetric $n \times n$ complex matrices, or equivalently, the space of pure complex quadratic forms on $\mathbf{C}^n$. The inverse of this isomorphism is given by sending a complex symmetric form $B(w) \equiv \sum_{ij} b_{ij} w_i w_j$ to $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ defined by $A(w) = \mathrm{Re}\, B(w)$. This gives the coordinate expression

$$\left(D^2_z u\right)^{\rm skew}(w) = \mathrm{Re}\left(\sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial z_j}(z) w_i w_j\right) = \mathrm{Re}\, B(w) \tag{3.7a}$$

which shall be abbreviated to:

$$\left(D^2_z u\right)^{\rm skew} = \mathbf{Re}\left(\frac{\partial^2 u}{\partial z_i \partial z_j}\right). \tag{3.7b}$$

Cautionary Note. Keep in mind that (3.7b) denotes the real part of the quadratic form $\mathrm{Re}\, B(w)$ in (3.7a), and **not** the real part of the matrix $\left(\frac{\partial^2 u}{\partial z_i \partial z_j}\right)$. Note also that

$$\mathrm{Re}\, B(w) = 0 \qquad \iff \qquad B(w) = 0.$$

The implication $\Rightarrow$ follows since $\mathrm{Re}\, B(\sqrt{i}w) = \mathrm{Re}\, iB(w) = -\mathrm{Im}\, B(w)$.

Now the Lagrangian case can be presented in a manner analogous to the complex case. We first describe the Lagrangian analogue of (3.2). For this a standard canonical form is needed, which for future reference we state explicitly.

LEMMA 3.1. *For each $B \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ which is skew hermitian (i.e., $BJ = -JB$) there exists a unitary basis $e_1, ..., e_n$ of $\mathbf{C}^n$ and numbers $\lambda_j \geq 0,\ j = 1, ..., n$ so that B takes the canonical form*

$$B \ \equiv\ \lambda_1 \left(P_{e_1} - P_{Je_1}\right) + \cdots + \lambda_n \left(P_{e_n} - P_{Je_n}\right), \tag{3.8}$$

where P_v denotes orthogonal projection onto the line generated by v.

PROOF. Since B is skew hermitian, if e is an eigenvector with eigenvalue λ, then Je is an eigenvector with eigenvalue $-\lambda$. ☐

As with any unitary basis, note that

$$\begin{aligned} W \ &\equiv\ \mathrm{span}\,\{e_1, ..., e_n\} \qquad \text{and} \\ JW \ &\equiv\ \mathrm{span}\,\{Je_1, ..., Je_n\} \qquad \text{are Lagrangian} \end{aligned} \tag{3.9}$$

Using this fact along with (3.8), we will answer the natural question: which real symmetric forms A have the property that their restriction to every Lagrangian n-plane has trace zero?

Let $\mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n)$ denote the space of traceless hermitian symmetric forms. Note that given $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$,

$$A \in \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n) \qquad \iff \qquad A^{\mathrm{skew}} = 0 \quad \text{and} \quad \mathrm{tr}A = 0.$$

In particular, the decomposition of $\mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ in (3.3) further decomposes as

$$\mathrm{Herm}^{\mathrm{sym}}(\mathbf{C}^n) = [I] \oplus \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n). \tag{3.3$'$}$$

LEMMA 3.2. *Suppose $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. Then*

$$\mathrm{tr}\left(A\big|_W\right) = 0 \quad \forall\, W \in \mathrm{LAG} \qquad \iff \qquad A \in \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n).$$

Said differently, $E \equiv \mathcal{P}(\mathrm{LAG}) \cap (-\mathcal{P}(\mathrm{LAG})) = \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n)$ is the generalized equation defining Lagrangian pluriharmonic functions, cf. Def. 2.3.

PROOF. ($\Rightarrow$) If $W = \mathrm{span}\,\{e_1, ..., e_n\}$ is the Lagrangian plane in Lemma 3.1 for $B = A^{\mathrm{skew}}$, then since each $\lambda_j \geq 0, j = 1, ..., n$,

$$\lambda_1 + \cdots + \lambda_n \ =\ \mathrm{tr}\left(A\big|_W\right) \ =\ 0 \qquad \Rightarrow \qquad A^{\mathrm{skew}} \ =\ 0.$$

Moreover, for any $W \in \mathrm{LAG}$

$$\mathrm{tr}A \ =\ \mathrm{tr}\left(A\big|_W\right) + \mathrm{tr}\left(A\big|_{JW}\right), \tag{3.10}$$

proving that $\mathrm{tr}A = 0$.

($\Leftarrow$) Conversely, if $A \in \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n)$, then $\langle Ae, e\rangle = \langle AJe, Je\rangle$, so that $\mathrm{tr}\left(A\big|_W\right) = \mathrm{tr}\left(A\big|_{JW}\right)$ for all $W \in \mathrm{LAG}$. This proves, by (3.10) that $\mathrm{tr}\left(A\big|_W\right) = 0\ \forall W \in \mathrm{LAG}$. ☐

The Lagrangian Hessian. Because of Lemma 3.2, the orthogonal complement of $\mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)$ is now our focus. This space is exactly

$$\mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)^{\perp} \ = \ [\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n) \tag{3.11}$$

where $[\mathrm{I}] \equiv \mathbf{R} \cdot \mathrm{I}$ denotes the line through the identity. The projection of $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ onto this space, which will be denoted $A^{\mathrm{Lag}} \equiv \pi(A)$ is given by

$$A^{\mathrm{Lag}} \ \equiv \ \frac{(\mathrm{tr} A)}{2n} \cdot \mathrm{I} + A^{\mathrm{skew}} \tag{3.12}$$

and is called the **Lagrangian part of** A.

Corollary 3.3. *For any $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ and $W \in \mathrm{LAG}$ one has that*

$$\mathrm{tr}\left(A\big|_W\right) \ = \ \mathrm{tr}\left(A^{\mathrm{Lag}}\big|_W\right) \ = \ \tfrac{1}{2}\mathrm{tr}(A) + \mathrm{tr}\left(A^{\mathrm{skew}}\big|_W\right) \tag{3.13}$$

Definition 3.4 (The Lagrangian Hessian). Suppose u is a smooth function defined on an open subset of $\mathbf{R}^n$. The **Lagrangian hessian** of u is the Lagrangian part of the second derivative of u, that is:

$$\mathrm{Hess}^{\mathrm{Lag}}(u) \ = \ (D^2u)^{\mathrm{Lag}} \ = \ \frac{\Delta u}{2n} \cdot \mathrm{I} + \left(D^2u\right)^{\mathrm{skew}} \tag{3.14}$$

where $(D^2u)^{\mathrm{skew}} \equiv \frac{1}{2}(D^2u + JD^2uJ)$.

Note that by Lemma 3.2, u is lagrangian pluriharmonic if and only if $\mathrm{Hess}^{\mathrm{Lag}}(u) \equiv 0$. In particular, the notion of Lagrangian plurisubharmonic only depends on the Lagrangian hessian. Also note that just as the complex hessian can be expressed in $z, \bar{z}$ coordinates as $(D^2u)^{\mathrm{sym}} \cong \left(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}\right)$, the Lagrangian hessian at a point z can be written as the quadratic form

$$\left(D^2_z\right)^{\mathrm{Lag}} \ \cong \ \left(\frac{\Delta u}{2n}\right)\mathrm{I} + \mathbf{Re}\left(\frac{\partial^2 u}{\partial z_i \partial z_j}\right) \tag{3.15}$$

From (3.13) we see that

$$\inf_{W \in \mathrm{LAG}} \mathrm{tr}\left(A\big|_W\right) \ = \ \inf_{W \in \mathrm{LAG}} \mathrm{tr}\left(A^{\mathrm{Lag}}\big|_W\right) \tag{3.16}$$

which of course applies to $A = D^2u$.

The canonical form for the Lagrangian Hessian. As in the complex case, once the key part of the second derivative has been identified, it can be put in canonical form.

Corollary 3.5. *For each $H \in [\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)$, there exists a unitary basis $e_1, Je_1, ...e_n, Je_n$ for $\mathbf{C}^n$ consisting of eigenvectors of H with corresponding eigenvalues*

$$\frac{\mathrm{tr} H}{2n} + \lambda_1, \ \frac{\mathrm{tr} H}{2n} - \lambda_1, \ \ldots \ \frac{\mathrm{tr} H}{2n} + \lambda_n, \ \frac{\mathrm{tr} H}{2n} - \lambda_n. \tag{3.17}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are the eigenvalues of H^{skew}.

Proof. Note that $H = \frac{\mathrm{tr} H}{2n}\mathrm{I} + H^{\mathrm{skew}}$ and H^{skew} has the canonical form (3.8). □

Perhaps not surprisingly we have the following.

Theorem 3.6. *For each $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$*

$$\inf_{W \in \mathrm{LAG}} \mathrm{tr}\left(A\big|_W\right) \;=\; \frac{\mathrm{tr} A}{2} - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \tag{3.18}$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ are the non-negative eigenvalues of A^{skew}.

We shall give an indirect proof with a lemma which will be useful later. First, for any $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, we make the abbreviations: $\mu \equiv \frac{1}{2}\mathrm{tr} A$, $B \equiv A^{\mathrm{skew}}$, and $H \equiv A^{\mathrm{Lag}} \equiv \frac{\mu}{n} I + B =$ the Lagrangian part of A. In the following choose the eigenstructure given by Corollary 3.5 for $H \equiv A^{\mathrm{Lag}}$.

Lemma 3.7. *Given $H \equiv \frac{\mu}{n}\mathrm{I} + B$ with $B \in \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)$, extend H as a derivation D_H on $\Lambda^n_{\mathbf{R}}\mathbf{C}^n$. Then the eigenvectors of D_H are the $\binom{2n}{n}$ real axis n-planes in $\mathbf{R}^{2n} = \mathbf{C}^n$ (in terms of the B-eigenstructure). Each such axis n-plane ξ has a unique decomposition $\xi = \alpha \wedge \eta$ where α is an axis complex p-plane and η is an isotropic axis q-plane (with $2p + q = n$). The corresponding eigenvalue for D_B is $\mathrm{tr}(B\big|_{\mathrm{span}\,\eta})$, and for D_H it is $\mu + \mathrm{tr}(B\big|_{\mathrm{span}\,\eta})$.*

Proof. First note that for each of the complex p-planes α we have $D_B \alpha = 0$ since $D_B(e_i \wedge Je_i) = 0$. Any isotropic axis q-plane η is of the form

$$\eta \;=\; \{e_{i_1} \text{ or } Je_{i_1}\} \;\wedge\; \cdots \;\wedge\; \{e_{i_q} \text{ or } Je_{i_q}\} \tag{3.19}$$

for some increasing multi-index $I = (i_1, ..., i_q)$ of length q. With $+$ corresponding to the choice of e_{i_j} and $-$ corresponding to the choice of Je_{i_j}, we have

$$D_B \eta \;=\; (\pm\lambda_{i_1} \pm \cdots \pm \lambda_{i_q})\eta \tag{3.20}$$

Therefore the eigenvalues of D_B are

$$\pm\lambda_{i_1} \pm \cdots \pm \lambda_{i_q} \qquad \text{for all} \;\; I = (i_1, ..., i_q), \;\; 0 \leq q \leq n. \tag{3.21}$$

This proves that the eigenvalues of D_H are

$$\mu \pm \lambda_{i_1} \pm \cdots \pm \lambda_{i_q} \qquad \text{for} \;\; 0 \leq |I| = q \leq n. \tag{3.22}$$

since $D_{\frac{\mu}{n}\mathrm{I}}\xi = \mu\xi$ for all axis n-planes ξ. □

Proof of Theorem 3.6. It is now obvious that the minimum eigenvalue of D_H equals

$$\lambda_{\min} \;=\; \mu - (\lambda_1 + \cdots + \lambda_n) \tag{3.23}$$

and the corresponding eigenvector is $\xi = Je_1 \wedge \cdots \wedge Je_n$. Of course

$$\lambda_{\min}(D_H) \;=\; \inf_L \mathrm{tr}\left(D_H\big|_L\right) \tag{3.24}$$

where the inf is taken over all lines L in $\Lambda^n_{\mathbf{R}}\mathbf{C}^n$. Finally note that each $W \in \mathrm{LAG}$ determines the line $L(W)$ in $\Lambda^n_{\mathbf{R}}\mathbf{C}^n$ through the simple n-form obtained by wedging a basis for W, and that

$$\text{(3.25)} \qquad \mathrm{tr}\left(D_H\big|_{L(W)}\right) \;=\; \mathrm{tr}\left(H\big|_W\right).$$

(In fact (3.25) is true with H replaced by any $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ and W any subspace.) Therefore,

$$\text{(3.26)} \qquad \lambda_{\min}(D_H) \;=\; \inf_{W\in\mathrm{LAG}} \mathrm{tr}\left(H\big|_W\right). \qquad \square$$

Combining (3.13), (3.26) and (2.3) in Definition 2.1 yields

THEOREM 3.8. *With $H \equiv A^{\mathrm{Lag}}$, the Lagrangian part of $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, one has*

$$A \in \mathcal{P}(\mathrm{LAG}) \qquad \iff \qquad D_H \;\geq\; 0.$$

In terms of smooth functions, u is Lagrangian plurisubharmonic if and only if at each point z in its domain,

$$D_H \;\geq\; 0 \qquad \text{where} \quad H \;\equiv\; \left(D^2_z u\right)^{\mathrm{Lag}} \quad \text{is the Lagrangian Hessian of } u.$$

REMARK 3.9. The operator $f(D^2u)$ defined by

$$\text{(3.27)} \qquad f(A) \;=\; \frac{\mathrm{tr}A}{2} - \lambda_1^+ - \cdots - \lambda_n^+,$$

where $\lambda_1^+, ..., \lambda_n^+ \geq 0$ are the non-negative eigenvalues of A^{skew}, defines the subequation $\mathcal{P}(\mathrm{LAG})$ by $f(A) \geq 0$, and the equation $\partial\mathcal{P}(\mathrm{LAG})$ by also requiring that $f(A) = 0$. As discussed in [**HL**$_5$, §3] (see also [**HL**$_4$, Rmk. 14.11]), each subequation $G \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is defined as $G = \{A : g(A) \geq 0\}$ for a unique operator $g : \mathrm{Sym}^2(\mathbf{R}^n) \to \mathbf{R}$ with the property that $g(A+tI) = g(A)+t$, which we call the canonical operator for the subequation G. Since $g \equiv \frac{1}{n}f$ with f defined by (3.27) has this property, the operator $\frac{1}{n}f$ is the canonical operator for $G \equiv \mathcal{P}(\mathrm{LAG})$.

4. Lagrangian pluriharmonic functions

In this section we explore the analogues of the pluriharmonic functions in pluripotential theory in our context of Lagrangian geometry. (In a more general context both are examples of *"edge functions"* – see Remark 4.4 below.) Unlike their cousins, Lagrangian pluriharmonics do not exist on general manifolds. Even in euclidean space our first result (Theorem 4.1) explains the limited nature of these functions. However, in spite of this result, the Lagrangian pluriharmonics do play an important role by providing an equivalent way of defining the dually Lag-psh functions (Theorem 4.6), which is formulated in classical language.

Recall that by Definition 2.5 a function $h \in C(\Omega)$, $\Omega^{\mathrm{open}} \subset \mathbf{C}^n$, is **Lagrangian pluriharmonic** if both h and $-h$ are Lag-psh. By Proposition 2.7, this implies that $\Delta h = 0$ and hence h is real analytic. Much more is true.

THEOREM 4.1. *The space of Lag-pluriharmonic functions on a connected open subset Ω of $\mathbf{C}^n$ consists of the traceless hermitian degree 2 polynomials:*

$$h(z) \ = \ c + \sum_{k=1}^{n}(b_k z_k + \bar{b}_k \bar{z}_k) + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} z_i \bar{z}_j \tag{4.1}$$

where $c \in \mathbf{C}, b \in \mathbf{C}^n$, and $A = (a_{ij})$ satisfies $A = \overline{A}^t$.

PROOF. The elementary proof is preceded by some elementary observations. One can also refer to a Lag-pluriharmonic as a solution to the edge equation

$$D_z^2 h \ \in \ E \ \equiv \ \mathcal{P}(\mathrm{LAG}) \cap (-\mathcal{P}(\mathrm{LAG})) \tag{4.2}$$

Recall from Section 3 that $E = \mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)$, and since $E^\perp = [\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)$, it is defined by the vanishing of the Lagrangian hessian $(D_z^2 h)^{\mathrm{Lag}}$. Since $\Delta h = 0$, by (3.15) this is equivalent to

$$(D_z^2 h)^{\mathrm{skew}} \ \cong \ \mathbf{Re}\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\right) \ = \ 0,$$

which is equivalent to $\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\right) = 0$ (see the Cautionary Note after (3.7)), This reduces the proof of Theorem 4.1 to the following lemma. $\square$

LEMMA 4.2. *If h is a smooth function satisfying*

$$(D_z^2 h)^{\mathrm{skew}} \ \cong \ \mathbf{Re}\left(\frac{\partial^2 h}{\partial z_i \partial z_j}\right) \ = \ 0, \quad \text{or equivalently} \quad \frac{\partial^2 h}{\partial z_i \partial z_j} \ = \ 0 \ \ \forall\, i,j,$$

near a point z, then (4.1) holds in a neighborhood of z.

PROOF. As noted above $\frac{\partial^2 h}{\partial z_i \partial z_j} = 0$ for all i,j is equivalent to $(D_z^2 h)^{\mathrm{skew}} = 0$. Next note that

$$\begin{aligned} 4\frac{\partial^2 h}{\partial z_i \partial z_j} &= \left(\frac{\partial}{\partial x_i} - i\frac{\partial}{\partial y_i}\right)\left(\frac{\partial h}{\partial x_j} - i\frac{\partial h}{\partial y_j}\right) \\ &= \left(\frac{\partial^2 h}{\partial x_i \partial x_j} - \frac{\partial^2 h}{\partial y_i \partial y_j}\right) - i\left(\frac{\partial^2 h}{\partial x_i \partial y_j} - \frac{\partial^2 h}{\partial y_i \partial x_j}\right). \end{aligned}$$

Therefore,

$$\frac{\partial^3 h}{\partial x_i \partial x_j \partial x_k} \ = \ \frac{\partial^3 h}{\partial y_i \partial y_j \partial x_k} \ = \ -\frac{\partial^3 h}{\partial y_i \partial x_j \partial y_k} \ = \ -\frac{\partial^3 h}{\partial x_i \partial x_j \partial x_k}$$

all vanish. Similarly,

$$\frac{\partial^3 h}{\partial y_i \partial y_j \partial y_k} \ = \ \frac{\partial^3 h}{\partial x_i \partial x_j \partial y_k} \ = \ -\frac{\partial^3 h}{\partial x_i \partial y_j \partial x_k} \ = \ -\frac{\partial^3 h}{\partial y_i \partial y_j \partial y_k}$$

all must vanish. Thus all the third partial derivatives vanish, proving that h is of degree 2. Since the component $(D_z^2 h)^{\mathrm{skew}} = 0$, we have $D^2 h \in \mathrm{Herm}^{\mathrm{sym}}(\mathbf{C}^n)$, which proves (4.1) since $\Delta h = 0$. $\square$

REMARK 4.3. As noted above $E = \mathcal{P}(\mathrm{LAG}) \cap (-\mathcal{P}(\mathrm{LAG})) = \mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)$ is a generalized equation (see Definition 2.4(d)). By contrast $\mathrm{Herm}^{\mathrm{sym}}(\mathbf{C}^n)$ is *not* a generalized equation since the smallest subequation F containing $\mathrm{Herm}^{\mathrm{sym}}(\mathbf{C}^n)$ is $F = \mathrm{Herm}^{\mathrm{sym}}(\mathbf{C}^n) + \mathcal{P}$, which equals all of $\mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, because $\mathcal{P} \cap \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n) = \{0\}$.

REMARK 4.4 (Edge functions). For any convex cone subequation $\mathcal{P}^+$, the *edge* is defined to be $E \equiv \mathcal{P}^+ \cap (-\mathcal{P}^+)$. (See Appendix A for more details.) In general, one can define, using viscosity test functions, solutions of this (generalized) equation. It is natural to refer to such functions as *edge functions*, or in the geometric cases (i.e., $\mathcal{P}^+ \equiv \mathcal{P}(\mathbb{G})$ with $\mathbb{G}$ a closed subset of a Grassmannian) as $\mathbb{G}$-*pluriharmonic functions*. In the four cases: $\mathcal{P}^+ = \mathcal{P}(G_{\mathbf{R}}(1, \mathbf{R}^n)), \mathcal{P}(G_{\mathbf{C}}(1, \mathbf{C}^n)), \mathcal{P}(G_{\mathbf{H}}(1, \mathbf{H}^n))$ and $\mathcal{P}(\mathrm{LAG})$, the edges are: $E = \{0\}, \mathrm{Herm}_{\mathbf{C}}^{\mathrm{skew}}(\mathbf{C}^n), \mathrm{Herm}_{\mathbf{H}}^{\mathrm{sym}}(\mathbf{C}^n)^{\perp}$, and $\mathrm{Herm}_0^{\mathrm{sym}}$ respectively. The edge functions, or pluriharmonics, are: affine functions for $\mathcal{P}(G_{\mathbf{R}}(1, \mathbf{R}^n))$. $\mathrm{Re}\{f(z)\}$ with f holomorphic for $\mathcal{P}(G_{\mathbf{C}}(1, \mathbf{C}^n))$, and in the case $\mathcal{P}(\mathrm{LAG})$, the traceless degree 2 polynomials (Theorem 4.1). Thus the Lagrangian case is more like the convex case than the complex case in that the space of edge functions is finite dimensional, whereas the space of real parts of holomorphic functions is infinite dimensional.

The Lag-pluriharmonic functions are closely related to the dually Lag-psh functions (i.e., the subharmonics for the dual subequation $\widetilde{\mathcal{P}}(\mathrm{LAG})$).

DEFINITION 4.5. An upper semi-continuous function u is "sub" the traceless hermitian degree-2 polynomials on an open subset $X \subset \mathbf{C}^n$ if for all domains $\Omega \subset\subset X$ and all traceless hermitian degree-2 polynomials h,

$$u \le h \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad u \le h \quad \text{on} \quad \overline{\Omega}. \tag{4.3}$$

THEOREM 4.6. *A function u is dually* Lag-*psh* $\iff$ *u is "sub" the traceless hermitian degree-2 polynomials* $\iff$ *u is locally "sub" the traceless hermitian degree-2 polynomials.*

PROOF. Suppose u is $\widetilde{\mathcal{P}}(\mathrm{LAG})$-subharmonic on X and h is a traceless hermitian degree-2 polynomial. Since $-h$ is $\mathcal{P}(\mathrm{LAG})$-subharmonic, (4.3) follows from comparison (see Thm. 6.2 in [$\mathbf{HL}_{12}$]).

The proof of the converse illustrates the importance of Proposition A.2 in the Appendix. Suppose now that u is not $\widetilde{\mathcal{P}}(\mathrm{LAG})$-psh on X. Then (see Lemma 2.4 in [$\mathbf{HL}_4$]) there exists $z_0 \in X$, a quadratic polynomial φ, and $\alpha > 0$ such that

$$u(z) \le \varphi(z) - \alpha|z - z_0|^2 \quad \text{near } z_0 \text{ with equality at } z_0, \tag{4.4a}$$

but

$$D^2_{z_0}\varphi \notin \widetilde{\mathcal{P}}(\mathrm{LAG}), \quad \text{i.e.,} \quad -D^2_{z_0}\varphi \in \mathrm{Int}\mathcal{P}(\mathrm{LAG}). \tag{4.4b}$$

By Proposition A.2(4) we have $\mathrm{Int}\mathcal{P}(\mathrm{LAG}) = \mathrm{Int}\mathcal{P} + \mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)$. Thus

$$-D^2_{z_0}\varphi = P + B \quad \text{with} \quad P > 0 \quad \text{and} \quad B \in \mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n). \tag{4.4c}$$

The traceless hermitian degree-2 polynomial h satisfies

$$\begin{aligned} h(z) \ &\equiv\ \varphi(z_0) + \langle D_{z_0}\varphi, z - z_0\rangle - \tfrac{1}{2}\langle B(z-z_0), z-z_0\rangle \\ &=\ \varphi(z) - \tfrac{1}{2}\langle D^2_{z_0}\varphi, z-z_0\rangle - \tfrac{1}{2}\langle B(z-z_0), z-z_0\rangle \\ &=\ \varphi(z) + \tfrac{1}{2}\langle P(z-z_0), z-z_0\rangle. \end{aligned}$$

Therefore, (4.4) implies

$$u(z) \ \le\ h(z) - \alpha|z-z_0|^2 \tag{4.5}$$

near z_0 with equality at z_0. This implies that u is not sub the function h on any small ball about z_0. Hence, u is not locally "sub" the traceless hermitian degree-2 polynomials. ☐

REMARK 4.7. Theorem 4.6 has analogues for the subequations $\mathcal{P}$ and its complex analogue. For the first case it says that u is $\widetilde{\mathcal{P}}$-subharmonic if and only if it is "sub" the affine functions (see [**HL**$_2$]). For the complex Monge-Ampère subequation $\mathcal{P}_{\mathbf{C}} \equiv \{A : A - JAJ \ge 0\}$ on $\mathbf{C}^n$, it says that u is $\widetilde{\mathcal{P}}_{\mathbf{C}}$-subharmonic if and only if u is "sub" the pluriharmonic functions (see Prop. 5.14 in [**HL**$_9$]).

Fix an open set $X \subset \mathbf{C}^n$ and a compact subset $K \subset X$. Then the Lagrangian hull $\widehat{K} \subset X$ is defined in Definition 9.1, and if $\widehat{K} = K$, then K is called *Lagrangian convex* in X.

THEOREM 4.8 (Contact Sets for Lag-Harmonics). *Let H be a Lag-harmonic function on a connected open set $X \subset \mathbf{C}^n$. Suppose f is Lag-pluriharmonic (a traceless Hermitian polynomial) with $f \le H$ on a compact set $K \subset X$. Consider the* contact set

$$\Sigma \ \equiv\ \{x \in K : f(x) = H(x)\}$$

Then

$$\Sigma \ \subset\ \widehat{\Sigma \cap \partial K}.$$

In particular, if K is Lagrangian convex in X, then

$$\Sigma \ =\ \widehat{\Sigma \cap \partial K}.$$

PROOF. This follows from [**HL**$_{15}$] where quite general theorems of this sort are proved. ☐

REMARK 4.9. Theorem 4.8 has analogues for the subequations $\mathcal{P}$ and $\mathcal{P}_{\mathbf{C}}$. The real case was first done in [**O**] and [**OS**]. For the complex case we have that H is complex Monge-Ampère harmonic and f is the real part of a complex polynomial. Both cases are covered by [**HL**$_{15}$].

REMARK 4.10. In Section 11 the Dirichlet problem is solved for the Lagrangian Monge-Ampère operator by the Perron method. That is, for a domain Ω and a function $\varphi \in C(\partial\Omega)$, one takes the upper envelope of the

$u \in \mathcal{P}_{\text{Lag}}(\Omega) \cap \text{USC}(\overline{\Omega})$ with $u \le \varphi$ on $\partial\Omega$. Once the theorem is proved, one observes that smaller families will also work. For example, one can use $u \in \mathcal{P}_{\text{Lag}}(\Omega) \cap C(\overline{\Omega})$ with $u \le \varphi$ on $\partial\Omega$.

Now in $\mathbf{C}^n$ we have the Lag-pluriharmonics, and one might wonder whether the solution can be realized using just those functions. This is in fact the case. There is a general theorem which applies to all *basic edge subequations*, of which LAG is one. Details appear in [**HL**$_{14}$].

5. A Lagrangian operator of Monge-Ampère-type

In this section we introduce a nonlinear partial differential operator $\mathrm{M}_{\text{Lag}}(D^2u)$ which defines $\partial\mathcal{P}(\text{LAG})$ and the subequation $\mathcal{P}(\text{LAG})$, but also has other branches. This puts Lagrangian potential theory in a very special category of nonlinear equations whose constraint set is naturally defined using a polynomial operator.

As in the previous section, given $A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, let

$$\mu \;\equiv\; \frac{\text{tr}A}{2}, \qquad \text{and} \qquad \lambda_1, ..., \lambda_n \;\ge\; 0$$

denote the non-negative eigenvalues of the skew-hermitian part $A^{\text{skew}} \equiv \frac{1}{2}(A + JAJ)$ of A.

Definition 5.1 (The Lagrangian MA-operator). This operator

$$\mathrm{M}_{\text{Lag}} : \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) \to \mathbf{R}$$

is defined by

$$\mathrm{M}_{\text{Lag}}(A) \;\equiv\; \prod_{\pm}^{2^n} (\mu \pm \lambda_1 \pm \cdots \pm \lambda_n)\,. \tag{5.1}$$

Note (n=1). In this case $\mathrm{M}_{\text{Lag}}(A) = \mu^2 - \lambda^2$ where $\mu = \frac{1}{2}\text{tr}A$ and $\pm\lambda$ are the eigenvalues of A^{skew}. One can compute that

$$\mathrm{M}_{\text{Lag}}(A) \;=\; \det A \tag{5.2}$$

is the standard Monge-Ampère operator on $\mathbf{R}^2$.

Proposition 5.2 (n $\ge$ 2). *$\mathrm{M}_{\text{Lag}}(A)$ is a polynomial of degree 2^n on $\text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. In fact, after setting $\lambda_0 \equiv \mu$, it is a symmetric polynomial of degree 2^{n-1} in the variables $(\lambda_0^2, \lambda_1^2, ..., \lambda_n^2)$. (Note that $\lambda_1^2, ..., \lambda_n^2$ are the eigenvalues of $\left(A^{\text{skew}}\right)^2$ where each λ_j^2 occurs with multiplicity 2).*

PROOF.

$$
\begin{aligned}
\mathrm{M}_{\mathrm{Lag}}(A) &\equiv \prod_{\pm}^{2^n} (\mu \pm \lambda_1 \pm \cdots \pm \lambda_n) \\
&= \prod_{\pm}^{2^{n-1}} [(\mu \pm \lambda_1 \pm \cdots \pm \lambda_{n-1}) + \lambda_n] \\
&\quad \times \prod_{\pm}^{2^{n-1}} [(\mu \pm \lambda_1 \pm \cdots \pm \lambda_{n-1}) - \lambda_n] \\
&= \prod_{\pm}^{2^{n-1}} \left[(\mu \pm \lambda_1 \pm \cdots \pm \lambda_{n-1})^2 - \lambda_n^2\right].
\end{aligned}
$$

This last expression is fixed by $\mu \mapsto -\mu$ and by interchanging μ and λ_1. Since the first expression is fixed by the permutation group π_n acting on $(\lambda_1, ..., \lambda_n)$ and by $\mathbf{Z}_2^n$ acting on $(\lambda_1, ..., \lambda_n)$ by $\pm$, this proves the proposition. $\square$

Proposition 5.2 gives the polynomial $\mathrm{M}_{\mathrm{Lag}}(A)$ a particularly nice structure. We illustrate this in the first two cases.

Example. (n=2).

$$
\begin{aligned}
\mathrm{M}_{\mathrm{Lag}}(A) &= \lambda_0^4 + \lambda_1^4 + \lambda_2^4 - 2\left(\lambda_0^2\lambda_1^2 + \lambda_0^2\lambda_2^2 + \lambda_1^2\lambda_2^2\right) \\
&= \mu^2 - 2(\lambda_1^2 + \lambda_2^2)\mu^2 + (\lambda_1^2 - \lambda_2^2)^2
\end{aligned}
$$

Example. (n=3).

$$
\begin{aligned}
\mathrm{M}_{\mathrm{Lag}}(A) &= \big[\lambda_0^4 + \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\big(\lambda_0^2\lambda_1^2 + \lambda_0^2\lambda_2^2 + \lambda_0^2\lambda_3^2 + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 \\
&\qquad + \lambda_2^2\lambda_3^2\big)\big]^2 - (8\lambda_0\lambda_1\lambda_2\lambda_3)^2 \\
&= \big[\mu^4 + \lambda_1^4 + \lambda_2^4 + \lambda_3^4 - 2\mu^2\big(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\big) - 2\big(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 \\
&\qquad + \lambda_2^2\lambda_3^2\big)\big]^2 - (8\mu\lambda_1\lambda_2\lambda_3)^2
\end{aligned}
$$

Central to the discussion of operators of this type is Gårding's theory of hyperbolic polynomials. (We refer the reader to ([**Gå**], [$\mathbf{HL}_5$], [$\mathbf{HL}_7$]) for details.) To begin this discussion we need the following.

LEMMA 5.3.

$$
\mathrm{M}_{\mathrm{Lag}}\left(\frac{t}{n}\mathrm{I} + A\right) = \prod_{\pm}^{2^n} (t + \mu \pm \lambda_1 \pm \cdots \pm \lambda_n). \tag{5.3}
$$

PROOF. Note that $\frac{1}{2}\mathrm{tr}(\frac{t}{n}\mathrm{I} + A) = t + \frac{\mathrm{tr}A}{2}$, and $\frac{t}{n}\mathrm{I} + A$ has the same skew-hermitian part as A. $\square$

COROLLARY 5.4. $\mathrm{M_{Lag}}$ *is a Gårding polynomial hyperbolic in the direction* $\frac{1}{n}\mathrm{I}$ *with* $\mathrm{M_{Lag}}(\frac{1}{n}\mathrm{I}) = 1$*. The Gårding eigenvalues of* $\mathrm{M_{Lag}}(\frac{t}{n}\mathrm{I}+A)$*, which by definition are the negatives of the roots of this polynomial in* t*, are*

$$\Lambda_{\pm\pm\cdots\pm}(A) \;\equiv\; \mu \pm \lambda_1 \pm \cdots \pm \lambda_n. \tag{5.4}$$

PROPOSITION 5.5. *The subequation* $\mathcal{P}(\mathrm{LAG})$ *is exactly the closed Gårding cone defined by the inequalities* $\Lambda_{\pm\pm\cdots\pm}(A) \geq 0$*.*

PROOF. With $\lambda_1, ..., \lambda_n \geq 0$ denoting the non-negative eigenvalues of A^{skew}, the minimum Gårding eigenvalue is $\Lambda_{\min}(A) = \mu - \lambda_1 - \cdots - \lambda_n$. Theorem 3.6 states that $\Lambda_{\min} \geq 0$ defines $\mathcal{P}(\mathrm{LAG})$. ☐

COROLLARY 5.6. *The Gårding polymonial* $\mathrm{M_{Lag}}$ *is a Gårding/Dirichlet operator in the terminology of* [**HL**$_5$]*, that is, the associated closed Gårding cone* $\mathcal{P}(\mathrm{LAG})$ *satisfies* $\mathcal{P} \subset \mathcal{P}(\mathrm{LAG})$*.*

PROOF. One has $\mathcal{P} \subset \mathcal{P}(\mathrm{LAG})$, since $P \geq 0 \;\Rightarrow\; \mathrm{tr}(P\big|_W) \geq 0$ for any subspace W. ☐

Note that (5.2) is not all that surprising since, when $n = 1$, $\mathrm{LAG} = G(1, \mathbf{R}^2)$ and so $\mathcal{P}(\mathrm{LAG}) = \mathcal{P}$, the subequation which defines convex functions.

$\mathrm{M_{Lag}}(D^2u)$ **as a polynomial differential operator.** While Proposition 5.2 gives $\mathrm{M_{Lag}}(A)$ a nice structure (symmetrically intertwining μ^2 and the λ_j^2's,) it somewhat obscures $\mathrm{M_{Lag}}(D^2u))$ as a differential operator. We now give two expressions for $\mathrm{M_{Lag}}(D^2u)$ in terms of the second coordinate partial derivatives of u. By Proposition 5.2, there exists homogeneous symmetric polynomials $s_k(\lambda) \equiv s_k((A^{\mathrm{skew}})^2)$ of degree k in $\lambda_1^2, ..., \lambda_n^2$ (degree $2k$ in $\lambda_1, ..., \lambda_n$) for $k = 1, ..., 2^{n-1}$, such that:

$$\mathrm{M_{Lag}}(A) \;=\; \mu^{2^n} + s_1\mu^{2^n-2} + s_2\mu^{2^n-4} + \cdots + s_{2^{n-1}} \tag{5.5}$$

With an abuse of notation, if we set

$$s_k(u) \;\equiv\; s_k\left(\left((D^2u)^{\mathrm{skew}}\right)^2\right),$$

then

$$\mathrm{M_{Lag}}(D^2u) = (\Delta u)^{2^n} + s_1(u)(\Delta u)^{2^n-2} + s_2(u)(\Delta u)^{2^n-4} + \cdots + s_{2^{n-1}}(u) \tag{5.5$'$}$$

Rather than focus on computing what these functions $s_k(\lambda)$ are explicitly, we move on to our second expression for $\mathrm{M_{Lag}}(D^2u)$.

Each $s_k(\lambda)$ can be written as a polynomial expression in the basic functions

$$\tau_\ell \;\equiv\; \lambda_1^{2\ell} + \cdots + \lambda_n^{2\ell}$$

Now to compute the operator $\mathrm{M_{Lag}}(D^2u)$ we recall that $\mu = \frac{1}{2}\Delta u$ and that $\lambda_1^2, ..., \lambda_n^2$ are the eigenvalues (with multiplicity 2) of $[(D^2u)^{\text{skew}}]^2$. Given a symmetric polynomial $\sigma(\lambda)$ in the λ_j^2 we set

$$\sigma(u) \;\equiv\; \sigma(\lambda_1^2, ..., \lambda_n^2).$$

In particular, we have

$$\tau_\ell(u) \;=\; \tfrac{1}{2}\mathrm{tr}\left\{[(D^2u)^{\text{skew}}]^{2\ell}\right\}$$

Calculation shows the following.

Example. (n=2).

$$\mathrm{M_{Lag}}(A) \;=\; \mu^4 - 2\tau_1\mu^2 + (2\tau_2 - \tau_1^2)$$

$$\mathrm{M_{Lag}}(D^2u) \;=\; (\Delta u)^4 - 2\tau_1(u)(\Delta u)^2 + 2\tau_2(u) - \tau_1(u)^2$$

Example. (n=3).

$$\mathrm{M_{Lag}}(A) \;= \mu^8 - 4\tau_1\mu^6 + (4\tau_2 + 2\tau_1^2)\mu^4 + \tfrac{4}{3}[-16\tau_3 + 18\tau_2\tau_1 - 5\tau_1^3]\mu^2 + (2\tau_2 - \tau_1^2)^2$$

$$\mathrm{M_{Lag}}(D^2u) \;=\; (\Delta u)^8 - 4\tau_1(u)(\Delta u)^6 + \{4\tau_2(u) + 2\tau_1(u)^2\}(\Delta u)^4 + \cdots$$

Branches of $\mathrm{M_{Lag}}(D^2u)$

It is a general fact (see [**HL**$_5$]) that any operator defined by a Gårding/Dirichlet polynomial of degree d on $\mathrm{Sym}^2(\mathbf{R}^N)$ has d **branches**, i.e., d subequations defined by requiring:

$$\Lambda_k(A) \;\geq\; 0,$$

where $\Lambda_1(A) \leq \Lambda_2(A) \leq \cdots \leq \Lambda_d(A)$ are the ordered Gårding eigenvalues of A. The smallest such subequation, defined by $\Lambda_1(A) \equiv \Lambda_{\min}(A) \geq 0$, is convex and is called the *Gårding cone*. It provides the *monotonicity property*

$$\{\Lambda_1(A) \geq 0\} + \{\Lambda_k(A) \geq 0\} \;\subset\; \{\Lambda_k(A) \geq 0\}. \tag{5.6}$$

Definition 5.7. By Corollary 5.4 the subequation $\mathcal{P}(\mathrm{LAG})$ is the Gårding cone associated to the Gårding/Dirichlet polynomial (5.1) with eigenvalues given by (5.4). This gives us $d = 2^n$ branches

$$\mathcal{P}_k(\mathrm{LAG}) \;\equiv\; \{A : \Lambda_k(A) \geq 0\}, \qquad k = 1, 2, ...$$

of the equation $\mathrm{M_{Lag}}$, each monotone with respect to the principle branch $\mathcal{P}(\mathrm{LAG}) = \mathcal{P}_1(\mathrm{LAG})$ (and so, in particular, they are subequations). The largest branch $\mathcal{P}_d(\mathrm{LAG})$ is the dual subequation $\widetilde{\mathcal{P}(\mathrm{LAG})}$.

This monotonicity of the branches with respect to $\mathcal{P}(\mathrm{LAG})$, has interesting consequences. For example, it implies the following removable singularity result.

Theorem 5.8 ([**HL**$_{10}$]). *Let $\Omega \subset \mathbf{R}^n$ be a domain and $\Sigma \subset \Omega$ a closed subset of locally finite Hausdorff $(n-2)$-measure. Then any function u which* *is* $\mathcal{P}_k(\mathrm{LAG})$*-subharmonic on* $\Omega - \Sigma$ *and is locally bounded above at points of* Σ*, extends to a* $\mathcal{P}_k(\mathrm{LAG})$*-subharmonic function on all of* Ω*. Similarly, if* u *is* $\mathcal{P}_k(\mathrm{LAG})$*-harmonic on* $\Omega - \Sigma$ *and continuous on* Ω*, then* u *is* $\mathcal{P}_k(\mathrm{LAG})$*-harmonic on* Ω*.*

NOTE 5.9 (Unitary Invariance). The operator $\mathrm{M}_{\mathrm{Lag}}$ on $\mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ is $\mathrm{U}(n)$-invariant, i.e.,

$$\mathrm{M}_{\mathrm{Lag}}(gAg^{-1}) = \mathrm{M}_{\mathrm{Lag}}(A) \quad \text{for all} \ \ g \in \mathrm{U}(n) \ \ \text{and} \ \ A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n).$$

This follows rather straightforwardly form the definition of $\mathrm{M}_{\mathrm{Lag}}$. It follows that each of the subequations

$$\mathcal{P}_k(\mathrm{LAG}) \quad \text{is} \ \ \mathrm{U}(n) \ \text{invariant, for} \ \ 1 \le k \le 2^n.$$

This fact is important because it shows that the subequations $\mathcal{P}_k(\mathrm{LAG})$ make sense on any Gromov manifold (see §7).

6. Two intrinsic definitions of the Lagrangian (Monge-Ampère) operator

In this section we present two constructions of the operator $\mathrm{M}_{\mathrm{Lag}}(D^2u)$. Each gives us an intrinsic construction of the operator when we pass from $\mathbf{C}^n$ to general Gromov manifolds.

6.1. Constructing $\mathrm{M}_{\mathrm{Lag}}$ as the determinant of a derivation. Consider $\mathbf{C}^n = (\mathbf{R}^{2n}, J)$ as above. Then any symmetric matrix $H \in \mathrm{Sym}^2(\mathbf{R}^{2n})$ prolongs to a map

$$D_H : \Lambda^n \mathbf{R}^{2n} \longrightarrow \Lambda^n \mathbf{R}^{2n}$$

as a derivation.

OBSERVATION 6.1. *Suppose that* H *can be diagonalized by a hermitian orthonormal basis* $e_1, Je_1, ..., e_n, Je_n$*. Then* D_H *preserves the subspace* $S_H \subset \Lambda^n \mathbf{R}^{2n}$ *spanned by the* 2^n *vectors*

$$\xi_{\pm\pm\cdots\pm} \ \equiv \ (e_1 \text{ or } Je_1) \wedge (e_2 \text{ or } Je_2) \wedge \cdots \wedge (e_n \text{ or } Je_n)$$

In fact these vectors are eigenvectors of D_H*.*

We can now apply this observation to the Lagrangian part $H \equiv A^{\mathrm{Lag}} = \frac{1}{2n}(\mathrm{tr}A)\mathrm{I} + A^{\mathrm{skew}}$ of any matrix $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, and straightforward calculation shows that

$$\mathrm{M}_{\mathrm{Lag}}(A) \ = \ \det \left\{ D_{A^{\mathrm{Lag}}} \big|_{S_{A^{\mathrm{Lag}}}} \right\}. \tag{6.1}$$

With notation as in Observation 6.1, the eigenvector $\xi_{\pm\pm\cdots\pm}$ of D_H has eigenvalue precisely $\mu \pm \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n$.) In particular,

$$\begin{gathered}\mathrm{M}_{\mathrm{Lag}}(A) \ \ \text{is a factor of the determinant of} \\ D_{A^{\mathrm{Lag}}} \ \ \text{acting on} \ \ \Lambda^n \mathbf{R}^{2n}.\end{gathered} \tag{6.2}$$

We note that although (6.1) does not provide an intrinsic definition of $\mathrm{M}_{\mathrm{Lag}}(A)$ (since $S_{A^{\mathrm{Lag}}}$ depends on A), the factor $\mathrm{M}_{\mathrm{Lag}}(A)$ in (6.2) is intrinsic.

The assertion (6.2) can be somewhat strengthened by considering the subspace

$$\Lambda^n_{\text{prim}} \;=\; \{\varphi \in \Lambda^n : \omega \wedge \varphi = 0\}$$

LEMMA 6.2. *The subspace $\Lambda^n_{\text{prim}} \subset \Lambda^n \mathbf{R}^{2n}$ is invariant under $D_{A^{\text{Lag}}}$ for any $A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. Thus, in particular,*

$$\text{(6.3)} \qquad \mathrm{M}_{\text{Lag}}(A) \ \text{ is a factor of the determinant of } \ D_{A^{\text{Lag}}} \ \text{ acting on } \ \Lambda^n_{\text{prim}}.$$

PROOF. Let $e_1, Je_1, ..., e_n, Je_n$ be a hermitian orthonormal basis which diagonalizes A^{skew}. The since $\omega = e_1 \wedge Je_1 + \cdots + e_n \wedge Je_n$, we have $D_{A^{\text{skew}}}(\omega) = 0$, and so $D_{A^{\text{Lag}}}(\omega) = \frac{\text{tr}(A)}{n}\omega$. We can assume $\text{tr}(A) \neq 0$ by adding a multiple of the identity if necessary. Then if $\psi \in \Lambda^n_{\text{prim}}$, we have

$$0 \;=\; D_{A^{\text{Lag}}}(\omega \wedge \psi) \;=\; D_{A^{\text{Lag}}}(\omega) \wedge \psi + \omega \wedge D_{A^{\text{Lag}}}(\psi) \;=\; \omega \wedge D_{A^{\text{Lag}}}(\psi),$$

and so $D_{A^{\text{Lag}}}(\psi) \in \Lambda^n_{\text{prim}}$. $\square$

6.2. Constructing M_{Lag} as the determinant of a spinor endomorphism. In this subsection we present a construction of the operator $\mathrm{M}_{\text{Lag}}(D^2u)$ in terms of spinors – more specifically in terms of the irreducible representations of Spin^c_{2n}. This gives a second intrinsic construction of the operator, which is useful when we pass from $\mathbf{C}^n$ to general Gromov manifolds. We break the construction into two steps.

The first step is to define a natural algebraic map

$$\text{(6.4)} \qquad \Phi : \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) \;\longrightarrow\; \Lambda^2_{\mathbf{R}}\mathbf{C}^n.$$

Given $A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$, abbreviate

$$B \;\equiv\; A^{\text{skew}} \;=\; \tfrac{1}{2}(A + JAJ).$$

Since $Be = \lambda e$ implies $BJe = -\lambda Je$, the square $B^2 \geq 0$ has a unique positive square root $E \equiv \sqrt{B^2}$ which satisfies $Ee = |\lambda| e$ and $EJe = |\lambda| Je$. Note that $JE = EJ$ and that E is hermitian symmetric. Hence,

$$\text{(6.5)} \qquad EJ \in \text{Skew}_{\mathbf{R}}(\mathbf{C}^n) \;\equiv\; \Lambda^2_{\mathbf{R}}\mathbf{C}^n, \qquad \text{i.e.,} \ \ (EJ)^t = -EJ.$$

DEFINITION 6.3. Adopting these notations the map Φ is defined by:

(a) $\Phi(A) \;\equiv\; EJ \;=\; \sqrt{B^2}J, \quad B \;\equiv\; \frac{1}{2}(A + JAJ),$

or using Lemma 3.1,

(b) $\Phi(A) \;\equiv\; \lambda_1 e_1 \wedge Je_1 + \cdots + \lambda_n e_n \wedge Je_n.$

To see that (a) and (b) are equal, apply the canonical form for $B \in \text{Herm}^{\text{sym}}(\mathbf{C}^n)$ (with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$), and note that

$$\text{(6.6)} \qquad E \;\equiv\; \lambda_1(P_{e_1} + P_{Je_1}) + \cdots + \lambda_n(P_{e_n} + P_{Je_n}),$$

and second that

$$\text{(6.7)} \qquad EJ \;=\; \lambda_1(Je_1 \circ e_1 - e_1 \circ Je_1) + \cdots + \lambda_n(Je_n \circ e_n - e_n \circ Je_n)$$

Remark 6.4. Consider the blocking $\mathbf{C}^n \equiv W \oplus JW$ defined by (3.9). First $J \equiv \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, so that

$$B \equiv \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}, \qquad E \equiv \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \quad \text{and} \qquad EJ \equiv \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix}. \tag{6.8}$$

Note that $\lambda \geq 0$ is diagonal. Also note that $B = B^+ - B^-$ with

$$B^+ \ \equiv\ \tfrac{1}{2}(B+E) \ =\ \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \ \geq\ 0 \qquad \text{and}$$
$$B^- \ \equiv\ \tfrac{1}{2}(B-E) \ =\ \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} \ \geq\ 0$$

the positive and negative parts of B.

For the second step adopt the notation $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ and $B \equiv A^{\mathrm{skew}}$ as before. Now using the first step, set $\widetilde{\mathbf{B}} \equiv \Phi(A)$ and consider $\widetilde{\mathbf{B}}$ as an element in the Clifford algebra

$$\widetilde{\mathbf{B}} \ \in\ \Lambda^2\mathbf{R}^{2n} \ \subset\ \mathrm{Cl}_{2n}.$$

Recall that

$$\widetilde{\mathbf{B}} \ =\ \sum_{k=1}^n \lambda_k e_k J e_k.$$

Consider now the (unique) irreducible complex representation $\$ \cong \mathbf{C}^{2^n}$ of Cl_{2n}. This extends naturally to a representation of $\mathbf{Cl}_{2n} = \mathrm{Cl}_{2n} \otimes_{\mathbf{R}} \mathbf{C}$. Note that the elements

$$ie_kJe_k \in \mathbf{Cl}_{2n} \quad \text{satisfy} \quad (ie_kJe_k)^2 \ =\ 1.$$

We set

$$\pi_k^+ \equiv \tfrac{1}{2}(1+ie_kJe_k) \qquad \text{and} \qquad \pi_k^- \equiv \tfrac{1}{2}(1-ie_kJe_k).$$

Then

$$(\pi_k^+)^2 \ =\ \pi_k^+, \quad (\pi_k^-)^2 \ =\ \pi_k^- \qquad \text{and} \qquad \pi_k^+ + \pi_k^- \ =\ 1.$$

Also we have that the $\pi_k^\pm$ commute with all the $\pi_\ell^\pm$ for $k \neq \ell$ and

$$\pi_k^\pm e_k \ =\ -e_k\pi_k^\pm \qquad \text{for all} \ \ k.$$

Thus Clifford multiplication by the projectors $\pi_k^\pm$ decompose the Spinor space

$$\$ = \equiv\ \pi_k^+\$ \ \oplus\ \pi_k^-\$$$

into the $+1$ and -1 eigenspaces under multiplication by ie_kJe_k. These two spaces are isomorphic under multiplication by e_k and they are invariant under multiplication by $ie_\ell Je_\ell$ for all $\ell \neq k$. Hence we get a decomposition

$$\$ \ =\ \bigoplus_{\text{all } \pm} \$_{\pm\pm\cdots\pm} \qquad \text{where} \qquad \$_{\pm\pm\cdots\pm} \ \equiv\ \pi_1^\pm\pi_2^\pm\cdots\pi_n^\pm\$ \ =\ \bigcap_{k=1}^n \pi_k^\pm\$.$$

Now each $\mathbb{S}_{\pm\pm\cdots\pm}$ has dimension one and is an eigenspace for each ie_kJe_k. This follows because the $\pi_k^{\pm}$ all commute and $\pi_k^+e_k = -e_k\pi_k^-$, and so one can do a dimension count (see [**LM**, page 43 ff.]). In particular each

$$\text{(6.9)} \qquad \mathbb{S}_{\pm\pm\cdots\pm} \ \text{ is an eigenspace for } \ i\widetilde{\mathbf{B}} \ \text{ with eigenvalue} \\ \pm\lambda_1 \pm \lambda_2 \pm \ \cdots \ \pm \lambda_n.$$

Thus we have:

PROPOSITION 6.5. *As an endomorphism of $\mathbb{S}$ by Clifford multiplication, the element $i\widetilde{\mathbf{B}}$ has*

$$\det(i\widetilde{\mathbf{B}}) \ = \ \prod_{\text{all } \pm} (\pm\lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n)$$

Furthermore, for any real number μ, under Clifford multiplication, the element $\mu 1 + i\widetilde{\mathbf{B}}$ has

$$\det(\mu 1 + i\widetilde{\mathbf{B}}) \ = \ \prod_{\text{all } \pm} (\mu \pm \lambda_1 \pm \lambda_2 \pm \cdots \pm \lambda_n)$$

7. Lagrangian potential theory on Gromov manifolds

In this section we carry the previous discussion over to a general context which is relevant to symplectic geometry.

DEFINITION 7.1. By a **Gromov manifold** we mean a triple (X, ω, J) where (X, ω) is a symplectic manifold and J is an almost complex structure on X satisfying the conditions:

$$\text{(7.1)} \qquad \omega(v, w) \ = \ \omega(Jv, Jw) \qquad \text{and} \qquad \omega(Jv, v) \ > \ 0$$

for all non-zero tangent vectors v and w at each point of X. On such a manifold there is a natural riemannian metric $\langle \cdot, \cdot \rangle$ defined by

$$\text{(7.2)} \qquad \langle v, w \rangle \ \equiv \ \omega(Jv, w) \qquad \text{with} \quad \langle Jv, Jw \rangle \ = \ \langle v, w \rangle.$$

REMARK 7.2. It is a result of Gromov (see [**G**], [**MS**]) that any compact symplectic manifold admits an almost complex structure J satisfying (7.1).

REMARK 7.3. Note that on a Gromov manifold there is a well defined notion of a Lagrangian submanifold. There are also local J-holomorphic curves passing through any point with any prescribed complex tangent [**NW**]. One easily checks that if $F : X \to X$ is a symplectomorphism, (a diffeomorphism with $F^*\omega = \omega$), then:

(a) F maps Lagrangian submanifolds to Lagrangian submanifolds,

(b) The transported almost complex structure $\widetilde{J} \equiv F_* \circ J \circ (F^{-1})_*$ again satisfies the conditions (7.1),

(c) F maps J-holomorphic curves to $\widetilde{J}$-holomorphic curves.

We now recall that using the riemannian metric, one can define a Hessian (or second derivative) which will allow us to carry over the foregoing material to Gromov manifolds.

DEFINITION 7.4. Given a smooth function $u \in C^\infty(X)$, the **hessian** of u is a section of $\mathrm{Sym}^2(T^*X)$ defined on vector fields v, w by

$$(\mathrm{Hess} f)(v, w) \;\equiv\; vwf - (\nabla_v w)f \tag{7.3}$$

where ∇ denotes the Levi-Civita connection for the metric $\langle\cdot,\cdot\rangle$.

We note that this Hessian gives a canonical splitting of the 2-jet bundle of X:

$$J^2(X) \;=\; \mathbf{R} \oplus T^*X \oplus \mathrm{Sym}^2(T^*X), \tag{7.4}$$

(see [**HL**$_4$] or [**HL**$_8$]). Furthermore, using the metric and J we identify

$$T^*X \;\cong\; TX$$

and obtain the decomposition

$$\mathrm{Sym}^2(T^*X) \;=\; \mathbf{R} \oplus \mathrm{Herm}_0^{\mathrm{sym}}(TX) \oplus \mathrm{Herm}^{\mathrm{skew}}(TX) \tag{7.5}$$

corresponding exactly to the combined decompositions (3.3) and (3.3′).

Observe now that there is a well defined subbundle $\mathrm{LAG} \subset G(n, TX)$ of the Grassmann bundle of tangent n-planes, which consists of the Lagrangian tangent n-planes. This bundle is invariant under the group of symplectomorphisms of (X, ω). It embeds naturally into the bundle $\mathrm{Sym}^2(TX)$ by associating to W, the orthogonal projection of T_xX onto W.

With this said it should be clear that all of the algebraic considerations of the previous sections 2 and 3 apply in the obvious way to this context. In particular, we have a well-defined subequation of the 2-jet bundle:

$$\mathcal{P}(\mathrm{LAG}) \;\equiv\; \left\{J^2(u) : \mathrm{tr}\left(\mathrm{Hess}(u)\big|_W\right) \geq 0 \;\forall\; W \in \mathrm{LAG}\right\} \;\subset\; J^2(X),$$

as well as its dual subequation defined fibrewise by $\widetilde{\mathcal{P}}(\mathrm{LAG}) \equiv -(\sim \mathrm{Int}\mathcal{P}(\mathrm{LAG}))$, and the equation $\partial\mathcal{P}(\mathrm{LAG}) = \mathcal{P}(\mathrm{LAG}) \cap (-\widetilde{\mathcal{P}}(\mathrm{LAG}))$.

We can also carry over the definitions from Section 2. The primary two are:

DEFINITION 7.5. A function $u \in C^\infty(X)$ is **Lagrangian plurisubharmonic** if its 2-jet $J_x^2(u) \in \mathcal{P}(\mathrm{LAG})$ for all $x \in X$, or equivalently, if

$$\mathrm{tr}\left(\mathrm{Hess}(u)\big|_W\right) \geq 0 \;\forall\; W \in \mathrm{LAG}.$$

If, in addition, $-J_x^2(u) \in \widetilde{\mathcal{P}}(\mathrm{LAG})$ for all x, then u is called **Lagrangian harmonic**. Equivalently, u is Lagrangian harmonic if and only if $J_x^2(u) \in \partial\mathcal{P}(\mathrm{LAG})$ for all x.

All of the definitions in Section 2 extend to upper semi-continuous functions. The notion of a viscosity test function (Definition 2.4(a)) carries over directly to manifolds, and we have the following.

Definition 7.6. A function $u \in \mathrm{USC}(X)$ is **Lagrangian plurisubharmonic** if for each $x \in X$ and each test function φ for u at x, one has $J^2_x(\varphi) \in \mathcal{P}(\mathrm{LAG})$. If, in addition, for each $x \in X$ and each test function ψ for $-u$ at x, one has $J^2_x(\psi) \in \widetilde{\mathcal{P}}(\mathrm{LAG})$, then u is called **Lagrangian harmonic**. (This additional condition by itself defines the notion of **dually Lagrangian plurisubharmonic** for $v \equiv -u$.)

Theorem 7.7. *If M is a Lagrangian submanifold which is also a minimal submanifold, then the restriction $u\big|_M$ of a Lagrangian plurisubharmonic function u to M is subharmonic.*

Proof. If u is C^∞ it follows from the equation (see Proposition 2.10 in [**HL**$_1$])

$$\Delta\left(u\big|_M\right) \;=\; \mathrm{tr}_{TM}\{\mathrm{Hess} u\} - H_M(u)$$

where H_M is the mean curvature vector of M and Δ is the intrinsic Laplace-Beltrami operator on M with respect to the induced metric. For $u \in \mathrm{USC}(X)$ one must use the Restriction Theorem [**HL**$_9$, Thm. 6.4]. □

Note that (2.12a) easily carries over.

Lemma 7.8. *If $u \in \mathrm{USC}(X)$ is Lagrangian plurisubharmonic, then u is subharmonic.*

Proof. Let φ be a test function for u at $x \in X$, and choose a Lagrangian plane $W \subset T_xX$. Then $\mathrm{tr}\{\mathrm{Hess}_x\varphi\big|_W\} \geq 0$ and $\mathrm{tr}\{\mathrm{Hess}_x\varphi\big|_{W^\perp}\} \geq 0$ since $W^\perp$ is also Lagrangian. Hence, $\Delta(\varphi)_x = \mathrm{tr}\{\mathrm{Hess}_x\varphi\big|_W\} + \mathrm{tr}\{\mathrm{Hess}_x\varphi\big|_{W^\perp}\} \geq 0$, and we conclude that u is subharmonic in the viscosity sense. However, this notion coincides with all other notions of subharmonicity on a riemannian manifold. □

Note 7.9. In defining our Lagrangian plurisubharmonic functions here, we have used the riemannian Hessian. That same Hessian could also be used to define complex plurisubharmonic functions on our almost complex manifold. However, these complex psh functions do not always agree with the usual intrinsic ones, namely those whose restrictions to (pseudo) holomorphic curves are subharmonic (see Example 9.5 in [**HL**$_{11}$]). However, when the Kähler form of the almost complex structure is d-closed, as it is here, the two notions of complex psh functions do agree [**HL**$_{11}$, Thm. 9.1].

8. The Lagrangian Monge-Ampère operator on Gromov manifolds

On any Gromov manifold (X, J, ω) (in fact on any almost complex hermitian manifold) there is a well-defined Lagrangian Monge-Ampère Operator $\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u)$ defined exactly as in the euclidean case, but with D^2u replaced by the riemannian Hessian $\mathrm{Hess}\, u$. As before this operator has 2^n branches where $2n = \dim_{\mathbf{R}}(X)$. This operator can be defined intrinsically in two different ways.

For the first we consider the derivation

$$D \equiv D_{\mathrm{Hess}(u)^{\mathrm{Lag}}}$$

acting of the bundle $\Lambda^n T^*X$. Then as in (6.2) we see that

$$\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) \text{ is a factor of } \det\{D\}.$$

In fact as noted there we can restrict D to the bundle of primitive n-forms

$$\Lambda^n_{\mathrm{prim}}(X) \subset \Lambda^n T^*X$$

defined as the kernel of exterior multiplication $\omega\wedge : \Lambda^n T^*X \to \Lambda^{n+2}T^*X$ by the 2-form ω. Then, from Lemma 6.2 we have

$$\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) \text{ is a factor of } \det\left\{ D\big|_{\Lambda^n_{\mathrm{prim}}(X)} \right\}.$$

For the second construction let $\$ \longrightarrow X$ be any bundle of complex modules over the Clifford bundle $\mathbf{Cl}(X) \equiv Cl(X)\otimes_{\mathbf{R}} \mathbf{C}$. Assume further that $\$$ is pointwise irreducible, i.e., $\dim_{\mathbf{C}}(\$) = 2^n$. Then given any function $\mu \in C^\infty(X)$ and any section B of $\mathrm{Sym}^2(T^*X)$ which is skew hermitian ($BJ \equiv -JB$ on X), we consider the section of the Clifford bundle

$$\mu 1 + i\widetilde{\mathbf{B}} \in \Gamma(\mathbf{Cl}(X)).$$

in the notation of Section 6. Clifford multiplication gives a bundle map

$$(\mu 1 + i\widetilde{\mathbf{B}}) : \$ \longrightarrow \$$$

and we can take its determinant.

Now there always exists such a bundle of irreducible complex modules for $\mathbf{Cl}(X)$, namely

$$\$ \equiv \bigoplus_{q=0}^{n} \Lambda^{0,q}(X) \cong \Lambda^*_{\mathbf{C}} T^*_{\mathbf{C}}(X).$$

The Clifford multiplication is generated by letting a real tangent vector v act by $v\wedge - i_v$ where i_v is contraction. This gives the following.

THEOREM 8.1. *Let u be a C^2-function on a Gromov manifold X. Consider the section*

$$\tfrac{1}{2}\Delta u + i\widetilde{\mathbf{H}}_{\mathrm{skew}}(u)$$

acting by Clifford multiplication on the bundle $\$ \equiv \bigoplus_{q=0}^n \Lambda^{0,q}(X)$ of $(0,q)$ forms. Then the Lagrangian Monge-Ampère operator on X is the determinant of this bundle map:

$$\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) = \det\left\{ \tfrac{1}{2}\Delta u + i\widetilde{\mathbf{H}}_{\mathrm{skew}}(u) \right\}$$

NOTE 8.2. There are other choices, we could twist $\$$ with any complex line bundle. This does not change the determinant.

NOTE 8.3. There is a "quasi" form of this equation. We replace $\widetilde{\mathbf{B}} = \widetilde{\mathbf{H}}_{\mathrm{skew}}(u)$ with $\omega + \widetilde{\mathbf{B}}$.

Remark 8.4 (Branches). Each of the branches $\mathcal{P}_k(\mathrm{LAG})$, $1 \le k \le 2^n$ of the Lagrangian operator (see §5 and Definition 5.7) carries over to any Gromov manifold. This follows from the unitary invariance of $\mathcal{P}_k(\mathrm{LAG})$, which follows from the unitary invariance of $\mathrm{M}_{\mathrm{Lag}}$, as discussed in Note 5.9.

9. Lagrangian pseudoconvexity

In this section we investigate the Lagrangian analogue of the concepts of a pseudoconvex domain and a total real submanifold in complex analysis. Suppose $(X, \omega, J, \langle\cdot,\cdot\rangle)$ is a non-compact, connected Gromov manifold, and denote by $\mathrm{PSH}^\infty_{\mathrm{Lag}}(X)$ the cone of smooth Lagrangian plurisubharmonic functions on X. Recall also the succinct notation Lag-psh, etc. in Definition 2.5.

Definition 9.1. By the **Lagrangian hull** of a compact subset $K \subset X$ we mean the set

$$\widehat{K} \;\equiv\; \{x \in X : f(x) \le \sup_K f \quad \text{for all} \quad f \in \mathrm{PSH}^\infty_{\mathrm{Lag}}(X)\}$$

If $\widehat{K} = K$, then K is called **(Lagrangian) convex**.

Theorem 9.2. *The following are equivalent.*

1) If $K \subset\subset X$, then $\widehat{K} \subset\subset X$.

2) There exists a smooth Lag-psh proper exhaustion function f on X.

3) There exists a neighborhood N of ∞ in X and a smooth function v on N, which is Lag-psh, such that $\lim_{x\to\infty} v(x) = \infty$.

Definition 9.3. A Gromov manifold X satisfying the equivalent conditions of Theorem 9.2 is called **Lagrangian convex**.

Theorem 9.4. *The following are equivalent.*

1) $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$, and there exists $f \in C^\infty(X)$ which is strictly Lag-psh.

2) There exists a smooth strictly Lag-psh proper exhaustion function on X.

Definition 9.5. When X satisfies the equivalent conditions of Theorem 9.4, it is called **strictly Lagrangian convex**.

Theorem 9.6. *The following are equivalent.*

1) $K \subset\subset X \;\Rightarrow\; \widehat{K} \subset\subset X$, and there exists $f \in C^\infty(X)$ which is strictly Lag-psh outside a compact subset of X.

2) There exists a smooth Lag-psh proper exhaustion function on X which is strict outside a compact subset.

3) There exists a neighborhood N of ∞ in X and a smooth function v on N, which is strictly Lag-psh, such that $\lim_{x\to\infty} v(x) = \infty$.

DEFINITION 9.7. When X satisfies the equivalent conditions of Theorem 9.6, it is called **strictly Lagrangian convex at infinity**.

Theorems 9.2, 9.4 and 9.6 are proved (in greater generality) in §4 of [**HL**$_6$].

Cores. Given a function $f \in \mathrm{PSH}_{\mathrm{Lag}}(X)$, consider the open set

$$S(f) \;\equiv\; \{x \in X : f \;\text{ is strictly Lagrangian plurisubharmonic at }\; x\}$$

and the closed set

$$W(f) \;\equiv\; X - S(f).$$

Note that

$$W(\lambda f + \mu g) \;=\; W(f) \cap W(g)$$

for $f, g \in \mathrm{PSH}_{\mathrm{Lag}}(X)$ and $\lambda, \mu > 0$.

DEFINITION 9.8. The **core** of X is defined to be the intersection

$$\mathrm{Core}(X) \;\equiv\; \bigcap W(f)$$

over all $f \in \mathrm{PSH}_{\mathrm{Lag}}(X)$. The **inner core** is defined to be the set $\mathrm{InnerCore}(X)$ of points x for which there exists $y \neq x$ with the property that $f(y) = f(x)$ for all $f \in \mathrm{PSH}_{\mathrm{Lag}}(X)$.

Arguing exactly as in [**HL**$_1$] shows the following:

(9.1) $\qquad \mathrm{InnerCore}(X) \subseteq \mathrm{Core}(X)$

(9.2) $\qquad$ Every compact minimal Lagrangian submanifold is contained in $\mathrm{Core}(X)$

THEOREM 9.9. *Suppose X is Lagrangian convex. Then* $\mathrm{Core}(X)$ *is compact if and only if X is strictly Lagrangian convex at infinity. Furthermore,* $\mathrm{Core}(X) = \emptyset$ *if and only if X is strictly Lagrangian convex.*

Free submanifolds. In analogue with the concept of a totally real submanifold (one free of any complex tangent lines) in complex analysis, we introduce the following.

DEFINITION 9.10. A submanifold $M \subset X$ is said to be **(Lagrangian) free** if its tangent spaces contain no Lagrangian n-planes.

EXAMPLE 9.11. Note that any submanifold of dimension $< n$ is automatically free. Note also that any (almost) complex submanifold is also free.

As in [**HL**$_1$] free submanifolds can be used to construct huge families of strictly Lagrangian convex spaces. We begin with the following observation.

THEOREM 9.12. *Suppose X is strictly Lagrangian convex and of dimension $2n$. Then X has the homotopy-type of a CW-complex of dimension $\leq 2n-2$.*

Proof. Let $f : X \to \mathbf{R}^+$ be a strictly Lagrangian plurisubharmonic proper exhaustion function. By perturbing we may assume that f has non-degenerate critical points. The theorem follows if we show that each critical point has Morse index $\leq 2n-2$ (cf. [**M**]). If this fails, then there is a critical point x at which $\mathrm{Hess}_x f$ has at least $2n-1$ negative eigenvalues. This means there exists a subspace $W \subset T_x X$ of dimension $\geq 2n-1$ with $\mathrm{Hess}_x f\big|_W < 0$. However, any such W contains a Lagrangian n-plane W, and since f is strictly Lagrangian plurisubharmonic, we must have $\mathrm{tr}_W \mathrm{Hess}_x f > 0$, a contradiction. $\square$

The following results are proved by an adaptation of [**HW**$_1$, **HW**$_2$] and Theorems 6.4 and 6.5 in [**HL**$_1$].

Theorem 9.13. *Suppose M is a closed submanifold of X and let $f_M(x) \equiv \frac{1}{2}\mathrm{dist}(x, M)^2$ denote half the square of the distance to M. Then M is Lagrangian free if and only if the function f_M is strictly Lag-psh at each point in M (and hence in a neighborhood of M).*

Theorem 9.14. *Suppose M is any Lagrangian free submanifold of M. Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M such that:*

(a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

(b) Each neighborhood $U \in \mathcal{F}(M)$ is strictly Lagrangian convex.

(c) $\mathrm{PSH}_{\mathrm{Lag}}(V)$ is dense in $\mathrm{PSH}_{\mathrm{Lag}}(U)$ if $U \subset V$ and $V, U \in \mathcal{F}(M)$.

(d) Each compact set $K \subset M$ is $\mathrm{PSH}_{\mathrm{Lag}}(U)$-convex for each $U \in \mathcal{F}(M)$.

10. Lagrangian boundary convexity

Suppose $\Omega \subset\subset X$ is an open set with smooth boundary in a non-compact Gromov manifold X. The three global conditions in Definitions 9.3, 9.5 and 9.7 can be applied to the domain Ω since it is also a Gromov manifold. In this section we introduce local conditions on its boundary $\partial\Omega$ which are of a companion nature, and we prove a local to global result (Theorem 10.5) in the vein of the Levi problem in complex analysis.

A Lagrangian n-plane W at a point $x \in \partial\Omega$ will be called a **tangential Lagrangian** plane if $W \subset T_x\partial\Omega$. Let $LAG(\partial\Omega)$ denote the set of all such planes at all points of $\partial\Omega$.

Definition 10.1. Suppose that ρ is a **defining function** for $\partial\Omega$, that is, ρ is a smooth function defined on a neighborhood of $\overline{\Omega}$ with $\Omega = \{x : \rho(x) < 0\}$ and $\nabla\rho \neq 0$ on $\partial\Omega$. If

$$\mathrm{tr}_W \mathrm{Hess}\rho \;\geq\; 0 \quad \text{for all } W \in LAG(\partial\Omega), \tag{10.1}$$

then $\partial\Omega$ is called **Lagrangian convex**. If the inequality in (10.1) is strict for all $W \in LAG(\partial\Omega)$, then $\partial\Omega$ is called **strictly Lagrangian convex**. If $\mathrm{tr}_W \mathrm{Hess}\rho = 0$ for all W as in (10.1), then $\partial\Omega$ is **Lagrangian flat**.

Each of these conditions is a local condition on $\partial\Omega$, independent of the choice of ρ.

LEMMA 10.2. *Each of the three conditions in Definition 10.1 is independent of the choice of defining function* ρ. *In fact, if* $\overline{\rho} = f\rho$ *is another choice with* $f > 0$ *on* $\partial\Omega$, *then on* $\partial\Omega$

$$\text{(10.2)} \qquad \mathrm{tr}_W \mathrm{Hess}\overline{\rho} \;=\; f \cdot \mathrm{tr}_W \mathrm{Hess}\rho \quad \text{for all} \quad W \in LAG(\partial\Omega)$$

PROOF. Note that from (7.3) we have

$$\begin{aligned}\{\mathrm{Hess}(f\rho)\}(v,w) \;&=\; vw(f\rho) - (\nabla_v w)(f\rho) \\ &=\; f\mathrm{Hess}(\rho)(v,w) + (vf)(w\rho) + (wf)(v\rho) \\ &\qquad + \rho\mathrm{Hess}(f)(v,w)\end{aligned}$$

Since $\rho = 0$ on $\partial\Omega$ and $v\rho = 0$ for all $v \in T(\partial\Omega)$, the assertion follows. ☐

LEMMA 10.3. *Suppose* ρ *is a smooth real-valued function on* X, *and* $\psi : \mathbf{R} \to \mathbf{R}$ *is smooth on the image of* ρ. *Then*

$$\text{(10.3)} \qquad \mathrm{tr}_W \mathrm{Hess}\psi(\rho) \;=\; \psi'(\rho)\mathrm{tr}_W \mathrm{Hess}\rho + \psi''(\rho)|\nabla\rho \lrcorner W|^2$$

for all oriented tangent p*-planes* W.

PROOF. We first calculate that $\mathrm{Hess}\psi(\rho) = \psi'(\rho)\mathrm{Hess}\rho + \psi''(\rho)\nabla\rho \circ \nabla\rho$ and then note that $\mathrm{tr}_W(\nabla\rho \circ \nabla\rho) = |\nabla\rho \lrcorner v|^2$. ☐

COROLLARY 10.4. *With* $\delta = -\rho$ *and* $\rho < 0$, *one has*

$$\text{(10.4)} \qquad \mathrm{tr}_W \mathrm{Hess}(-\mathrm{log}\delta) \;=\; \frac{1}{\delta}\mathrm{tr}_W \mathrm{Hess}\rho + \frac{1}{\delta^2}|\nabla\rho \lrcorner W|^2$$

PROOF. Take $\psi(t) = -\log(-t)$ for $t < 0$, and note that $\psi'(t) = -1/t$ and $\psi''(t) = 1/t^2$, so that $\psi'(\rho) = 1/\delta$ and $\psi''(\rho) = 1/\delta^2$. ☐

We now come to the main result of this section, going from local to global.

THEOREM 10.5. *Let* $\Omega \subset\subset X$ *be a compact domain with smooth, strictly Lagrangian convex boundary. Suppose* ρ *is an arbitrary defining function for* $\partial\Omega$ *and let* $\delta \equiv -\rho$ *be the corresponding interior "distance function" to* $\partial\Omega$. *Then* $-\log\delta$ *is strictly Lag-psh outside a compact subset of* Ω. *Thus, in particular, the domain* Ω *is strictly Lagrangian convex at infinity.*

PROOF. At each point $x \in \Omega$ near $\partial\Omega$, we have that equation (10.4) holds for all n-planes W. Note that at $x \in \partial\Omega$, $|\nabla\rho \lrcorner W|^2$ vanishes if and only if W is tangential to $\partial\Omega$. For notational convenience we set

$$\cos^2\theta(W) \;=\; \frac{|\nabla\rho \lrcorner W|^2}{|\nabla\rho|^2} \;=\; \langle P_{\mathrm{span}\,\nabla\rho}, P_{\mathrm{span}\,W}\rangle$$

Then the inequality $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $G(p, T\partial\Omega) \subset G(p, TX)$. By restriction $|\cos\theta| < \epsilon$ defines a fundamental neighborhood system for $LAG \cap G(p, T\partial\Omega) \subset LAG$. The hypothesis of strict Lagrangian convexity for $\partial\Omega$ implies that there exists $\overline{\epsilon} > 0$ so that

$\mathrm{tr}_W\mathrm{Hess}\rho \geq \overline{\epsilon}$ for all Lagrangian planes W at points of $\partial\Omega$ with $|\cos\theta| < \epsilon$ for some $\epsilon > 0$. Consequently, we have by equation (10.4) that

$$\mathrm{tr}_W\mathrm{Hess}(-\log\delta) \ \geq \ \frac{\overline{\epsilon}}{2\delta}$$

near $\partial\Omega$ for all Lagrangian planes W with $|\cos\theta| < \epsilon$.

Now choose $M >> 0$ so that $\mathrm{tr}_W\mathrm{Hess}\rho \geq -M$ in a neighborhood of $\partial\Omega$ for all $W \in LAG$. Then, by (10.4)

$$\mathrm{tr}_W\mathrm{Hess}(-\log\delta) \ \geq \ -\frac{M}{\delta} + \frac{1}{\delta^2}|\nabla\rho\lrcorner W|^2.$$

If $|\cos\theta| \geq \epsilon$, this is positive in a neighborhood of $\partial\Omega$ in Ω. This proves that $-\log\delta$ is strictly Lagrangian plurisubharmonic near $\partial\Omega$. By Theorem 9.6 the domain Ω is strictly ϕ-convex at infinity. $\square$

Although a defining function for a strictly Lagrangian convex boundary may not be Lagrangian-plurisubharmonic, we do have the following.

Proposition 10.6. *Suppose $\Omega \subset\subset X$ has strictly Lagrangian convex boundary $\partial\Omega$ with defining function ρ. Then, for A sufficiently large, the function $\overline{\rho} \equiv \rho + A\rho^2$ is strictly Lag-psh in a neighborhood of $\partial\Omega$ and also a defining function for $\partial\Omega$.*

Proof. By Lemma 10.3 we have

$$\mathrm{tr}_W(\mathrm{Hess}\overline{\rho}) \ = \ (1 + 2A\rho)\mathrm{tr}_W(\mathrm{Hess}\rho) + 2A|\nabla\rho\lrcorner W|^2. \tag{10.5}$$

As noted in the proof of Theorem 10.5, there exist $\epsilon, \overline{\epsilon} > 0$ so that $\mathrm{tr}_W(\mathrm{Hess}\rho) \geq \overline{\epsilon}$ for $W \in LAG$ with $|\cos\theta(W)| < \epsilon$, by the strict boundary convexity. Therefore $\mathrm{tr}_W(\mathrm{Hess}\overline{\rho}) \geq (1+2A\rho)\overline{\epsilon}$ if $W \in LAG$ with $|\cos\theta(W)| < \epsilon$. Choose a lower bound $-M$ for $\mathrm{tr}_x(\mathrm{Hess}\rho)$ over all $W \in LAG$ for a neighborhood of $\partial\Omega$. Then by (10.5), $\mathrm{tr}_W(\mathrm{Hess}\rho) \geq -(1 + 2A\rho)M + 2|\nabla\rho|^2A\epsilon^2$ for $W \in LAG$ with $|\cos\theta(W)| \geq \epsilon$. For A sufficiently large, the right hand side is > 0 in some neighborhood of $\partial\Omega$. $\square$

This leads to the following, which will be useful in the next section.

Theorem 10.7. *Suppose $\Omega \subset\subset X$ has strictly Lagrangian convex boundary and that there exists a smooth strictly Lag-psh function u on $\overline{\Omega}$. Then Ω admits a strictly Lag-psh defining function.*

Note that in the case $\Omega \subset\subset X \equiv \mathbf{C}^n$ there many strictly Lag-psh functions on X.

Proof. By Proposition 10.6 there exists a smooth Lag-psh function ρ defined on a neighborhood of $\partial\Omega$. For $\delta_1 >> \delta_2 > 0$ sufficiently small, the function $\max\{\rho, -\delta_1+\delta_2 u\}$ is Lag-psh and equal to ρ near $\partial\Omega$. We now apply the maximum smoothing (see pages 373-4 in [**HL**$_3$]) to obtain the desired defining function. $\square$

The (strict) Lagrangian convexity of a boundary can be equivalently defined in terms of its second fundamental form. Note that if $Y \subset X$ is a smooth hypersurface with a chosen unit normal field n we have the associated **second fundamental form** II defined on TY by

$$II(v, w) \ \equiv \ \langle \nabla_v \widetilde{w}, n\rangle$$

where $\widetilde{w}$ is any extension of w to a tangent vector field on Y. For example, when $H = S^{n-1}(r) \subset \mathbf{R}^n$ is the euclidean sphere of radius r, oriented by the outward-pointing unit normal, we find that $II(V, W) = -\frac{1}{r}\langle V, W\rangle$.

Recall the following standard fact (cf. [**HL**$_1$, Lemma 5.11]).

LEMMA 10.8. *Suppose ρ is a defining function for Ω and let II denote the second fundamental form of the hypersurface $\partial\Omega$ oriented by the outward-pointing normal. Then*

$$\mathrm{Hess}\,\rho\,\big|_{T\partial\Omega} \ = \ -|\nabla\rho|\, II$$

and therefore

$$\mathrm{tr}_W \mathrm{Hess}\rho \ \ = \ \ -|\nabla\rho|\, \mathrm{tr}_W II$$

for all $W \in G(n, T\partial\Omega)$ and in particular for all $W \in LAG \cap G(n, T\partial\Omega)$.

REMARK. Recall that a defining function ρ for Ω satisfies $|\nabla\rho| \equiv 1$ in a neighborhood of $\partial\Omega$ if and only if ρ is the signed distance to $\partial\Omega$ (< 0 in Ω and > 0 outside of Ω). In fact any function ρ with $|\nabla\rho| \equiv 1$ in a riemannian manifold is, up to an additive constant, the distance function to (any) one of its level sets. In this case the lemma implies that

$$\mathrm{Hess}\rho \ = \ \begin{pmatrix} 0 & 0 \\ 0 & -II \end{pmatrix} \tag{10.6}$$

where II denotes the second fundamental form of the hypersurface $H = \{\rho = \rho(x)\}$ with respect to the normal $n = \nabla\rho$ and the blocking in (10.6) is with respect to the splitting $T_x X = \mathrm{span}\,(n_x) \oplus T_x H$.

As an immediate consequence of Lemma 10.8 we have

PROPOSITION 10.9. *Let $\Omega \subset X$ be a domain with smooth boundary $\partial\Omega$ oriented by the outward-pointing normal. Then $\partial\Omega$ is Lagrangian convex if and only if its second fundamental form satisfies*

$$\mathrm{tr}_W II \ \le \ 0 \tag{10.7}$$

for all Lagrangian planes W which are tangent to $\partial\Omega$, i.e., for all $W \in LAG(\partial\Omega)$. This can be expressed more geometrically by saying that

$$\mathrm{tr}_W B \quad \text{must be inward} - \text{pointing}$$

for all $W \in LAG(\partial\Omega)$, where $B_{v,w} \equiv (\nabla_v \widetilde{w})^{Nor}$ is the normal-vector valued second fundamental form.

Furthermore, $\partial\Omega$ is strictly Lagrangian convex if and only if

$$\mathrm{tr}_W II \ < \ 0 \qquad \forall\, W \in LAG(\partial\Omega), \tag{10.8}$$

or equivalently, $\mathrm{tr}_W B$ is non-zero and inward-pointing for all $W \in LAG(\partial\Omega)$.

Remark 10.10. If ρ is the signed distance to $\partial\Omega$, then equation (10.6) together with Lemma 10.8 can be used to simplify (10.4). An arbitrary n-plane W at a point can be put in the canonical form $W = (\cos\theta n + \sin\theta e_1) \wedge e_2 \wedge \cdots \wedge e_n$ with $n = \nabla\rho$ and $n, e_1, ..., e_n$ orthonormal. Then $\eta = e_1 \wedge \cdots \wedge e_p$ is the tangential projection of W. Note that $\mathrm{tr}_W \mathrm{Hess}\rho = -\sin^2\theta \mathrm{tr}_\eta II$ and that $|\nabla\rho \lrcorner W|^2 = \cos^2\theta$, so that (10.4) becomes

$$\mathrm{tr}_W \mathrm{Hess}(-\mathrm{log}\rho) \;=\; -\frac{1}{\delta}\sin^2\theta\, \mathrm{tr}_\eta II + \frac{1}{\delta^2}\cos^2\theta$$

Let n denote the outward unit normal field to $\partial\Omega$. Then at each point $x \in \partial\Omega$ we have an orthogonal decomposition

$$T_x X \;=\; \mathbf{R}n \oplus T_x\partial\Omega \;=\; \mathbf{R}n \oplus \mathbf{R}Jn \oplus \mathcal{H}$$

where $\mathcal{H} = T_x(\partial\Omega) \cap JT_x(\partial\Omega)$ is the (unique) maximal complex subspace in $T_x(\partial\Omega)$.

Proposition 10.11. *Let $W \subset T_x X$ be any Lagrangian n-plane at a point $x \in \partial\Omega$. Then W is of the form*

$$W \;=\; (n\cos\theta + Jn\cos\theta) \wedge W_0$$

where W_0 represents a Lagrangian $(n-1)$-plane in the complex subspace $\mathcal{H}$. In particular, every tangential Lagrangian n-plane is of the form

$$W \;=\; Jn \wedge W_0$$

Proof. Put W in canonical form $W = (\cos\theta n + \sin\theta e_1) \wedge e_2 \wedge \cdots \wedge e_n$ as above with $n, e_1, ..., e_n$ orthonormal. Set $e(\theta) = \cos\theta n + \sin\theta e_1$. If W is Lagrangian, then $e(\theta), Je(\theta), e_2, Je_2, ..., e_n, Je_n$ form an orthonormal basis of $\mathbf{R}^{2n}$. In particular, $e_2, Je_2, ..., e_n, Je_n$ are perpendicular to span $\{e(\theta), Je(\theta)\}$ = span $\{n, e_1\}$. Hence, span $\{n, e_1\}$ is a J-invariant. We conclude that $e_1 = \pm Jn$ and $\mathcal{H} = \mathrm{span}\,\{e_2, Je_2, ..., e_n, Je_n\}$. $\square$

Combined with Proposition 10.9 we conclude the following, which provides another way of describing Lagrangian boundary convexity.

Corollary 10.12. *Let II be the second fundamental form of $\partial\Omega$ with respect to the outer unit normal field n as above. Then $\partial\Omega$ is Lagrangian convex if and only if*

$$II(Jn, Jn) + \mathrm{tr}_{W_0} II \;\leq\; 0 \tag{10.9}$$

for all Lagrangian $(n-1)$-planes W_0 in the holomorphic tangent space $\mathcal{H}$. Furthermore, $\partial\Omega$ is strictly Lagrangian convex if and only if the inequality in (10.9) is strict for all W_0.

11. The Dirichlet problem for the Lagrangian Monge-Ampére equation

It is an important fact that Lagrangian harmonic functions exist in abundance locally on any Gromov manifold X. In fact the Dirichlet problem can always be solved on any domain $\Omega \subset\subset X$ with smooth boundary such that

(11.1)
Ω *admits a strictly Lagrangian plurisubharmonic defining function.*

By Theorem 10.7 this is equivalent to the fact that

$$(11.1') \qquad \begin{array}{l} \partial\Omega \ \textit{is strictly Lagrangian convex and there exists a smooth} \\ \textit{strictly Lagrangian plurisubharmonic function on } \ \overline{\Omega}. \end{array}$$

We make this assumption in each of the following theorems. Note that all sufficiently small metric balls about any point have this property, and also that any domain $\Omega \subset \mathbf{C}^n$ admits a smooth strictly Lag-psh function.

THEOREM 11.1 (The Homogeneous Dirichlet Problem). *For any continuous $\varphi \in C(\partial\Omega)$ there exists a unique function $u \in C(\overline{\Omega})$ such that*

(1) $u\big|_{\Omega}$ *is Lagrangian harmonic on Ω, and*

(2) $u\big|_{\partial\Omega} = \varphi$.

PROOF. This is a special case of Theorem 16.1 with $\mathbb{G} = \mathrm{LAG}$ in [**HL**$_4$]. (It follows as well from the more general Theorem 13.1 in [**HL**$_4$].) □

We can also treat the inhomogeneous equation for the Lagrangian Monge-Ampére operator:

$$\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) \ = \ \psi, \qquad u \ \ \mathrm{Lag}-\mathrm{psh}$$

with continuous inhomogeneous term ψ. Existence and uniqueness are easier when $\psi > 0$ and smooth, and we outline below the proof given in [**HL**$_4$]. The case where $\psi \geq 0$, that is, where $\mathrm{Hess}\, u$ is allowed to hit the boundary of $\mathcal{P}(\mathrm{LAG})$, and ψ is continuous, is more complicated, and we shall refer to [**HL**$_{13}$] for that case.

THEOREM 11.2 (The Inhomogeneous Dirichlet Problem). *Fix a continuous $\psi \geq 0$ on $\overline{\Omega}$. Then for any $\varphi \in C(\partial\Omega)$ there exists a unique function $u \in C(\overline{\Omega})$, which is Lagrangian plurisubharmonic on Ω, such that*

(1) $u\big|_{\Omega}$ *is the viscosity solution of* $\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) = \psi$ *on Ω, and*

(2) $u\big|_{\partial\Omega} = \varphi$.

PROOF. We shall first consider smooth $\psi > 0$. Note to begin that

$$\{A \in \mathcal{P}(\mathrm{LAG}) : \mathrm{M}_{\mathrm{Lag}}(A) \geq 1\}$$

is unitarily invariant. As a result this "universal" subequation determines the subequation $\mathrm{M}_{\mathrm{Lag}}(\mathrm{Hess}\, u) \geq 1$ on any Gromov manifold (see Chapter

5 in [**HL**$_4$].) Theorem 13.1 in [**HL**$_4$] then applies to give the version of Theorem 11.1 for the equation $\mathrm{M_{Lag}}(\mathrm{Hess}\, u) = 1$.

Now by conjugating $\mathrm{Sym}^2(T^*X)$ by

$$A \quad \mapsto \quad \left(\psi^{-2^{n/2}}\right) A \left(\psi^{-2^{n/2}}\right)^t = \psi^{-2^n} A,$$

we get a jet-equivalence of our equation $\mathrm{M_{Lag}}(\mathrm{Hess}\, u) = \psi$ with the equation $\mathrm{M_{Lag}}(\mathrm{Hess}\, u) = 1$. We can then apply Theorem 13.1′ in [**HL**$_4$].

For continuous $\psi \geq 0$ the reader is referred to [**HL**$_{13}$, Thm. 2.11]. See also Thm. 6.8 in Example 6.7. ☐

We now turn attention to the branches $\mathcal{P}_k(\mathrm{LAG})$ of $\mathrm{M_{Lag}}$ which determine subequations on any Gromov manifold X (see Remark 8.4 and Note 5.9).

Theorem 11.3 (The homogeneous Dirichlet problem for the branches). *Fix a continuous function $\varphi \in C(\partial\Omega)$ and any $k = 1, ..., 2^n$. Then there exists a unique function $u \in C(\overline{\Omega})$ such that*

(1) $u\big|_\Omega$ *is $\mathcal{P}_k(\mathrm{LAG})$-harmonic on Ω, and*

(2) $u\big|_{\partial\Omega} = \varphi$.

Proof. This is Theorem 13.1 in [**HL**$_4$]. We point our that $\mathcal{P}_k(\mathrm{LAG})$ is a riemannian $\mathrm{U}(n)$-subequation on X. Furthermore, the boundary is strictly $\mathcal{P}(\mathrm{LAG})$-convex, which implies strict $\mathcal{P}_k(\mathrm{LAG})$-convexity and strict dual $\mathcal{P}_k(\mathrm{LAG})$-convexity (= strict $\mathcal{P}_{2^n-k}(\mathrm{LAG})$-convexity) since $\mathcal{P}(\mathrm{LAG}) = \mathcal{P}_1(\mathrm{LAG}) \subset \mathcal{P}_k(\mathrm{LAG})$ for all k. So the boundary condition is satisfied. ☐

Note 11.4. Theorem 11.3 remains true under the weaker natural boundary convexity condition, (discussed in the proof above) that:

There exists a strictly $\mathcal{P}_\ell(\mathrm{LAG})$-subharmonic defining function for $\partial\Omega$ where $\ell = \min\{k, 2^n - k\}$.

This is equivalent (as above for $k = 1$) to $\partial\Omega$ being strictly $\mathcal{P}_\ell(\mathrm{LAG})$-convex and the existence of a strictly $\mathcal{P}_\ell(\mathrm{LAG})$-subharmonic function on $\overline{\Omega}$.

Note 11.5. There is also an operator $\mathrm{M}_{\mathrm{Lag},k} \equiv \Lambda_k \Lambda_{k+1} \cdots \Lambda_{2^n}$ on the subequation $\mathcal{P}_k(\mathrm{LAG})$, where $\Lambda_1 \leq \Lambda_2 \leq \cdots$ are the ordered eigenvalues of $\mathrm{M_{Lag}} = \mathrm{M}_{\mathrm{Lag},1}$ (see the end of Section 5), although it is not a polynomial operator unless $k = 1$. This operator is discussed, for example, in [**HL**$_{13}$]; see in particular, (6.3) and Prop. 6.11 in [**HL**$_{13}$]. The inhomogeneous Dirichlet Problem for $\mathrm{M}_{\mathrm{Lag},k}$ can be solved uniquely for any inhomogeneous term $\psi \geq 0$ under the boundary assumptions in Note 11.4, again by the Theorem 2.11 in [**HL**$_{13}$].

12. Ellipticity of the linearization

It is natural to consider the linearization of the operator $\mathrm{M_{Lag}}(\mathrm{Hess}\, u)$ on compact subsets of the interior of its domain, i.e., on compact subsets of $\mathrm{Int}\mathcal{P}(\mathrm{LAG})$. Of course for any weakly elliptic operator f (one where $f(A+P) \geq f(A)$ for $P \geq 0$ and A in its domain), the linearization $L(f)$ is weakly elliptic; and if f is uniformly elliptic, so is $L(f)$. However, there is a (not particular trivial) result that for operators defined by Gårding/Dirichlet polynomials, such as $\mathrm{M_{Lag}}$, the linearization at all interior points of the subequation is always positive definite. Details of this can be found in [**HL**$_{13}$] and [**HL**$_7$].

PROPOSITION 12.1. *Let $\Omega \subset X$ be a compact domain, and assume that u is a C^2 LAG-plurisubharmonic function on $\overline{\Omega}$ which satisfies the equation*

$$\mathrm{M_{Lag}}(\mathrm{Hess} u) \;=\; f \;>\; 0$$

on $\overline{\Omega}$. Then the linearization of $\mathrm{M_{Lag}}$ at this solution is uniformly elliptic.

This allows one to use Implicit Function Techniques to get smooth solutions for nearby boundary data.

Appendix A.

A more detailed presentation of the Lagrangian subequation. A more detailed algebraic discussion of the subequation $\mathcal{P}(\mathrm{LAG})$ is presented here. It includes a description of the extreme rays. The material in this appendix is self-contained and includes a second treatment of the results of Section 3.

First, for the reader's convenience, we summarize results that hold for any geometric subequation (see ... for more details). Start with any closed subset $\mathbb{G}$ (say $\mathbb{G} = \mathrm{LAG}$) of the Grassmannian $G(p, \mathbf{R}^N)$ of unoriented p-planes in $\mathbf{R}^N$. Identify $W \in \mathbb{G}$ with orthogonal projection $P_W \in \mathrm{Sym}^2(\mathbf{R}^N)$ onto W. We have the following concepts.

(A.1) **(The Subequation $\mathcal{P}^+$)** $A \in \mathcal{P}^+ \iff \mathrm{tr}\left(A\big|_W\right) = \langle A, P_W\rangle \geq 0 \;\; \forall\, W \in \mathbb{G}$.

(A.2) **(The Convex-Cone Hull $\mathcal{P}_+$)** $\mathcal{P}_+ \equiv$ the convex-cone hull of $\mathbb{G}$, denoted $\mathrm{CCH}(\mathbb{G})$.

(A.3) **(Polars)** $\mathcal{P}^+$ and $\mathcal{P}_+$ are polar cones in $\mathrm{Sym}^2(\mathbf{R}^N)$.

(A.4) **(The Edge E)** $E \equiv \mathcal{P}^+ \cap (-\mathcal{P}^+)$, that is,

$$A \in E \iff \mathrm{tr}\left(A\big|_W\right) = \langle A, P_W\rangle = 0 \;\; \forall\, W \in \mathbb{G}.$$

(A.5) **(The Span S)** $S \equiv \mathrm{span}\, \mathbb{G} \equiv \mathrm{span}\, \mathcal{P}_+$.

(A.6) **($E = S^\perp$)** $\mathrm{Sym}^2(\mathbf{R}^N) = E \oplus S$ is an orthogonal decomposition,

(A.7) **(Extreme Rays in $\mathcal{P}_+ \equiv \mathrm{CCH}(\mathbb{G})$)** The extreme rays in $\mathcal{P}_+$ are the rays through

P_W where $W \in \mathbb{G}$.

Note that (A.1), (A.2) and (A.5) are definitions, while (A.4) is a definition combined with an immediate consequence of (A.1). The polar facts (A.3) and(A.6) are easy. For (A.7) note that from the definition of $\mathcal{P}_+$ every extreme ray is generated by a P_W. On the other hand, all of the elements P_W lie on a sphere in the hyperplane $\{\mathrm{tr}A = p\}$, centered about $\frac{p}{n}\mathrm{I}$, and it then follows that every P_W is extreme.

The more interesting geometric cases tend to have a non-trivial edge. The edge E is a vector subspace of $\mathcal{P}(\mathbb{G})$ and it contains all other vector subspaces of $\mathcal{P}(\mathbb{G})$. It can be ignored in the following sense. Let $\pi : \mathrm{Sym}^2(\mathbf{R}^N) \to S$ denote orthogonal projection.

(A.8a) **(The Reduced Subequation $\mathcal{P}_0^\perp$)** $\mathcal{P}^+ = E \oplus \mathcal{P}_0^+$ defines $\mathcal{P}_0^+$, and

(A.8b) $\mathcal{P}_0^+ = \pi(\mathcal{P}^+) = \mathcal{P}^+ \cap S = \mathcal{P}_+^0$ (= the polar of $\mathcal{P}_+$ in S).

Now we give an explicit description of each of the objects above in the case at hand, namely $\mathbb{G} = \mathrm{LAG}$. We use the standard fact that the action of the unitary group on $[\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)$ has a cross-section D with each orbit intersecting D in a finite number of points.

(A.9) $D \equiv \left\{ H(t,\lambda) \equiv \frac{t}{2n}\mathrm{I} + \sum_j \lambda_j (P_{e_j} - P_{Je_j}) : (t,\lambda) \in \mathbf{R}^{n+1} \right\}$.

(The label D is used since as a $2n \times 2n$-matrix, each element of D is diagonal.)

The **Lagrangian part** of $A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ is defined to be

(A.10a) $A^{\mathrm{LAG}} \equiv \frac{1}{2n}(\mathrm{tr}A)\mathrm{I} + \frac{1}{2}(A + JAJ) \equiv \pi(A)$,

and the **skew-hermitian part** of A is

(A.10b) $A^{\mathrm{skew}} \equiv \frac{1}{2}(A + JAJ)$.

Let $W(\varepsilon) = W(\pm 1, ..., \pm 1) \equiv W(\varepsilon_1, ..., \varepsilon_n)$ be the axis Lagrangian n-plane defined by the condition that $e_j \in W(\varepsilon)$ if $\varepsilon_j = +1$ and $Je_j \in W(\varepsilon)$ if $\varepsilon_j = -1$. Then

(A.11) $P_{W(\varepsilon)} = \frac{1}{2}\mathrm{I} + \frac{1}{2}\sum_j \varepsilon_j (P_{e_j} - P_{Je_j})$ In particular, these 2^n axis Lagrangian planes $P_{W(\varepsilon)} \in D$ (with $t = n$ and $\lambda = \frac{\varepsilon}{2}$) comprise the vertices of the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in the $t = n$ hyperplane.

It is easy to see that $\mathbb{G} \cap D = \mathrm{LAG} \cap D$ consists of the 2^n points $P_{W(\varepsilon)}$. This proves

(A.12a) $\mathrm{span}\, G \cap D = D$, and hence

(A.12b) $S \equiv \mathrm{span}\, \mathbb{G} = [\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)$.

Thus, all of the facts in (A.1) through (A.8) are true with the edge E of the subequation $\mathcal{P}^+$ identified as

(A.13) $E = \mathrm{Herm}_0^{\mathrm{sym}}(\mathbf{C}^n)$, the space of traceless hermitian-symmetric forms, since this space equals $S^\perp = ([\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n))^\perp$.

Since the 2^n points $P_{W(\varepsilon)}$ are the vertices of a cube in the affine hyperplane $\{t = n\} \cap D$ in D, the convex hull, which is $\mathcal{P}_+ \cap D$, equals the cone on this cube. Consequently, by (A.11),

(A.14a) $\mathcal{P}_+ \cap D = \{H(t, \lambda) : \sup_j |\lambda_j| \leq \frac{t}{2n}\}$ with $P_{W(\varepsilon)} = H(n, \frac{\varepsilon}{2})$, so that

(A.14b) $P_{W(\varepsilon)} \in \partial(\mathcal{P}_+ \cap D)$ generate the extreme rays.

The polar of this cone $\mathcal{P}_+$ in D, which equals $\mathcal{P}^+ \cap D$, is now easy to compute. First compute that

(A.15) $\langle H(t, \lambda), P_{W(\varepsilon)} \rangle \;=\; \frac{t}{2} + \sum_j \varepsilon_j \lambda_j.$

This term is $\geq 0 \;\; \forall \;\; \varepsilon = (\pm 1, ..., \pm 1)$ $\iff$ $|\lambda_1| + \cdots + |\lambda_n| \leq \frac{t}{2}$. Therefore,

(A.16a) $\mathcal{P}^+ \cap D \;=\; \{H(t, \lambda) : |\lambda_1| + \cdots + |\lambda_n| \leq \frac{t}{2}\}.$

This proves that the subequation $\mathcal{P}^+$ is given by

(A.16b) $\mathcal{P}^+ = \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \frac{\mathrm{tr}A}{2} - \lambda_1^+ - \cdots - {}_n^+ \geq 0\}$, where $\lambda_1^+, ..., \lambda_n^+$ are the non-negative eigenvalues of A^{skew}.

Setting $t = 1$ in (A.16a), we see that $\mathcal{P}^+ \cap D$ is the cone on the set $|\lambda_1| + \cdots + |\lambda_n| \leq \frac{1}{2}$. Hence, its extreme rays are through $H(1, \frac{\alpha}{2})$ where α is one of the $2n$ unit vectors on an axis line in $\mathbf{R}^n$. That is:

(A.17a) The elements $H(1, \frac{\alpha}{2}) \equiv \frac{1}{2n}\mathrm{I} \pm \frac{1}{2}(P_{e_j} - P_{Je_j})$ (for some $\pm$ and $j = 1, ..., n$), generate the extreme rays in $\mathcal{P}^+ \cap D$.

On the other hand,

(A.17b) $P^{\mathrm{LAG}}_{e_j} \;=\; \frac{1}{2n}\mathrm{I} + \frac{1}{2}(P_{e_j} - P_{Je_j})$ and $P^{\mathrm{LAG}}_{Je_j} \;=\; \frac{1}{2n}\mathrm{I} - \frac{1}{2}(P_{e_j} - P_{Je_j}).$

This proves the following.

Theorem A.1. *The extreme rays in the reduced subequation $\mathcal{P}^+_0$ are through P^{LAG}_e, $e \neq 0$*

This has important special consequences for $\mathbb{G} \equiv \mathrm{LAG}$.

Proposition A.2. *Let $\mathbb{G} \equiv \mathrm{LAG}$. Then, regarding $\mathcal{P}^+$, one has:*

(1) $\mathcal{P}^+_0 \;=\; \pi(\mathcal{P}), \quad \mathrm{Int}\mathcal{P}^+_0 = \pi(\mathrm{Int}\mathcal{P}),$

(2) $\mathcal{P}^+ \;=\; E + \mathcal{P} \;\; = \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n) + \mathcal{P},$

(3) $\mathrm{Int}\mathcal{P}^+ \;=\; E + \mathrm{Int}\mathcal{P} \;\; = \mathrm{Herm}^{\mathrm{sym}}_0(\mathbf{C}^n) + \mathrm{Int}\mathcal{P}.$

While for $\mathcal{P}_+$ one has

(4) $\mathcal{P}_+ \;=\; \mathcal{P} \cap S \;\; = \mathcal{P} \cap ([\mathrm{I}] \oplus \mathrm{Herm}^{\mathrm{skew}}(\mathbf{C}^n)),$ and $\mathrm{Int}_{\mathrm{rel}}\mathcal{P}_+ = (\mathrm{Int}\mathcal{P}) \cap S.$

PROOF. Assertion (1) is immediate, and assertions (2) and (3) follow. Since $(\mathcal{P} \cap S)^0 = \mathcal{P}^0 + S^0 = \mathcal{P} + E$, which equals $\mathcal{P}^+$ by (2), this proves (4). For a second, more direct proof of (4) note that $H(t, \lambda) \geq 0 \iff |\lambda_j| \leq \frac{t}{2}, j = 1, .., n \iff H(t, \lambda) \in \mathcal{P}_+ \cap D$ by the definition of $H(t, \lambda)$ and (A.14) respectively. $\square$

REMARK A.3. By contrast to Proposition A.2, in the cases $\mathbb{G} = G_K(1, K^n) \subset \mathrm{Sym}^2_{\mathbf{R}}(K^n)$ for $K = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, the convex-cone hull $\mathcal{P}_+ \equiv \mathrm{CCH}(\mathbb{G})$ and the reduced subequation $\mathcal{P}^+_0 = \mathcal{P}^+ \cap S$ are self-polar in $S \equiv \mathrm{span}\, \mathbb{G}$.

Note that the edge E generates $\mathcal{P}^+$ as a subequations, i.e.,

(2) $\mathcal{P}^+ = E + \mathcal{P}$ in all four cases $\mathbb{G} = \mathrm{LAG}$ and $\mathbb{G} = G_K(1, K^n)$ for $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

REMARK A.4. For any subspace $E \subset \mathrm{Sym}^2(\mathbf{R}^n)$ with $E \cap \mathcal{P} = \{0\}$, the sum $E + \mathcal{P}$ is closed and hence a subequation. In fact, it has edge E, satisfies Theorem A.1, and enjoys all the properties in Proposition A.2. All of this is proven in [**HL**$_{14}$].

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