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Jim Simons is lecturing at IHÉS during the conference
for Jean-Pierre Bourguignon 60th birthday (2007)
with Jeff Cheeger, Mikhail Gromov and Blaine Lawson in the front row.

JAMES SIMONS
(1938–2024)

JEFF CHEEGER AND H. BLAINE LAWSON JR.

Jim Simons was in that order, a mathematician, a revolutionary in quantitative finance and a philanthropist whose support for fundamental research was unparalleled. He had three absolutely remarkable careers, three and a half if one counts a significant (apparently still classified) accomplishment during the relatively brief time that he worked as a code breaker at the Institute for Defense Analyses. In this memoir we will discuss Jim's mathematical work and it influence. As is well known, a portion of that work has had an extraordinary (totally unexpected) impact on modern physics.[1]

For both of the present authors, interactions with Jim at very early stages of our careers were of the utmost importance. This will be recounted in due course below.

Jim entered MIT in 1955 the age of 17. Already as an undergraduate, he was strongly influenced by Warren Ambrose and Is Singer. He graduated after three years and while still at MIT, began graduate work in differential geometry. After two years he moved to Berkeley, hoping to work with S-S. Chern, who, as turned out, was on leave that year. Undeterred, Jim finished his thesis in 1962 under Bertram Kostant. Kostant had mentioned to him the problem which became the subject of the thesis, but had advised him not to try it because it was likely to be too difficult. The problem concerned the holonomy groups of riemannian manifolds. The possible candidates for these Lie groups had been classified by Marcel Berger, although at the time, it was not known that all of the groups on Berger's list actually occurred. A case by case analysis had revealed that if the holonomy group acted irreducibly, then either it acted transitively on the unit sphere or the underlying riemannian manifold was a locally symmetric space of rank ≥ 2. The problem which Kostant mentioned was to find a uniform purely algebraic proof which exhibited the underlying reason for the transitivity of the action. Jim's thesis provided such a proof. The resulting paper, "On the transitivity of holonomy systems", was published in the Annals of Mathematics in 1962; [Sim, 1962].

Jim returned to MIT as a Moore Instructor in 1962. In 1963 he became an Assistant Professor at Harvard. There he taught a course on PDE out of Lars Hormander's book "Linear'Partial Differential Operators". One of us, Jeff Cheeger, an undergraduate, was a student in that course. As Jim who was teaching the course with the goal of learning the subject liked to recount (and Cheeger can attest) he was able to stay about a week ahead of the class. In the fall of 1964, both Cheeger and Simons found themselves in Princeton. Jim, who already was married with two children, had decided to work as a code breaker at the Institute for Defense Analyses (IDA). Cheeger was now graduate student at the university. He had gone to Princeton with the intention of specializing in topology, but by early 1965 had decided that instead, he wanted to give

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[1] A Memorial Collection, in the Notices of the American Mathematical Society (AMS) deals with the nonmathematical aspects of Jim's life and careers; https://www.ams.org/notices/202501/rnoti-p32.pdf. An extended discussion of Jim's mathematical work and its impact, will appear in an issue Bulletin of the AMS to be published in 2026 or early 2027. Included are articles by mathematicians and physicists. The reader may also enjoy the interview of the first author, https://www.simonsfoundation.org/2024/05/14/jim-simons-reflects-on-his-career-in-mathematics/, in Science Lives.

differential geometry a try. Since (luckily!) nobody on the Princeton faculty was currently working in that subject, he contacted Jim and asked if he could help him to learn differential geometry. Jim said "Sure." Cheeger's formal advisor at Princeton was Salomon Bochner who had done fundamental work in riemannian geometry about 15 years earlier, but his real teacher was Jim. After about a year, things had progressed to the point where Jim suggested a thesis problem. It was to turn what was known as Ambrose's isometry theorem into a comparison theorem. After Cheeger solved the problem, he realized that it suggested something very different, a finiteness theorem for manifolds with bounded sectional curvature, bounded diameter and volume bounded below. Jim was surprised and enthusiastic. To Cheeger's relief, Bochner was enthusiastic as well.

Minimal Varieties. Jim's first international splash was created by his work on minimal varieties which came to fruition in 1967. It had begun in 1962 after the completion of his PhD, and continued over the next 5 years including the time when he was working at IDA. A feature of working at IDA was that you could spend half of the time on your own research as long as you spent the other half on their work.

Jim was convinced that minimal submanifolds of general dimension could be used as a tool for research in Riemannian geometry, much like those of dimension one, the geodesics. This, however, would require an adequate understanding of their regularity properties.

To say that a submanifold of a Riemannian manifold is minimal means that it is a critical point of the volume function. Since the first variation of the volume function is given by the mean curvature (the trace of the second fundamental form) being minimal is equivalent to the vanishing of mean curvature. Jim began a systematic investigation of the subject. At certain point, he derived a fundamental second order nonlinear elliptic equation for the the second fundamental form, now known as the Simons equation, and gave some nice applications to isolation theorems for minimal submanifolds of the sphere. Eventually, this led to conversations with Fred Almgren at Princeton University.

Almgren told Jim about recent work on minimizing currents in codimension 1. There were several mathematicians who had proved the existence of solutions to the Plateau problem in a "weak" context. Herbert Federer and Wendell Fleming wrote a classic paper which gave solutions in all dimensions in the general context of Riemannian manifolds. These were achieved in the world of rectifiable currents, which have many good properties but are far from being n-dimensional submanifolds. In codimension 1, there was also the work of Ennio De Giorgi whose solutions were submanifolds outside a closed subset of n-dimensional Hausdorff measure zero.[2] However, for 2-dimensional minimizers in $\mathbb{R}^3$, or more generally, in Riemannian 3-manifolds, Fleming proved that all of the above weak solutions were actually embedded surfaces.[3] His procedure used tangent cones. In general, these are again mass minimizing. Furthermore, outside the origin, the tangent cone to the tangent cone splits as a line times a minimizing object of one lower dimension. Moreover, when the tangent cone at a point x is a linear subspace, the minimizing object is a submanifold in a neighborhood of x. Using tangent cones to a minimizing cone C of dimension 2 in $\mathbb{R}^3$, Fleming showed that C was a linear space and so, regularity holds. Almgren did the same thing for minimizing 3 dimensional cones in dimension 4. Here the proof was much more subtle. He first showed that the link of the cone was

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[2] By A different method, Reifenberg had treated the unoriented case.This was relatively easier since multiplicities are essentially 1. (One uses $\mathbb{Z}_2$-coefficients and not integers as in the oriented case.)

[3] Reifenberg had already proved regularity in the unoriented case.

a 2-sphere, and then he proved that any minimal 2-sphere in the Euclidean 3-sphere is totally geodesic, that is, equatorial. Almgren told Simons that what one wanted was the following. Let $D^n \subset \mathbb{R}^n$ denote the unit ball and let $C \subset \mathbb{R}^n$ be a cone on a codimension 1 minimal submanifold $X^{n-2} \subset S^{n-1}$, such that $C \cap D^n$ is absolutely volume-minimizing for its boundary $C \cap S^{n-1}$. Then $C \cap S^{n-1}$ is a totally geodesic sphere. Equivalently, C is a linear subspace $\mathbb{R}^{n-1} \subset \mathbb{R}^n$.

Being volume minimizing implies *stability*. By definition, stability means that the second variation of volume, a quadratic form which can be expressed in geometric terms, is ≥ 0 (as in the second derivative test in elementary calculus). Jim went to work on what Almgren had told him. By bringing in the Simons equation, in relatively short order he was able to show for $n < 7$, that if $C \cap S^{n-1}$ is not a totally geodesic sphere, then $C \cap D^n$ is not stable, and in particular not volume minimizing. As a consequence, interior regularity holds when the the ambient dimension satisfies $n < 7$. Furthermore, Jim's result established the Bernstein Conjecture for ambient dimension $n \leq 8$. This conjecture states that if the graph of a function, $u : \mathbb{R}^n \to \mathbb{R}$, is a minimal hypersurface in $\mathbb{R}^{n+1}$, then the graph is an affine subspace.[4]

Jim's instability proof failed in ambient dimension 8 for the following reason. His computation of the second variation of the volume contained a dimension dependent term which for ambient dimension ≥ 8 had the wrong sign. In fact, he showed that in $\mathbb{R}^8$ the cone on the Clifford torus, $S^3 \times S^3 \subset S^7$, is stable! This led him to conjecture that it might, in fact, be minimizing.

Almost immediately, Enrico Bombieri, Ennio De Georgi, and Enrico Guisti showed (after what must have been some intensive work) that this cone is absolutely volume minimizing; [BDGG, 1968]. Thus, for ambient dimension ≥ 8, interior regularity fails to hold. Their paper also showed that the Bernstein conjecture fails for abient dimension ≥ 9. The existence of two major results in geometric analysis which only held up to a certain specific dimension was unprecedented. Jim's paper, "Minimal varieties in Riemannian manifolds", appeared in the Annals in 1968. It received international acclaim. His original vision concerning a central role of minimal submanifolds in riemannian geometry had been vindicated.

Getting fired from IDA and becoming Chair at Stony Brook. Jim's tenure at IDA came to an abrupt end. He was opposed to the war in Vietnam and had told a reporter of the following plan. Until the war ended he would only do his own mathematical work. When the war ended, he would make up the work he owed to IDA of which he was scrupulously keeping track. His boss, after being informed of the scheme, decided he had to call the big boss, General Maxwell Taylor. When he got off the phone, he said to Jim: "You're fired!"

While this was going on there were ambitious plans to make the State University at Stony Brook into "the Berkeley of the east". The New York Governor, Nelson Rockefeller, had committed substantial resources to the project. The Noble Prize winning physicist, C-N. Yang (Frank Yang) had already been attracted to the university where he headed his own Institute. It was probably through Yang's connection with Chern, that Jim, despite his relatively young age, was suggested as a candidate to take over the Chairmanship of the math department. At his interview, Bentley Glass said something which Jim got a kick out of recounting: "Well Dr. Simons, you're the first person we've interviewed for this job who actually wants it."

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[4] Notably, over the previous 5 years during which Jim had been following his own path, he had not published anything on minimal varieties. There is no indication that he was at all focused on, or even aware of, the two major problems which he eventually solved.

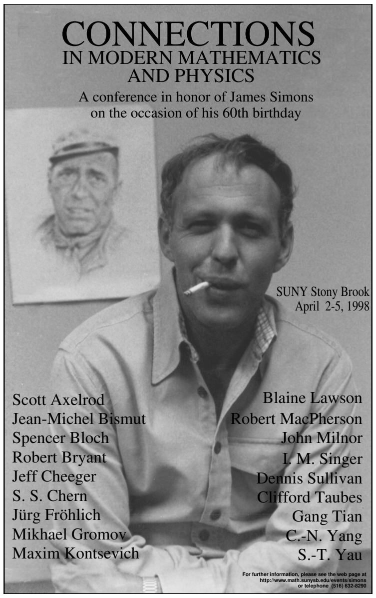
After an initial period in 1968 during which he got the lay of the land, Jim began hiring in earnest. In 1969 he made 10 new hires. The most prominent was Jim Ax who, with Simon Kochen, had just won the Cole Prize of the American Mathematical Society. Together with the presence of Jim, the hiring of Detlef Gromoll, Wolfgang Meyer, David Ebin and Cheeger instantly turned Stony Brook into a center of research in Differential Geometry. Other top notch hires included the functional analyst Ron Douglas and the young star of group representations Roger Howe. In 1970, another collection of strong hires was made. In 1974, with help from Tony Phillips, Misha Gromov, arrived from Russia. Jim was responsible for putting Stony Brook on the map as a top mathematics department. It was his first stint as a leader, though for the participants in this exciting enterprise, the term, "ring leader", would have seemed more appropriate.

Lawson-Simons. One of the first papers to exploit the insights in Jim's minimal varieties work was written at Stony Brook by Jim and Blaine Lawson. Their initial contact had occurred when Blaine was a graduate student at Stanford, and his advisor, Bob Osserman, gave him the preprint of Jim's minimal varieties paper which had just appeared. Blaine read the manuscript and sent Jim a list of comments. He also had further results related to Jim's which were independently found by Chern, Manfredo do Carmo and Shoshichi Kobayashi.

In 1971, the year after Blaine's two year postdoc at Berkeley, Jim invited him to spend the Spring semester at Stony Brook. During that Spring, Lawson and Simons proved a number of geometric results using minimizing currents. Some of the conclusions actually involved Stiefel-Whitney classes. Although in general dimensions, mass-minimizing currents can have singularities, this paper gave procedures for getting geometric formulas on these currents. One of the most interesting results was that any stable minimal current in complex projective space with the classical metric, is a positive algebraic cycle. Their paper was published in the Annals; [LawSim, 1973].

For Lawson, it was an extraordinary and beautiful experience. He was completely taken by Jim's openness with mathematical ideas, his inclusiveness, and his real support for the people around him, not just the mathematicians.

The Simons-Yang seminar. One of Jim Simons' incredibly important contributions to science came from a seminar organized by Frank Yang at his Institute for Theoretical Physics. Yang thought that mathematicians and physicists needed more communication, and the seminar was aimed at this. The basic rule was: if a mathematician was talking, only physicists could ask questions, and reciprocally, if a physicist was talking, only mathematicians could ask questions. After Jim came to Stony Brook, Yang asked him to give lectures in this program, and something completely unforeseen happened. They discovered that the notion of a connection on a principle bundle in geometry was the same thing as a Yang-Mills field in physics! The implications were enormous. There was a sea change in the relations of mathematicians and physicists at the highest levels of research, and it engendered a flood of new results.



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Chern-Simons invariants. Chern-Simons invariants were discovered in the course of Jim's attempt to find a local combinatorial formula for the signature in dimension 4, beginning in 1968.

Let X^{4k} denote a closed connected $4k$-manifold with orientation O. Consider the nondegenerate symmetric bilinear form given by the cup product pairing, $(x \cup y)([X^{4k}])$, where $x, y \in H^{2k}(X^{4k})$ and $[X^{4k}]$ denotes the orientation class in homology. The *signature*, $\mathrm{Sig}(X^{4k})$, is the dimension of a maximal subspace on which the pairing is positive definite minus the dimension of a maximal subspace on which it is negative definite.

A smooth triangulation, T, of X^{4k}, induces a triangulation $\hat{T}$ and orientation, $\hat{O}$, on the link, $L(v_\alpha)$, of every vertex v_α. By definition, the *link*, $L(v_\alpha)$, of v_α is the boundary, $\partial\mathrm{star}(v_\alpha)$, where *star*$(v_\alpha)$, a combinatorial $4k$-ball, denotes the union of all $4k$-simplices containing v_α. Thus, $(L(v_\alpha), \hat{T}, \hat{O})$ is a triangulated oriented $(4k-1)$-sphere. A *local combinatorial formula* for the signature is a formula of the form

$$\mathrm{Sig}(X^{4k}) = \sum_\alpha c_0(L(v_\alpha), \hat{T}, \hat{O}),$$

where c_0 is any numerical invariant of oriented, triangulated $(4k-1)$-spheres which depends only on the oriented simplicial isomorphism class of the triangulation.

Jim was inspired by Chern's 1944 intrinsic proof of what came to be known as the Chern-Gauss-Bonnet formula for the Euler characteristic $\chi(M^{2n})$. In that case, M^{2n} is a equipped with a riemannian metric and the formula states:

$$\chi(M^{2n}) = \int_{M^{2n}} P_\chi(\Omega).$$

Here P_χ is $\frac{1}{(2\pi)^n}$ times the Pfaffian, and the differential $2n$-form, $P_\chi(\Omega)$, is obtained by applying this to the curvature Ω. Since both the sign of $P_\chi(\Omega)$ and the integral depend on a local choice of orientation, the r.h.s. of the Chern-Gauss-Bonnet formula is globally well defined, even in the nonorientable case.

Let θ denote the riemannian connection. The key to Chern's proof was the observation that when pulled back to the unit sphere bundle, the form $P_\chi(\Omega)$, becomes the exterior derivative of an explicit form, $TP_\chi(\theta)$, whose restriction to the fibre is a multiple of the volume form, normalized to have integral 1 on the unit sphere. On M^{2n} as above, there always exist vector fields, V, which vanish only at a finite number of points, $x_1, \ldots, x_N$, and have standard behavior in a neighborhood of each x_i. This enables one to attach an integer, the *index* of V, to each x_i. According to the Hopf index theorem, for any such V, the sum of the indices is equal to the Euler characteristic. On the complement of the set $\{x_i\}$, the normalized vector field, $V/|V|$, is a section of the unit sphere bundle. From this, together with Stokes' theorem, the Chern-Gauss-Bonnet formula follows by what can be thought of as a residue calculation.

Given a smooth triangulation of a manifold, there is an essentially canonical vector field, V, vanishing precisely at the barycenters of the simplices of the first barycentric subdivision. On each such i-simplex, the index of V is $(-1)^i$, so in this special case, it is clear that the sum of the indices is the Euler characteristic. This may have played a role in Jim's original thinking since the context in which he was interested was a combinatorial one.

Jim planned to make use of the Hirzebruch signature theorem. For closed oriented 4-manifolds, it states that the signature is given by $\mathrm{Sig}(M^4) = \frac{1}{3}P_1(M^4)$ where $P_1(M^4)$ denotes the first Pontrjagin class evaluated on the orientation class in homology. He knew from Chern-Weil theory that this class can be represented by a characteristic differential 4-form, $P_1(\Omega)$, analogous to the form $P_\chi(\Omega)$. He was able to find a 3-form $TP_1(\theta)$ on the principle bundle whose exterior derivative is equal to pullback of the characteristic form $P_1(\Omega)$. In actuality, such "transgression forms", which came to be called *Chern-Simons forms*, were already known to Chern in much greater generality. However, prior to that point, they played only a supporting role in Chern-Weil theory. Jim's great discovery was that in and of themselves, they contained new highly interesting information.

Although (for good reason) Jim's plan to generalize Chern's proof to the signature case got stuck, he did realize that the 3-forms, $TP(\theta)$, were already defined on the frame bundle of any oriented compact riemannian 3-manifold, M^3, (even though $P_1(\Omega)$, a 4-form, vanishes identically). Since closed oriented compact 3-manifolds are parallelizable, choosing a parallelization, and integrating the pull back of $TP(\theta)$ over M^3 gives a number which depends on the riemannian metric. A different nonhomotopic choice of parallelization changes this number by an integer, so it is well defined in $\mathbb{R}/\mathbb{Z}$. It is the first example of a *Chern-Simons invariant*.

Jim was led to try to understand the significance of these new global geometric invariants. For example, he was able to show that they depend only on the conformal class of the metric. While on a plane flight with Cheeger to visit the University of Minnesota, he completed the computation for the case of a manifold immersed in $\mathbb{R}^4$ with the induced metric and and expressed disappointment that it was "uninteresting because its always zero". After thinking for a moment, Cheeger responded "Isn't that a theorem?"

Naturally, Jim told his friend Chern about what he had been doing. Chern immediately understood that the whole story could be vastly generalized, resulting in the fundamental paper [ChernSim, 1974]. In that paper a consequence of the plane flight story appears as Example 1: Since for $SO(3)$ with its constant curvature 1 metric, the Chern-Simons invariant is $1/2$, this riemannian manifold does not conformally immerse in $\mathbb{R}^4$, although it does immerse topologically and locally, it embeds isometrically.

Differential characters. Early on, with the special case of dimension 3 in mind, Jim became convinced that the Chern-Simons forms, $TP(\theta)$, of a principle bundle, $F \to E \xrightarrow{\pi} B$, with connection, should *always* have an incarnation which lives on the base. This led to his preprint "Characters associated to a connection", where the relevant new mathematical objects, *differential characters*, which do in fact live on the base, were first defined. However, the preprint was never published. Already in 1969, Jim had involved Cheeger in the project of local combinatorial formulas and what became Chern-Simons invariants. As a consequence, Cheeger made some significant contributions to theory of differential characters, including the ring structure, the possibility of representing differential characters by equivalence classes of forms with singularities (where the definition of the product structure is particularly transparent) and the corresponding product formula (Whitney sum formula) for the characters of the Whitney sum of two vector bundles with connection. So it was decided that together, they would write the foundational paper on differential characters. The resulting preprint was widely distributed in the form of lecture notes at the big Stanford Summer Institute on Global Analysis in 1973. Still in its original form, this work was only published in 1985; [ChSim, 1985]. Let us briefly indicate its content.

Let $\Lambda \subset \mathbb{R}$ denote a subring. The main interest is in the cases $0, \mathbb{Z}, Q$. Let $\wedge^k(M)$ denote the space of differential k-forms and $\wedge_0^{k+1}(M) \subset \wedge^{k+1}(M)$ denote the space of closed $(k+1)$-forms whose integral over any $(k+1)$-cycle lies in Λ. Given a smooth manifold M, there is a graded ring, *the ring of differential characters*, whose component in degree k is denoted $\hat{H}^k(M, \mathbb{R}/\Lambda)$. It is defined as follows.

An element, f of $\hat{H}^k(M, \mathbb{R}/\Lambda)$ is a homomorphism from singular k-cycles to $\mathbb{R}/\Lambda$, for which there exists $\omega \in \wedge_0^{k+1}(M)$, such that if the k-cycle, $b_k = \partial c_{k+1}$, is a boundary, then $f(b_k) = \int_{c_{k+1}} \omega \mod \Lambda$.

The product in ring structure on differential characters is denoted by $*$. Its definition is not obvious. In defining it, for technical reasons, one considers singular homology theory based on normalized cubical chains. The ring structure is *graded* in the sense that $\hat{H}^k(M, \Lambda) * \hat{H}^\ell(M, \Lambda) \to \hat{H}^{k+\ell+1}(M, \Lambda)$.

Since it is not difficult to see that ω is uniquely determined by f, we write $\omega = \delta_1(f)$. Clearly, there is a natural inclusion, $\wedge^k(M) \subset \hat{H}^k(M, \mathbb{R}/\Lambda)$. If $\psi \in \wedge^k(M)$, it follows easily from the relation below that $\delta_1(\psi) = d\psi$, the exterior derivative of ψ. One can also show that there is a uniquely determined $u \in H^{k+1}(M, \Lambda)$ whose image in $H^{k+1}(M, \mathbb{R})$ coincides with the real cohomology class of ω determined by the de Rham isomorphism. We write $u = \delta_2(f)$. Let $r : H^k(M, \Lambda) \to H^k(M, \mathbb{R})$ denote the natural map. Then

There is an exact sequence,

$$0 \to H^k(M, R)/r(H^k(M, \Lambda)) \to \hat{H}^k(M, \mathbb{R}/\Lambda) \xrightarrow{(\delta_1, \delta_2)} (\wedge_0^{k+1}(M), H^{k+1}(M, \Lambda)) \to 0\,.$$

Let $\mathcal{B}$ denote the Bockstein of the coefficient sequence $\Lambda \to \mathbb{R} \to \mathbb{R}/\Lambda$. If $f \in H^k(M, \mathbb{R}/\Lambda) \subset \hat{H}^k(M, R/\Lambda)$, then $\delta_2(f) = -\mathcal{B}(f)$.

Returning to the original motivation, the crucial point is that the Weil homomorphism lifts canonically to the ring of differential characters. Let G denote a Lie group with finitely many components and universal classifying space B_G. Let $G \to E \xrightarrow{\pi} B$ be a principle G-bundle with connection θ. The Weil homomorphism, w, maps invariant polynomials, P, on $\mathfrak{g}$ to characteristic forms on B. Let $u \in H^k(B_G, \Lambda)$ with $r(u)$ equal to the de Rham cohomology class of $w(P)$. Then there is a canonical $f \in \hat{H}^k(M, R/\Lambda)$ with $\delta_1(f) = w(P)$ and $\delta_2(f) = u$. In the obvious sense, the above correspondence commutes with pull back under connection preserving bundle maps. The pull back of f to the total space of a principle bundle with connection is the mod Λ reduction of the corresponding Chern-Simons form. The proofs of the above statements depend on the existence of universal bundles, but *not* on the existence of universal connections; see Theorem 2.2 of [ChSim, 1985].

The simplest case is that of the Euler class of an $SO(2)$ bundle with connection over a circle and angle of holonomy ϕ. In that case, the Euler form and the Euler class vanish and the value of the associated differential character on the circle is $\phi/2\pi$. This case is related the Aharonov-Bhom effect in quantum physics, in which the wave function of a charged particle passing around a long solenoid experiences a phase shift, even though the magnetic and electric fields (but not the potential) are negligible in the region through which the particle passes. After learning of this effect, Jim relayed the news to Cheeger: "Electromagnetism is a differential character! End of story!"

In the sense of obstruction theory, the Euler class is the primary obstruction to the existence of a non-vanishing section of a vector bundle with fibre, $\mathbb{R}^n$. As previously noted, the key point in Chern's proof of the Chern-Gauss-Bonnet formula was his observation that if the bundle has a connection, then the form $TP_\chi(\theta)$ on the principle bundle can be pushed down to the unit sphere bundle. Similarly, for a complex vector bundle with fibre, $\mathbb{C}^n$, the i-Chern class, c_i, is the primary obstruction to the existence of $n-i+1$ linearly independent sections. If in addition, the bundle carries a connection, then the Chern-Simons form on the principle bundle can be pushed down to the corresponding bundle with Stiefel manifold as fibre. This leads to the possibility of representing the corresponding the differential character, $\hat{c}_i$, as a differential form with singularities. Like the case of the Euler class, where it was necessary to choose a section with isolated zeros, the representative of $\hat{c}_i$ so obtained depends on a corresponding choice. Since Pontrjagin classes of a vector bundle with fibre $\mathbb{R}^n$ are, up to sign, the Chern classes of the complexified bundle, the discussion extends to Pontrjagin classes as well.

Included in [ChSim, 1973] were a number of basic results proved by Jim's student, John Millson, as part of his 1973 PhD thesis. For instance, Millson showed that for M a nonnegatively curved space form, the Pontrjagin differential characters, $\hat{p}_i$ take values in $Q/\mathbb{Z}$. For the case of lens spaces, he gave an explicit formula for these invariants. Together with the corresponding conformal nonimmersion theorems, he concluded that for each k, there are infinitely many $(4k-1)$-dimensional lens spaces which smoothly immerse in $\mathbb{R}^{4k}$ but do not conformally immerse in $\mathbb{R}^{6k-1}$; compare Theorem 2.2 of [ChernSim, 1974].

The case of compact manifolds of constant negative curvature is much deeper. As in the positive case, the characteristic forms vanish and the characters, $\hat{p}_i$, are $\mathbb{R}/\mathbb{Z}$ cohomology classes. However, since now, the fundamental group is infinite, there is no rationality conclusion. In fact, it seems likely that in most instances, the values are irrational.

There are additional cases in which the characteristic forms vanish identically. One such occurs in the context of foliations. Raul Bott had defined a family of connections on the normal bundle to the leaves of a foliation, $\mathcal{F}$, whose leaves have codimension n. All of these connections have the property that the Pontrjagin forms $P_k(\Omega)$ of the normal bundle vanish identically if $k > n$. This led in [ChSim, 1973] to the definition of cohomolgy classes, $\hat{p}_i(\mathcal{F}) \in H^{4i-1}(M, \mathbb{R}/\mathbb{Z})$, $2i > n$, and $\hat{Q}^{2k-1}(\mathcal{F}) \in H^{2k-1}(M, \mathbb{R})$, $k > n$. These classes are natural under smooth maps which are transverse to $\mathcal{F}$ and are cobordism invariants. For all i with $2i > n$, there are examples for which the $\hat{p}_i$ classes don't vanish. The $\hat{Q}$ classes were defined independently by others, including the Gobidbillon-Vey invariant for $k = 2$.

In general, a bundle with connection is called *flat* if the holonomy group is discrete, or equivalently, if the curvature vanishes identically. In particular, we obtain $\mathbb{R}/\mathbb{Z}$ cohomology classes, $H^{2k-1}(M,\mathbb{R}/\mathbb{Z}) \subset \hat{H}^{2k}(M,\mathbb{R}/\mathbb{Z})$. These can be viewed as $\mathbb{R}/\mathbb{Z}$ characteristic classes of the group, G, equipped with the *discrete topology*, which we denote by G°.

In the case, $k = 1$ discussed above, the value of the differential character in $\hat{H}^1(S^1,\mathbb{Z})$ on the orientation class, $[S^1]$, is equal to the normalized angle of holonomy. It can take any value mod $\mathbb{Z}$, depending on the particular flat connection which is chosen. On the contrary, for $k \geq 2$, it was shown that given a continuous 1-parameter family of flat connections, the classes in $H^{2k-1}(M,\mathbb{R}/\mathbb{Z}) \subset \hat{H}^{2k}(M,\mathbb{Z})$ are independent of the parameter.

Cheeger and Simons, conjectured and eventually proved: For $G = SO(2,\mathbb{C}), SO(2,\mathbb{R})$, $SO(2n-1,1,\mathbb{R})$, the groups, $H_{2n-1}(G^\circ)$, have infinite rank. Their initial approach involved trying to show that on a suitable collection of cycles, the corresponding $\mathbb{R}/\mathbb{Z}$ characteristic classes took countably many irrational values which are linearly independent over the rationals. They were able to relate this to the following conjecture. Consider a $(2k-1)$-simplex In the unit sphere, $S^{2k-1} \subset \mathbb{R}^{2k}$, with totally geodesic faces and rational dihedral angles. Although in all known cases, the volume of such a simplex is a rational fraction of the volume of the sphere, they conjectured that typically the values are irrational and that the vector space over the rationals which they generate has infinite rank.[5]

Cheeger and Simons got nowhere on this conjecture which, for a time, became quite depressing. Indeed, the conjecture remains open to this day. Eventually they realized that for the desired application, a different method would work. This involved so called *real continuous* characteristic classes in $H^{2k-1}(G^\circ,\mathbb{R})$. The trick was to evaluate these on a certain countable family of $(2k-1)$-cycles which were constructed with the aid of algebraic number theory. Roughly speaking, these cycles could be moved around independently using algebraic automorphisms of the complex numbers whose action on the complex numbers is not continuous. Both Cheeger and Simons gave invited lectures on the subject of Chern-Simons invariants and differential characters at the 1974 International Congress of Mathematicians held in Vancouver. Cheeger's talk dealt with the case of flat bundles discussed above.

Contemporaneous work of Atiyah-Patodi-Singer on the index theorem for manifolds with boundary was immediately recognized as being closely related to Chern-Simons invariants and differential characters.

In 1976, Simons, left academia and began a second career which ultimately resulted in his having revolutionized quantitative finance. Also in 1976, along with Bill Thurston, Jim was awarded the Veblen Prize of the American Mathematical Society for his ground breaking work on minimal varieties. The committee consisted of Chern, Rob Kirby and John Milnor. Somewhat later, Chern said to Cheeger that they had hoped that winning prize would encourage Jim to stay in mathematics. According to Jim, someone told him that eventually, Chern had accepted that he was leaving and had remarked, "Well, he wasn't Hilbert."

At the time Jim left academic mathematics, there were two as yet unresolved questions which were on his mind. One was to characterize connections on vector bundles with the property that the characteristic forms vanish identically. Another was to construct an analog of differential characters for K-theory, a generalized cohomology theory.

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[5] Fundamental work on these issues, actually goes back to Ludwig Schläfli in 1852 (Theorie der vielfachen Kontinuität) which was only published in 1901.

Much later, Michael Hopkins and Isadore Singer showed that an analog of differential characters, called *differential cohomology*, can be defined for any generalized cohomology theory. As a consequence, at present, differential characters are sometimes referred to as *ordinary differential cohomology*.

Simons-Sullivan. Starting the late 1990's, several developments took place which eventually led to joint work of Simons and Dennis Sullivan in the 2000's. In 1998, Geometry Festival held at Stony Brook was devoted to a celebration of Jim's 60th birthday.[6] The first two talks were by Lawson and Cheeger. The former recounted Jim's work on minimal varieties while the latter recounted his work on Chern-Simons invariants and differential characters. During the conference, Dennis learned from Jim about the question concerning the vanishing of characteristic forms and was struck by the fact that such a natural question had not been addressed in the literature.

After Jim had left academia, he continued off and on to ponder the unresolved questions which were on his mind when he left. Every once in a while, he would call up Cheeger to explain his latest thoughts. One such conversation, which took place in the late 1990's was particularly significant. In it Jim explained an idea for functorially characterizing differential characters in terms of a certain diagram. It seemed to Jim that it depended on knowing whether homology classes could be represented by submanifolds. Jeff told him that this was "almost true" and that the right person to consult on the matter was Dennis.

Yet another factor which added to Jim's rekindled interest in characters was a conversation with Blaine which took place around 2005. In it, he mentioned to Jim his joint work with Reese Harvey and John Zweck, in which they proved that on a compact manifold, differential characters obeyed duality, a combination of Poincarè duality and Pontryagin Duality. Jim was enthusiastic. Prior to this, Lawson and Harvey had discovered a theory of special currents they called *sparks*. These appear naturally in their theory of singular connections.They showed that differential characters can be defined as equivalence classes of sparks. This gave canonical isomorphisms between the differential character groups that occur from many quite distinct areas. Since there was no mention of multiplication and since the isomorphisms were canonical, it followed that the ring structure in the original Cheeger-Simons work that the characters associated with these theories carries over to this general context.

The question concerning the vanishing of characteristic forms which Simons had pondered was resolved in work of Simons and Sullivan. Their work also included the above mentioned axiomatic characterization of differential characters and a particularly nice model of differential K-theory; [SimSul, 2008], [SimSul, 2010a], [SimSul, 2010b], [SimSul, 2012].

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[6] Videos of all of the lectures are available on YouTube; here are links to the lectures of Lawson and Cheeger: https://www.youtube.com/watch?v=NhEOb4m9R3U, https://www.youtube.com/watch?v=7FSJC1STQ9c

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