Homework 5

Due Monday Feb. 26 at the beginning of class

1. Let f be holomorphic and non-constant on a domain Ω. Show that:

   (i) For all $a \in \mathbf{C}$ the set $\{z \in \Omega : f(z) = a\}$ is discrete in Ω.

   (ii) If $\overline{\Omega}$ is compact and f extends to a holomorphic function on a nbhd. of $\overline{\Omega}$, then $\{z \in \Omega : f(z) = a\}$ is finite.

2. (**The permanence of analytic continuation**). Prove that:

   If f is holomorphic in a domain (an open and connected set) Ω, and if there exists $\{|z - z_0| < \epsilon\} \subset \Omega$ where f is identically zero, then $f \equiv 0$ in all of Ω.

   If the zeros of f have an accumulation point in Ω, then $f \equiv 0$ in Ω.

3. Prove that:

   (i) If $f : \mathbf{C} \to \mathbf{C}$ is holomorphic and has a non-essential singularity at ∞, then f is a polynomial.

   (ii) The functions e^z and $\sin(z)$ have essential singularities at ∞.

4. Determine explicitly the largest disk about 0 whose image under the mapping $w = z^2 + z$ is one-to-one. Do the same for $w = e^z$.

5. Let $\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane.

   (i) Show that for any $a \in \mathbf{H}$, the mapping

$$w = \frac{z - a}{z - \bar{a}}$$

gives a holomorphic equivalence $\mathbf{H} \longrightarrow \Delta$ where $\Delta = \{|z| < 1\}$.

   (ii) Let $f : \mathbf{H} \to \mathbf{H}$ be holomorphic. Show that for any $z_0 \in \mathbf{H}$,

$$\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{z - z_0}{z - \overline{z_0}} \right| \qquad \forall\, z \in \mathbf{H}.$$

6. Show by using Schwarz's Lemma that every one-to-one holomorphic mapping of an open circular disk onto another is given by a linear transformation (of S^2).

7. If γ is a piecewise smooth arc contained in Δ, the integral

$$\int_\gamma \frac{|dz|}{1 - |z|^2}$$

is called the **noneuclidean length** (or **hyperbolic length**) of γ. Show that every holomorphic function $f : \Delta \to \Delta$ maps every γ to an arc with noneuclidean length $\leq$ that of γ. Deduce that the linear transformations of Δ to itself (cf. Problem 1 of Homework 1) preserve the noneuclidean length of all arcs.

Definition. A function $f : \Omega \to \mathbf{C} \cup \{\infty\}$ on a domain Ω is **meromorphic** if for each $a \in \Omega$ either

(i) f is holomorphic on $\{|z-a| < \epsilon\}$ for some $\epsilon > 0$, or

(ii)

$$f(z) \;=\; \frac{1}{(z-a)^m} g(z), \qquad m > 0$$

where g is holomorphic on $\{|z-a| < \epsilon\}$ for some $\epsilon > 0$, and $g(a) \neq 0$. In this case a is called a **pole** of f and m is called **the order** of this pole.

8. Show:

(i) If f is meromorphic on Ω, then $\frac{1}{f}$ is also meromorphic on Ω, and the poles of f are isolated in Ω.

(ii) f is meromorphic in Ω if and only if $f : \Omega \to \mathbf{C} \cup \{\infty\} = S^2$ is a holomorphic map.

9. Let f be a meromorphic function on a neighborhood of $\overline{\Delta}$ with zeros: $a_1, ..., a_N$ poles: $b_1, ..., b_P$ in Ω, both counted to multiplicity. Assume there are no zeros or poles on $\partial\Omega$. Show that

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{f'(z)}{f(z)}\, dz \;=\; N - P.$$

What happens if Δ is replaced by a bounded domain Ω with piece-wise smooth boundary?