Homework 4

Due Monday Feb. 19 at the beginning of class

1. Compute the real integral

$$\int_{-\infty}^{\infty} \frac{1}{x^4+1}\, dx$$

2. Prove that the Laurent series is unique.

3. Show that the Laurent series at the origin of the function

$$f(z) \;=\; \frac{1}{e^z - 1}$$

is

$$f(z) \;=\; -\frac{1}{2} + \frac{1}{z} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

where the numbers B_k, known as Bernoulli numbers, are all positive. (These numbers are very important in topology and number theory.)
Calculate B_1 and B_2.

4. Express the Laurent series at the origin of $\cot(z)$ in terms of the Bernoulli numbers.

5. Consider the domain $\Omega = \mathbf{C} - [\mathbf{0}, \infty)$. Let $g : \Omega \to \mathbf{C}$ be defined by

$$g(z) \;=\; \log(z) \;=\; \log|z| + i \arg(z)$$

where $0 < \arg(z) < 2\pi$. What is $g(\Omega)$?

Using g, show that any holomorphic function f on $\{0 < |z-a| < r\}$ with values in Ω must extend holomorphically at a or have a pole at a.

6. Suppose f is a holomorphic function on a domain Ω and that γ is a piecewise smooth, **closed** curve in Ω. For $w_0 \notin \text{Image}(\gamma)$, consider

$$I(w_0) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w_0}\, dz.$$

Using the transformation $w = f(z)$ give an interpretation of $I(w_0)$.

7. Let f be a holomorphic function on a domain $\Omega \subset \mathbf{C}$. Show that the zeros of f are isolated points in Ω.

8. Let $\Omega \subset \mathbf{C}$ be a bounded domain with piecewise smooth boundary. Let f be a holomorphic function on a neighborhood of $\overline{\Omega}$, and suppose that f has no zeros on $\partial\Omega$. Show that

$$f(z) = (z-a_1)^{n_1} \cdots (z-a_k)^{n_k} g(z)$$

where g is holomophic and nowhere zero on a neighborhood of $\overline{\Omega}$.
Compute

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(z)}{f(z)}\, dz$$