Homework 1

Due Monday Feb. 5 at the beginning of class

1. Let $f : S^2 \to S^2$ be a mapping given by the rational function

$$\frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}$$

where the numerator and denominator have no common roots and $a_n b_m \neq 0$. What is the degree of f? Take the definition of degree given in class.

2. Give a direct proof of Green's Theorem for the rectangle:

$$\Omega \;=\; \{(x, y) : 0 \leq x \leq r \;\text{ and }\; 0 \leq y \leq R\}.$$

3. For $r > 0$ compute the line integral

$$\int_{|z|=r} \frac{1}{z}\, dz$$

where the circle $|z| = r$ is oriented in the counterclockwise direction.

4. Let γ be the oriented arc

$$\gamma(t) \;=\; (2 + \cos t)e^{2it} \qquad \text{for } \; 0 \leq t \leq 2\pi.$$

Compute

$$\int_{\gamma} \frac{1}{z}\, dz$$

(Hint: Use Problem 3.)

5. Given a smooth function $f = u + iv$, we define

$$df \;=\; du + i dv$$

where

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \qquad \text{and} \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$$

Show that

$$df \;=\; \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \overline{z}} d\overline{z}$$

6. Adopt the algebraic convention that:

$$dx\,dx \;=\; dy\,dy \;=\; 0 \qquad \text{and} \qquad dx\,dy \;=\; -dy\,dx.$$

Show that if $\omega = p\,dx + q\,dy$, then

$$d\omega \;=\; dp\,dx + dq\,dy \;=\; \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx\,dy$$

Similarly, if $\omega = P\,dz + Q\,d\overline{z}$, then

$$d\omega \;=\; dPdz + dQ\,d\overline{z} \;=\; \left(\frac{\partial Q}{\partial z} - \frac{\partial P}{\partial \overline{z}}\right) dz\,d\overline{z}$$

7. Show that Green's Theorem says (under the same algebraic convention) that

$$\int_{\partial\Omega} f(z)\,dz \;=\; \int_{\Omega} \frac{\partial f}{\partial \overline{z}} d\overline{z}\,dz$$

8. Consider the function $\log(z-a)$ on an open set in $\mathbf{C} - \{a\}$ where a branch of $\arg(z-a)$ can be (and is) defined. Show that

$$d\,\log(z-a) \;=\; \frac{1}{z-a}\,dz$$

Consider a smooth closed curve $\gamma : [0,1] \to \mathbf{C} - \{a\}$, $\gamma(0) = \gamma(1)$. What is the value of

$$\frac{1}{2\pi i}\int_{\gamma} \frac{1}{z-a}\,dz?$$