Homework 12

Due Monday April 23th at the beginning of class.

1. Let $\Omega \subset \mathbf{C}^2$ be a domain. A smooth function $f : \Omega \to \mathbf{C}$ is **holomorphic** if its tangent linear map $df_x : T_x\mathbf{C}^2 \cong \mathbf{C}^2 \to \mathbf{C}$ is complex linear at each point (i.e., it commutes with scalar multiplication by i). A smooth map from Ω to $\mathbf{C}^2$ is holomorphic if its two coordinate functions are holomorphic.

(i) Set

$$\Delta^2 = \{(z,w) \in \mathbf{C}^2 : |z| < 1, |w| < 1\} \qquad \text{and} \qquad I^4 = [-1,1] \times \cdots \times [-1,1]$$

where the splitting of I^4 is according to the real and imaginary coordinates axes. Does there exist a homeomorphism $F : \overline{\Delta^2} \to \overline{I^4}$ which is holomorphic on Δ^2? Why?

(ii) Let $f : \overline{\Delta^2} \to \mathbf{C}$ be a continuous map which is holomorphic in Δ^2. Show that

$$\sup_{\overline{\Delta^2}} |f| \;\leq \sup_{|z|=|w|=1} |f(z,w)| \qquad \text{(the max on the } 2-\text{torus)}.$$

(iii) Show that there does not exist a homeomorphism $F : \overline{\Delta^2} \to \overline{B}$ which is holomorphic on Δ^2, where B is the unit ball about the origin in $\mathbf{C}^2$.

2. Show that with proper choice of branches the function

$$f(z) \;=\; \int_0^z \frac{1}{\sqrt{\zeta(1-\zeta^2)}}\, d\zeta$$

maps the upper half plane onto a square.

3. Let H be the upper half plane. Let T be an equilateral triangle in $\mathbf{C}$ with vertices A, B and C.

(1) Why does there exist a holomorphic automorphism

$$f : H \to T$$

continuous on the closure, which sends 0, 1, ∞ to A, B, C?

(2) Why is f unique?

(3) Using Shwarz reflection, extend the mapping f^{-1} to the entire complex plane.