<u>Prop. 2</u> Fix $x \in X$. Then $\exists\, U^{\text{open}} \ni x$ and $\varepsilon > 0$ s.t.

① Any two points of U are endpoints of a unique geodesic of length $< \varepsilon$.

② This geodesic $\gamma(x_1, x_2, t)$ depends smoothly on the endpoints x_1 x_2 and on the parameter t
In particular, if $v_1 = \frac{\partial \gamma}{\partial t}(x_1, x_2, 0)$
Then (x_1, v_1) is smooth in (x_1, x_2)

③ $\forall y \in U$

$$\exp_y : \{ v \in T_y X : \|v\| < \varepsilon \} \xrightarrow{\cong} U_y$$

is a diffeomorphism onto an open set

$$U_y \supset U.$$

Theorem 3. Let $\gamma = \gamma(x_1, x_2)$ be a geodesic from Prop. 2. Then γ is the unique shortest curve joining its endpoints.

Cor 1 $K^{cpt} \subset X$

Then $\exists\ \varepsilon > 0$ st any two points $x, y \in K$ with

$$d(x,y) < \varepsilon$$

are joined by a unique geodesic γ of length $< \varepsilon$.

Furthermore

γ is minimal and

γ depends C^∞ on its endpoints.

Pf $\forall\ x \in K$ choose $\cancel{W_x}$ $U = U(x)$, ε_x as in Prop. 2. Choose r_x st

$$B(x, 2r_x) \subset \cancel{W_x}\ U(x)$$

Cover K by a finite no. of

$$B(x_j, r_j)$$

(note)

$\varepsilon \equiv \min\{r_j\}$.

QED

If $x, y \in K$ with

$d(x,y) < \varepsilon$

Then $\exists\, x_j$ st $d(x, x_j) < r_j$.

$\therefore$ $d(x_j, y) < d(x_j, x) + d(x, y)$

$< r_j + \varepsilon < 2r_j$

$\therefore$ $x, y \in \tilde{U}(x_j)$

and Prop 2 + Thm 3 apply

qed.

Cor 2 Let $\sigma : [0, L] \to X$ be a piecewise C^1 curve parameterized by arc-length. If σ is length-minimizing, The σ is a geodesic.

Pf Cor 1 with $K = \sigma([0, L])$
$\Longrightarrow \quad \forall t, \ \sigma\big|_{[t-\varepsilon, t+\varepsilon]}$ is a geodesic, ged

Geodesic Convexity

Def ~~The~~ A set $U \subset X$ is called geodesically convex if $\forall\, y_1, y_2 \in U$ There exists a unique minimal geodesic γ in X from y_1 to y_2 and

$$\gamma \subset U$$

Prop. # Fix $x \in X$. Then

① $B(x,r)$ is geodesically convex $\forall\, r$ suff. small

② $\exists\, r_0 > 0$ s.t. $B(y, r_0)$ is geodesically convex $\forall\, y \in B(x, r_0)$ and all $0 < r < r_0$.

For proof we need:

Def $u \in C^2(X)$. The riemannian Hessian of u is the section

$$\mathrm{Hess}(u) \text{ of } \mathrm{Sym}^2(T^*X)$$

defined by

$$\{\mathrm{Hess}(u)\}(V,W) = VWu - \nabla_V W u$$

for v-fields V, W.

Note A tensor in V

$$(\mathrm{Hess}\, u)(V, W) = (\mathrm{Hess}\, u)(W, V)$$

since
$$VWu - \nabla_V W u - WVu + \nabla_W V u$$
$$= ([V, W] - \nabla_V W - \nabla_W V)u$$
$$= T_{VW} u = 0$$

$\therefore$ A tensor in W and symmetric

Discuss: Only need Tor $= 0$
However, in general Hess u
only makes sense at cr. points

Lemma

$$(\mathrm{Hess}\, u)(V, W) = \langle \nabla_V (\nabla u), W \rangle$$

Pf Fix $V, W \in T_x X$. Extend to local v-fields V, W.

$$(\mathrm{Hess}\, u)(V, W) = VWu - (\nabla_V W)u$$
$$= V(\langle W, \nabla u \rangle) - \langle \nabla_V W, \nabla u \rangle$$
$$= \langle \nabla_V W, \nabla u \rangle + \langle W, \nabla_V (\nabla u) \rangle - \langle \nabla_V W, \nabla u \rangle$$

$$= \langle \nabla_V (\nabla u), W \rangle$$

an eq^{n} of tensors. qed

<u>Lemma b</u> If $\gamma(s)$ is a geodesic,

$$\frac{d^2}{ds^2} u(\gamma(s)) = (\mathrm{Hess}\, u)\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)$$

<u>Pf</u>

$$\frac{d}{ds} u(\gamma(s)) = \langle \nabla u, \frac{d\gamma}{ds} \rangle$$

$$\frac{d^2}{ds^2} u(\gamma(s)) = \langle \nabla_{\frac{d\gamma}{ds}} \nabla u, \frac{d\gamma}{ds} \rangle + \langle \nabla u, \underset{0}{\underset{\shortparallel}{\frac{D}{ds}\frac{d\gamma}{ds}}} \rangle$$

$$= (\mathrm{Hess}\, u)\left(\frac{d\gamma}{ds}, \frac{d\gamma}{ds}\right)$$

by Lemma a. qed

Lemma c Let $u(x) = \frac{1}{2} d^2(x, x_0)$.

Then
$$(\mathrm{Hess}_{x_0} u)(v,v) = \|v\|^2$$

i.e. as a sym. map $TX \to TX$
$$\mathrm{Hess}_{x_0} u \cong \mathrm{Id}.$$

Pf Fix $V \in T_{x_0}X$. Assume (wlog) $\|V\| = 1$.

Let $\gamma(s)$ be good. with
$$\gamma(0) = x_0 \qquad \frac{d\gamma}{ds}(0) = V$$

Now $u(\gamma(s)) = \frac{1}{2} s^2$.

$\therefore$ by Lemma b.
$$(\mathrm{Hess}_{x_0} u)(v,v) = \frac{d^2}{ds^2} u(\gamma(s)) \Big|_{s=0}$$
$$= 1.$$

qed

Lemma d Let $\Omega \overset{\text{open}}{\subset} X$.

Suppose

① $\gamma : [a,b] \longrightarrow \Omega$ a geodesic

② $u \in C^2(\Omega)$ satisfies

$$\mathrm{Hess}_x u > 0 \quad \forall x \in \Omega$$

Then $u(\gamma(s))$ is strictly convex on $[a,b]$. In particular

- no interior maxima

Pf Lemma b.

Proof of Prop 3

Fix x_0.

Let U (and $\varepsilon > 0$) be from Prop 2,

Assume w.l.o.g.

$$U = B(x_0, 4r) \quad \text{some } r > 0$$

By shrinking r, assume by Lem. C

$$\mathrm{Hess}\left(\tfrac{1}{2} d(x, x_0)^2\right) > 0 \text{ on } U.$$

We know for any $x, y \in B(x_0, r)$

$\exists!$ good. γ joining x to y.

and ① $L(\gamma) = d(x, y)$

② $\gamma \subset B(x_0, 4r)$

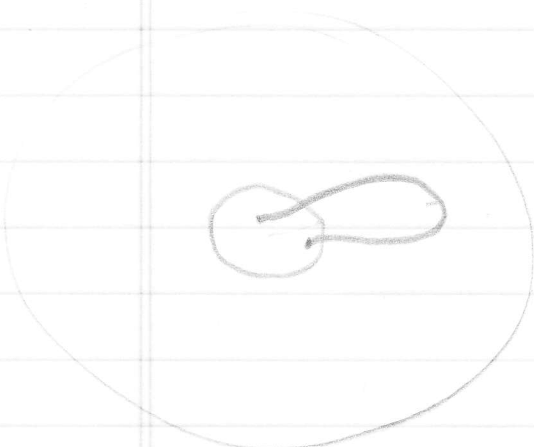
(Thm 3)

Now $u = \frac{1}{2} d^2(x, x_0)$ is convex on γ and $\therefore$ cannot have interior max. $\therefore$ $u|_\gamma \leq \max(u(x), u(y))$.

$$u\big|_{\gamma} \le \max(u(x), u(y))$$

$$\therefore \quad \operatorname{dist}(\gamma(t), x_0) \le \max\{\operatorname{dist}(x, x_0), \operatorname{dist}(y, y_0)\} < r$$

$$\therefore \quad \gamma \subset B(x_0, r)$$



This proves part 1.

<u>Proof of part 2</u> Lemma c $\Longrightarrow$

$\exists r > 0$ s.t.

$$\operatorname{Hess}_x\left\{\tfrac{1}{2} d^2(x,y)\right\} > 0$$

$$\forall\, x, y \in B(x_0, 2r)$$

Hence if $\quad y \in B(x_0, r)$

Then $\quad B(y, r) \subset B(x_0, 2r)$

$\therefore \quad \text{Hess}_x \left\{ \frac{1}{2} d^2(x, y) \right\} > 0$

$$\forall x \in B(y, r)$$
$$\forall y \in B(x_0, r)$$

We assume $r \leq$ previous r

s.t. $\quad B(x_0, 8r) \subset U$

$\therefore \quad B(y, 4r) \subset U$

Now every $B(y, r)$ is good convex for $y \in B(x_0, r)$

by same argument.

qed

Theorem 5 For each compact subset $K \subset X$, $\exists\, \varepsilon > 0$ s.t. $\forall x \in K$, $B(x, \varepsilon)$ is geod.-convex.

Proof (~~Exercise~~) A compactness argument like proof of. ~~Thm~~. Cor 1.

$\forall x \in K$, $\exists\, r_x > 0$ st

$$\forall y \in B(x, 2r_x),\ B(y, 2r_x) \text{ is } g\text{-convex}$$

Take finite cover $B(x_j, r_j)$ note.

$$\varepsilon \equiv \min_j \{r_j\}$$

Fix $x \in K$ and suppose $y \in B(x, \varepsilon)$

$\exists\, x_j$ st. $d(x, x_j) < r_j$.

$$\begin{aligned} d(y, x_j) &< d(y, x) + d(x, x_j) \\ &< \varepsilon + r_j < 2r_j. \end{aligned}$$

$\therefore$ $B(y\ \varepsilon) \subset B(y, 2r_j)$ is g-convex.

qed

Def For $x \in X$, the convexity radius ~~at~~ of X at x is

$$\text{conv-rad}_X(x) = \sup\{r : B(x,r) \text{ is geod.-convex}\}$$

For $E \subset X$ a subset

$$\text{conv-rad}(E) = \inf_{x \in E} \text{conv-rad}(x)$$

Thm 5 If E cpt, then

$$\text{conv-rad}(E) > 0.$$