ON EQUIVARIANT ALGEBRAIC SUSPENSION

H. Blaine Lawson, Jr., P. C. Lima-filho, and M.-L. Michelsohn

Abstract.

Equivariant versions of the Suspension Theorem [L_1] for algebraic cycles on projective varieties are proved. Let G be a finite group, V a projective G-module, and $X \subset \mathbb{P}_{\mathbb{C}}(V)$ an invariant subvariety. Consider the algebraic join $\Sigma\!\!\!/^{V_0} X = X \# \mathbb{P}_{\mathbb{C}}(V_0)$ of X with the regular representation $V_0 = \mathbb{C}^G$ of G. The main result asserts that algebraic suspension induces a G-homotopy equivalence $\mathcal{Z}^s(X) \longrightarrow \mathcal{Z}^s(\Sigma\!\!\!/^{V_0} X)$ of topological groups of algebraic cycles of codimension-s for all $s \leq \dim X - e(X)$ where $e(X)$ is the maximal dimension of g-fixed point sets in $\Sigma\!\!\!/^{V_0} X$ for $g \neq 1$. This leads to a Stability Theorem for equivariant algebraic suspension. The methods also yield a Quaternionic Suspension Theorem for cycles in $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$ under the antiholomorphic involution corresponding to scalar multiplication by the quaternion j. From this the homotopy type of spaces of quaternionic cycles is completely determined.

1991 *Mathematics Subject Classification.* Primary 14C05; Secondary 55P47, 55R45
Key words and phrases. Chow varieties, equivariant algebraic suspension, quaternionic suspension.
All authors were supported by the NSF.

Table of Contents

§**1. Introduction.** In 1989 a Suspension Theorem for algebraic cycles on complex projective varieties was formulated and proved in [L_1]. This result proved to be the key to the development of an interesting theory for varieties. It also led to some new constructions and results in algebraic topology. (See [L_2] for a survey.) The point of this paper is to enunciate and prove an Algebraic Suspension Theorem in the presence of symmetry. The result is substantially more difficult than the non-equivariant analogue in [L_1]. It is also not universally valid; there are delicate necessary conditions involving the dimension of the cycles and the geometry of the action. However, the theorem yields a stable result which is foundational for building equivariant theories based on cycles and for applications to algebraic topology.

The methods also yield a quaternionic suspension theorem which is of independent geometric interest.

The first steps towards equivariant theories based on algebraic cycles were taken in [LM_1] where algebraic suspension was shown to be a homotopy equivalence after localizing away from the order of the group (i.e., after inverting all primes which divide the order of G). Such localization was necessary as shown by example. In this form the suspension has substantial content. However, in the world of finite groups, inverting the primes dividing $|G|$ often kills the interesting invariants of G, and it would be quite useful if one could show that some form of suspension for algebraic cycles was a full G-homotopy equivalence.

Such a theorem is the main result of this paper. We show that the suspension of algebraic cycles by the **regular representation** of G is a full G-homotopy equivalence provided that the dimension of the cycles is sufficiently large with respect to the dimension of the g-fixed point set for each $g \neq 1$ in G. In particular, if G acts freely, then suspension to the regular representation is a G- homotopy equivalence for cycles of any dimension ≥ 0. This result is sharp as shown by many examples (in §4). It also leads to a basic stability result.

We shall now be more explicit. Fix a finite group G and a finite dimensional complex G-module V, and consider a G-invariant algebraic subvariety $X \subset \mathbb{P}(V)$. For each $s \leq \dim(X)$, let $\mathcal{Z}^s(X)$ denote the free abelian group generated by the irreducible subvarieties of codimension-s in X. This group carries a natural topology (cf. [L_1], [Li_5]), and G acts naturally on $\mathcal{Z}^s(X)$ by continuous group automorphisms.

We now let V_0 denote the regular representation of G, and we define the V_0**-suspension** of X to be the G-invariant subset $\Sigma^{V_0} X = X \# \mathbb{P}(V_0) \subset \mathbb{P}(V \oplus V_0)$ consisting of the union of all lines joining X to $\mathbb{P}(V_0)$ in $\mathbb{P}(V \oplus V_0)$. (Here # denotes the *algebraic join*, [L1], [H]). Suspension extends naturally to algebraic cycles on X and defines a continuous G-equivariant homomorphism

$$\Sigma\!\!\!/^{V_0} : \mathcal{Z}^s(X) \longrightarrow \mathcal{Z}^s(\Sigma\!\!\!/^{V_0} X)$$

Our first main results are the following. Consider the numerical invariants

$$e_0(X) = \max_{g \in G - \{1\}} \dim X^g \qquad \text{and} \qquad e(X) = \max_{g \in G - \{1\}} \dim (\Sigma\!\!\!/^{V_0} X)^g$$

of the action of G on X, where $X^g \stackrel{\text{def}}{=} \{x \in X : g(x) = x\}$.

Theorem A. (The Equivariant Suspension Theorem) *If $s \leq \dim X - e(X)$, then suspension to the regular representation*

$$\Sigma\!\!\!/^{V_0} : \mathcal{Z}^s(X) \longrightarrow \mathcal{Z}^s(\Sigma\!\!\!/^{V_0} X)$$

is a G-homotopy equivalence.

Theorem B. (Stability) *Let $q > 1$ be the smallest prime divisor of $|G|$. Then the suspension homomorphisms*

$$\mathcal{Z}^s(\Sigma\!\!\!/^{mV_0} X) \xrightarrow{\Sigma\!\!\!/^{V_0}} \mathcal{Z}^s(\Sigma\!\!\!/^{(m+1)V_0} X)$$

are G-homotopy equivalences for all non-negative integers m such that

$$m \geq \tfrac{q}{(q-1)|G|}\Big(s + e_0(X) - \dim X\Big) + \tfrac{1}{(q-1)}$$

Our last set of results concerns "quaternionic subvarieties". Let $\mathbb{H}$ denote the quaternions with standard basis $\{1, i, j, k\}$, and write $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j$ where $\mathbb{C} = \text{span}_{\mathbb{R}}\{1, i\}$. Multiplication on the left by j in $\mathbb{H}^n$ induces a complex antilinear map $j : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ with $j^2 = -\text{Id}$. This induces an antiholomorphic involution

$$\overline{j} : \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n) \longrightarrow \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$$

which is fixed-point free (and which is covered by an antilinear map of $\mathcal{O}(1)$ whose square is -Id). For each $s < 2n - 1$ the map $\overline{j}$ induces an involution

$$\overline{j}_* : \mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)) \longrightarrow \mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n))$$

whose fixed-points $\mathcal{Z}^s_{\mathbb{H}}(n)$ can be considered the "quaternionic cycles" of codimension-s. The quotient group $\widetilde{\mathcal{Z}}^s_{\mathbb{H}}(n) = \mathcal{Z}^s_{\mathbb{H}}(n)/(1 + \overline{j}_*)\mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n))$ by the $\overline{j}_*$-averaged cycles, can be considered the group of "irreducible quaternionic cycles" of codimension-s. It is a topological 2-torsion group. When the cycle-dimension is 1 for example, this group is the $\mathbb{Z}/2\mathbb{Z}$-vector space generated by the irreducible $\overline{j}_*$-invariant curves, which are, in a sense, the non-orientable projective algebraic curves.

Theorem C. (The Quaternionic Suspension Theorem) *For each $s < 2n-1$ the $\overline{j}_*$-equivariant map*

$$\Sigma\!\!\!/^{\mathbb{H}} : \mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)) \longrightarrow \mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^{n+1}))$$

given by suspension to $\mathbb{H}$, is a $(\mathbb{Z}/2\mathbb{Z})$-homotopy equivalence. It induces homotopy equivalences

$$\Sigma\!\!\!/^{\mathbb{H}} : \mathcal{Z}^s_{\mathbb{H}}(n) \longrightarrow \mathcal{Z}^s_{\mathbb{H}}(n+1) \qquad \text{and} \qquad \Sigma\!\!\!/^{\mathbb{H}} : \widetilde{\mathcal{Z}}^s_{\mathbb{H}}(n) \longrightarrow \widetilde{\mathcal{Z}}^s_{\mathbb{H}}(n+1).$$

There is also a theorem for quaternionic suspension of $\mathcal{Z}^s(\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n \oplus \mathbb{C}))$. These results allow a determination of the homotopy type of $\mathcal{Z}^s_{\mathbb{H}}$ and $\widetilde{\mathcal{Z}}^s_{\mathbb{H}}$. The spaces $\mathcal{Z}^s_{\mathbb{H}}$ fit in a surprizing and beautiful way with the classical theory of representations and characteristic classes.

Using results in this paper the authors have succeeded in calculating the coefficients of the new equivariant cohomology theories established in [LLM$_1$]. The results also play a role in defining cohomology operations on the morphic cohomology groups introduced in [FL$_1$].

Since writing this paper the authors have learned of independent results of Jacob Mostovoy on quaternion cycles. Among other things he has obtained Theorem 6.4 over the rationals.

§**2. Suspension to the regular representation.** Consider a finite group G of order γ, and let V be a finite dimensional G-module. Then G acts naturally on the projective space $\mathbb{P}(V)$, and for each $p \geq 0$, G also acts by automorphisms on the topological monoid $\mathcal{C}_p(\mathbb{P}(V))$ of effective p-cycles on $\mathbb{P}(V)$. Hence it also acts by automorphisms on the naïve topological group completion $\mathcal{Z}_p(\mathbb{P}(V))$.

Let $V_0 = \mathbb{C} \cdot G$ denote the regular representation of G. There is a continuous G-equivariant homomorphism

$$\Sigma\!\!\!/^{V_0} : \mathcal{Z}_p(\mathbb{P}(V)) \longrightarrow \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0))$$

given by associating to a cycle c in $\mathbb{P}(V)$ its algebraic suspension $\Sigma\!\!\!/^{V_0}(c) = c\#\mathbb{P}(V_0)$. This map is always a homotopy equivalence (cf. [L$_1$]). However, it is far from being a G-homotopy equivalence in general. This is discussed in detail in §4. The main result of this section asserts that if the dimension p is large enough that every $(p+1)$-cycle in $\mathbb{P}(V \oplus V_0)$ is forced to meet the subset $\mathbb{P}(V \oplus V_0)_{\text{free}}$ where G acts freely, then $\Sigma\!\!\!/^{V_0}$ is a G-homotopy equivalence.

This produces the "stable" result that after sufficiently many iterations, the maps $\Sigma\!\!\!/^{V_0}$ become G-homotopy equivalences.

Before stating the main result we examine the condition referred to above. Note that $\mathbb{P}(V \oplus V_0)_{\text{free}} = \mathbb{P}(V \oplus V_0) - \Sigma$ where

$$\begin{aligned} \Sigma &= \{x \in \mathbb{P}(V \oplus V_0) \ : \ G_x \neq \{1\}\} \\ &= \{x \in \mathbb{P}(V \oplus V_0) \ : \ \exists g \in G, g \neq 1 \text{ and } gx = x\} \\ &= \bigcup_{g \in G - \{1\}} \mathbb{P}(V \oplus V_0)^g \end{aligned}$$

where X^g denotes the fixed-point set of g on X.

Definition 2.1. Let Γ_G denote the set of conjucagy classes of subgroups of G. Given a finite dimensional G-module V, let $\mathrm{d}_V : \Gamma_G \to \mathbb{Z}$ denote the (equivariant) **dimension function** of $\mathbb{P}(V)$ defined as $\mathrm{d}_V([H]) = \dim_{\mathbb{C}} \mathbb{P}(V)^H$, where $[H]$ denotes the conjugacy class of the subgroup H, and the dimension of an algebraic subset is defined to be the maximal dimension of its irreducible components. We then define

$$e(V) = e_G(V) \overset{\text{def}}{=} \max_{[H]\neq\{1\}\in\Gamma_G} \mathrm{d}_{V\oplus V_0}([H]). \tag{2.1.1}$$

In the following technical result we use the same letter W to denote both a G-module W and its restriction $\mathrm{Res}^G_H(W)$ to a subgorup $H \leq G$.

Lemma 2.2. *The function $e(V)$ has the following properties.*

(1) *If $H < G$, then $e_H(V) \leq e_G(V)$.*

(2)
$$e_G(V) = \max_{g\neq 1\in G} \dim\Big\{\mathbb{P}(V\oplus V_0)^g\Big\},$$
where $\mathbb{P}(V\oplus V_0)^g$ denotes the set of points fixed by $g \in G$. Furthermore, one just needs to maximize over those points $g \in G$ of prime order.

(3) *If G is abelian, then*
$$e(V) \ = \max_{\rho\in\widehat{G}} \dim(V_\rho)$$
where $\widehat{G}$ denotes the character group of G and where $V_\rho \subset V$ is the simultaneous eigenspace with character ρ. If, in addition, G is cyclic of prime order then
$$e(V) = \ \dim\{\mathbb{P}(V)^g\} + 1 \qquad \text{for any } \ g \neq 1.$$

(4) *For any G let q be the smallest prime divisor of $\gamma = |G|$. Then*
$$e(V) \ \geq \ \tfrac{1}{q}\{\dim V + \gamma\} - 1$$
and for all $m > 0$,
$$e(V\oplus mV_0) \ \leq \ \dim V + \tfrac{\gamma}{q}(m+1) - 1$$

Proof. Statement (1) is obvious. Given subgroup $H < G$ and $h \in H$, let $< h >$ denote the subgroup generated by h. Then $\mathbb{P}(V\oplus V_0)^H \subset \mathbb{P}(V\oplus V_0)^{<h>} = \mathbb{P}(V\oplus V_0)^h$, and the first assertion of (2) follows. Note that since $\dim \mathbb{P}(V\oplus V_0)^g \leq \mathbb{P}(V\oplus V_0)^{g^k}$ for any k, it suffices to maximize over elements g of prime order in G.

For (3) recall that $V_0 \cong \bigoplus_\rho \ell_\rho$ where ℓ_ρ is the 1-dimensional eigenspace with character ρ. Then for any $g \neq 1$ we have

$$\mathbb{P}(V\oplus V_0)^g \ = \ \coprod_{\rho\in\widehat{G}} \mathbb{P}((V\oplus V_0)_\rho) \ = \ \coprod_{\rho\in\widehat{G}} \mathbb{P}(V_\rho\oplus\ell_\rho),$$

and (3) follows immediately from assertion (2).

To prove (4), consider $g \in G$ of prime order q. Then we have

$$\mathbb{P}(V \oplus V_0)^g \;=\; \coprod_{\rho \in \widehat{\mathbb{Z}/q}} \mathbb{P}((V \oplus V_0)_\rho) \;=\; \coprod_{\rho \in \widehat{\mathbb{Z}/q}} \mathbb{P}\left(V_\rho \oplus \tfrac{\gamma}{q}\ell_\rho\right),$$

from which the first assertion of (4) follows directly. For the second we note that

$$\mathbb{P}(V \oplus (m+1)V_0)^g \;=\; \coprod_{\rho \in \widehat{\mathbb{Z}/q}} \mathbb{P}\left(V_\rho \oplus (m+1)\tfrac{\gamma}{q}\ell_\rho\right),$$

and

$$\max_{\operatorname{ord} g \text{ prime}} \; \max_{\rho} \; \dim\left(V_\rho \oplus (m+1)\tfrac{\gamma}{\operatorname{ord} g}\ell_\rho\right) \;\le\; \dim V + (m+1)\tfrac{\gamma}{q} \qquad \blacksquare$$

We would like to point out that one can also give the following interpretation to the dimension functions that we considered, based on the previous lemma. Let χ_V denote the character of V, and for any subgroup $H < G$, let $\widehat{H}$ denote the collection of equivalence classes of 1-dimensional representations of H, identifiable with the "character group" of H, consisting of all group homomorphisms $H \to \mathbb{C}^*$. Then

$$\mathbb{P}(V)^H = \mathbb{P}(\operatorname{Res}^G_H(V))^H = \coprod_{\rho \in \widehat{H}} \mathbb{P}(\operatorname{Res}^G_H(V)_\rho).$$

On the other hand, $\dim \operatorname{Res}^G_H(V)_\rho = \left(\chi_{\operatorname{Res}^G_H(V)} \mid \chi_\rho\right)_H$, where $(\cdot|\cdot)_H$ denotes the usual inner product for class functions on H. Hence, Fröbenius reciprocity

$$\left(\chi_{\operatorname{Res}^G_H(V)} \mid \chi_\rho\right)_H = \left(\chi_V \mid \chi_{\operatorname{Ind}^G_H(\rho)}\right)_G$$

implies that

$$e(V) = \max\left\{ (\chi_V \mid \chi_\xi)_G + [G:H] \mid \xi \text{ monomial induced by } H \right\}.$$

Recall that a G-module ξ is said to be monomial if it has the form $\xi = \operatorname{Ind}^G_H(\rho)$, where ρ is a 1-dimensional H-module for some subgroup $H < G$.

2.3. Equivariant Algebraic Suspension Theorem I (Suspension to the regular representation). *Let G be a finite group of order γ and denote by $V_0 = \mathbb{C}^G$ the regular representation of G. Let V be any finite dimensional complex G-module. Then for each $p \ge e(V)$ the suspension homomorphism*

$$\Sigma\!\!\!/^{V_0} : \mathcal{Z}_p(\mathbb{P}(V)) \;\longrightarrow\; \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0))$$

is a G-homotopy equivalence.

Proof. Consider the open G-invariant submonoid

$$\mathcal{TC}_{p+\gamma} \stackrel{\text{def}}{=} \{c \in \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \ : \ \dim(|c| \cap \mathbb{P}(V) = p\}$$

of effective cycles of dimension $p+\gamma$ which meet $\mathbb{P}(V)$ in proper dimension, and let $\mathcal{ZC}_{p+\gamma}$ denote its naïve topological group completion. Algebraic suspension gives an equivariant embedding

$$\Sigma\!\!\!/^{V_0} : \mathcal{C}_p(\mathbb{P}(V)) \ \hookrightarrow \ \mathcal{TC}_{p+\gamma}$$

which extends to an injective homomorphism

$$\Sigma\!\!\!/^{V_0} : \mathcal{Z}_p(\mathbb{P}(V)) \ \hookrightarrow \ \mathcal{TZ}_{p+\gamma}$$

of group completions. The theorem is a consequence of the following two assertions.

Assertion 1. The image $\Sigma\!\!\!/^{V_0}[\mathcal{Z}_p(\mathbb{P}(V))]$ is a G-equivariant deformation retract of $\mathcal{TZ}_{p+\gamma}$. (This assertion holds for any $p \geq 0$.)

Assertion 2. The inclusion $j : \mathcal{TZ}_{p+\gamma} \subset \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0))$ is a G-homotopy equivalence.

Proof of Assertion 1. This is a straightforward application of pulling to the normal cone. Consider the continuous family of G-equivariant automorphsims

$$\Phi_t : \mathbb{P}(V \oplus V_0) \ \longrightarrow \ \mathbb{P}(V \oplus V_0) \qquad \text{for} \ \ 0 < t < \infty$$

defined by setting $\Phi_t([v : v_0]) = [tv : v_0]$ for $v \in V$ and $v_0 \in V_0$. This induces a family of G-equivariant continuous monoid automorphisms

$$(\Phi_t)_* : \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \ \longrightarrow \ \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \qquad \text{for} \ \ 0 < t < \infty.$$

When restricted to $\mathcal{TC}_{p+\gamma}$, this family extends continuously to $t = 0$. (See [L_1], [FL_2], or [Fu] for example.) When restricted to the image $\Sigma\!\!\!/^{V_0}\mathcal{Z}_p \stackrel{\text{def}}{=} \Sigma\!\!\!/^{V_0}[\mathcal{Z}_p(\mathbb{P}(V))]$, this family is the identity for all $t \geq 0$. One checks directly (cf. [L_1], [FL_2]) that Image $(\Phi_0)_* = \Sigma\!\!\!/^{V_0}\mathcal{Z}_p$ for all t. Thus the restriction of $(\Phi_t)_*$ to $\mathcal{TC}_{p+\gamma}$ gives an equivariant deformation retraction down to $\Sigma\!\!\!/^{V_0}\mathcal{Z}_p$. Since each $(\Phi_t)_*$ for $t \geq 0$ is a monoid homomorphism, this family extends to the group completions to give the desired result.

Proof of Assertion 2. This is based on the following Proposition. We abbreviate notation by setting $\mathcal{C}_{p+\gamma} = \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0))$.

Proposition 2.4. *Let $X \subset \mathcal{C}_{p+\gamma}$ be any compact subset. Then for all integers d sufficiently large, there exists a continuous family $\Psi_t : \mathcal{C}_{p+\gamma} \to \mathcal{C}_{p+\gamma}$ of G-equivariant continuous monoid homomorphisms defined for $0 \leq t \leq 1$ with the property that*

(1) $\Psi_0 = d^\gamma \cdot Id$, *and*
(2) $\Psi_t(X) \subset \mathcal{TC}_{p+\gamma}$ *for all* $t > 0$.

Proof. We enlarge our G-space to $V \oplus V_0 \oplus V_0$ with $V \oplus V_0$ embedded as the first two factors. Consider the invariant subspaces

$$V_0' \subset V_0 \oplus V_0 \qquad \text{and} \qquad V_0'' \subset V_0 \oplus V_0$$

where V_0' is the diagonal and $V_0'' = \{0\} \times V_0$. Let

$$\pi' : \mathbb{P}(V \oplus V_0 \oplus V_0) - \mathbb{P}(V_0') \longrightarrow \mathbb{P}(V \oplus V_0)$$

be the equivariant projection with vertex V_0'.

For each positive integer $d > 0$ let Div_d denote the space of effective divisors of degree d on $\mathbb{P}(V \oplus V_0 \oplus V_0)$, and consider the subset $\mathcal{U}_d' \subset \mathrm{Div}_d$ of those divisors D for which

$$\left\{ \bigcap_{g \in G} g|D| \right\} \cap \mathbb{P}(V_0') \ = \ \emptyset. \tag{2.4.1}$$

Let $\mathcal{U}_d'' \subset \mathrm{Div}_d$ be defined similarly with V_0' replaced by V_0'', and set $\mathcal{U}_d = \mathcal{U}_d' \cap \mathcal{U}_d''$.

Lemma 2.5. *The set $\mathcal{U}_d$ is Zariski open and non-empty.*

Proof. A straightforward argument (cf. [H]) shows that the set where (2.4.1) fails is Zariski closed. Thus $\mathcal{U}_d'$ is Zariski open, and by the same reasoning so is $\mathcal{U}_d''$. Thus $\mathcal{U}_d$ is Zariski open. To see that it is non-empty we consider the linear divisor

$$\Delta \ \stackrel{\text{def}}{=} \ d \cdot \left\{ V \oplus V_0 \oplus \bigoplus_{g \neq 1} \mathbb{C} \cdot g \right\} \tag{2.5.1}$$

and observe that

$$\bigcap_{g \in G} g|\Delta| = d^{\gamma}(V \oplus V_0). \quad \blacksquare$$

We now observe that for $D \in \mathcal{U}_d$ the intersection $\bigcap g|D|$ has proper dimension by (2.4.1), for otherwise it would meet $\mathbb{P}(V_0')$. Consequently, the intersection product

$$\mathbb{D} \ \stackrel{\text{def}}{=} \ \prod_{g \in G} g_* D$$

is well-defined and continuous on $\mathcal{U}_d$ (cf. [Fu]). Furthermore, for cycles c on $\mathbb{P}(V \oplus V_0)$, consider the suspension $\Sigma^{V_0''}(c) = c \# \mathbb{P}(V_0'')$ in $\mathbb{P}(V \oplus V_0 \oplus V_0)$.

Lemma 2.6. *The intersection product $\Sigma^{V_0''}(c) \bullet \mathbb{D}$ is well-defined and continuous on $\mathcal{C}_p(V \oplus V_0) \times \mathcal{U}_d$.*

Proof. By [Fu] it suffices to show that for any subvariety $Y \subset \mathcal{C}_p(V \oplus V_0)$, the intersection $\Sigma^{V_0''}(Y) \cap |\mathbb{D}|$ has proper dimension. For this it suffices to show that if Y is a point y, then this intersection is finite. However, this last assertion is obvious since $\Sigma^{V_0''}(y)$ is a linear subspace of dimension γ, and $\Sigma^{V_0''}(y) \cap |\mathbb{D}|$ is a subvariety which misses the hyperplane $\mathbb{P}(V_0'')$. ∎

To each divisor $D \in \mathcal{U}_d$ we now construct a continuous, G-equivariant monoid endomorphism

$$\Psi_D : \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \longrightarrow \mathcal{C}_{p+\gamma}(\mathbb{P}(V \oplus V_0))$$

by setting

$$\Psi_D(c) = \pi'_* \left\{ \Sigma^{V_0''}(c) \bullet \mathbb{D} \right\}. \tag{2.6.1}$$

By Lemma 2.6 these endomorphisms depend continuously on $D \in \mathcal{U}_d$ and for $D = \Delta$ (defined in (2.4.1)) we have

$$\Psi_\Delta(c) = d_\gamma \cdot c. \tag{2.6.2}$$

Given a subvariety $Y \subset \mathbb{P}(V \oplus V_0)$ we have $\Psi_D(Y) \cap \mathbb{P}(V) = \pi'\{\Sigma^{V_0''}(Y) \bullet \mathbb{D}\} \cap \mathbb{P}(V)$. From this we see that $\Psi_D(Y)$ meets $\mathbb{P}(V)$ properly if

$$\widetilde{Y} \overset{\text{def}}{=} (\pi')^{-1}(\mathbb{P}(V)) \cap \Sigma^{V_0''}(Y)$$

meets $\mathbb{D}$ properly. Now projection from V_0'' induces an equivariant isomorphism

$$F : \mathbb{P}(V \oplus V_0') = (\pi')^{-1}(\mathbb{P}(V)) \longrightarrow \mathbb{P}(V \oplus V_0)$$

and $\widetilde{Y} = F^{-1}(Y)$. Thus given Y we have

$$\begin{aligned} \mathcal{B}_Y &\overset{\text{def}}{=} \{D \in \mathcal{U}_d \ : \ \Psi_D(Y) \notin \mathcal{TC}_{p+\gamma}\} \\ &= \{D \in \mathcal{U}_d \ : \ \Psi_D(Y) \text{ does not meet } \mathbb{P}(V) \text{ properly}\} \\ &= \{D \in \mathcal{U}_d \ : \ \mathbb{D} \text{ does not meet } \widetilde{Y} \text{ properly}\}. \end{aligned} \tag{2.6.3}$$

We note that by the upper-semicontinuity of dimension, the subset $\mathcal{B}_Y$ is Zariski closed.

Lemma 2.7. *Let $Y \subset \mathbb{P}(V \oplus V_0)$ be an irreducible subvariety of dimension $p + \gamma$. Then the function*

$$f(d) = \operatorname{codim}(\mathcal{B}_Y) \text{ in } \mathcal{U}_d$$

satisfies $\lim_{d\to\infty} f(d) = \infty$.

Proof. We adopt the following notational conventions. Given any subvariety $Z \subset \mathbb{P}^M$ and any point $z \in Z$ we denote by Z_z the germ of Z at z. We also denote by $\mathcal{O}_{Z,z}$ the local ring of germs at z of functions regular in a neighborhood of z on Z, and by $\mathfrak{m}_{Z,z}$ the maximal ideal in $\mathcal{O}_{Z,z}$. For convenience we shall drop the reference to Z when $Z = \mathbb{P}(V \oplus V_0 \oplus V_0)$.

We shall use the last characterization of $\mathcal{B}_Y$ in (2.6.3). Fix a regular point $y \in \widetilde{Y}$. We say that $D \in \mathcal{U}_d$ is **bad** at y if $\dim(|\mathbb{D}| \cap \widetilde{Y})_y > p$, and we denote by $\mathcal{B}_y$ the subset of all such divisors. Note that $\mathcal{B}_y$ is a constructible subset of the linear subspace

$$\mathrm{Div}_{d,y} \stackrel{\mathrm{def}}{=} \{D \in \mathrm{Div}_d \ : \ gy \in D \ \forall \ g \in G\}.$$

Note that $\mathrm{Div}_{d,y} \ = \ \mathbb{P}(\mathcal{M}_{d,y})$ where

$$\mathcal{M}_{d,y} \stackrel{\mathrm{def}}{=} \{\sigma \in \mathcal{O}(d) \ : \ \sigma(gy) = 0 \ \forall g \in G\}$$

Suppose now that $y \in \mathbb{P}(V \oplus V_0')_{\mathrm{free}}$. Write $G = \{g_j\}_{j=1}^{\gamma}$ with $g_1 = 1$. Then the elements $y_j \equiv g_j(y)$, $j = 1, \ldots, \gamma$ are mutually distinct.

Sublemma 2.8. *Fix an integer $\ell > 0$ and let $y_1, \ldots, y_\gamma$ be distinct points in $\mathbb{P}^M = \mathbb{P}(V \oplus V_0 \oplus V_0)$. Then the map*

$$\mathcal{O}(d) \xrightarrow{\ \rho\ } \left(\mathcal{O}_{y_1}/\mathfrak{m}_{y_1}^{\ell+1}\right) \times \cdots \times \left(\mathcal{O}_{y_\gamma}/\mathfrak{m}_{y_\gamma}^{\ell+1}\right)$$

defined by sending σ to the classes of its germs $([\sigma_{y_1}], \ldots, [\sigma_{y_\gamma}])$, is surjective for all $d \geq d_\ell$ where d_ℓ depends only on ℓ, γ and M. Hence, the map

$$\mathcal{M}_{d,y} \xrightarrow{\ \rho\ } \left(\mathfrak{m}_{y_1}/\mathfrak{m}_{y_1}^{\ell+1}\right) \times \cdots \times \left(\mathfrak{m}_{y_\gamma}/\mathfrak{m}_{y_\gamma}^{\ell+1}\right)$$

is also surjective.

Proof. By general position we may choose an affine chart $\mathbb{C}^M \subset \mathbb{P}^M$ with coordinates $(\zeta_1, \ldots, \zeta_M)$ having the following properties:

(1) $\{y_1, ..., y_\gamma\} \subset \mathbb{C}^M$,
(2) $y_\gamma = 0$,
(3) $\{y_1, ..., y_{\gamma-1}\} \cap \{\zeta \ : \ \zeta_i = 0\} \ = \ \emptyset$ for $i = 1, ..., M$.

We want to show that there exists an integer d_γ such that

$$\rho_{\gamma,d} : \mathbb{C}[\zeta]_d \stackrel{\mathrm{def}}{=} \{p(\zeta) \in \mathbb{C}[\zeta] \ : \ \deg p \leq d\} \ \longrightarrow \ \prod_{i=1}^{\gamma} \left(\mathcal{O}_{y_i}/\mathfrak{m}_{y_i}^{\ell+1}\right)$$

is surjective for all $d \geq d_\gamma$. We proceed by induction. The statement is clear for $\gamma = 1$ Assume it is proved for $\gamma - 1$, and let $K_d = \ker(\rho_{\gamma-1,d})$. We want to find $d_\gamma > d_{\gamma-1}$ so that $\rho : K_d \longrightarrow \mathcal{O}_0/\mathfrak{m}_0^{\ell+1}$ is surjective for all $d \geq d_\gamma$.

For any multi-index α with $|\alpha| > \ell$ we have $P_{i,\alpha}(\zeta) \stackrel{\text{def}}{=} (\zeta - y_i)^\alpha = 0$ in $\mathcal{O}_{y_i}/\mathfrak{m}_{y_i}^{\ell+1}$ and so

$$P \stackrel{\text{def}}{=} \prod_{i=2}^{\gamma} P_{i,\alpha(i)} \ \in \ K_\ell$$

for any such choice of $\alpha(2), ..., \alpha(\gamma)$ with $|\alpha(2)| + \cdots + |\alpha(\gamma)| \leq d_{\gamma-1}$. Choose one such P and note that $P(0) \neq 0$ by our assumption (3).

Set $\hat{K}_\ell \equiv \text{span}\{\zeta^\alpha P(\zeta) \ : \ |\alpha| \leq \ell\} = \mathbb{C}[\zeta]_\ell \cdot P(\zeta)$, and note that $\hat{K}_\ell \subset K_\ell$. Furthermore, the map $\rho : \hat{K}_\ell \longrightarrow \mathcal{O}_0/\mathfrak{m}_0^\ell$ is evidently surjective. Thus, $d_\gamma = d_{\gamma-1} + \ell$ will work. ■

Sublemma 2.9. *Let Z be a germ of a subvariety of $\mathbb{P}^M$ at a point z. Fix a function $f \in \mathfrak{m}_{Z,z}$ and consider the (germ of the) subvariety $Y = Z \cap \{f = 0\}$ at z. Then for any $\ell > 0$ the ring $\mathcal{O}_{Y,z}/\mathfrak{m}_{Y,z}^{\ell+1}$ depends only on the class of f in $\mathcal{O}_{Z,z}/\mathfrak{m}_{Z,z}^{\ell+1}$*

Proof. Note that $\mathcal{O}_{Y,z}/\mathfrak{m}_{Y,z}^{\ell+1}$ is just the quotient of $\mathcal{O}_{Z,z}/\mathfrak{m}_{Z,z}^{\ell+1}$ by the principal ideal (f). ■

We now begin our main estimate of the codimension of $\mathcal{B}_Y$. To begin we fix ℓ and consider the composition

$$\mathcal{M}_{d,y} \stackrel{\rho}{\longrightarrow\!\!\!\!\rightarrow} \prod_{i=1}^{\gamma} \left(\mathfrak{m}_{y_i}/\mathfrak{m}_{y_i}^{\ell+1}\right) \xrightarrow{p \times pg_2^* \times pg_3^* \times \cdots \times pg_\gamma^*} \left(\mathfrak{m}_{\widetilde{Y},y}/\mathfrak{m}_{\widetilde{Y},y}^{\ell+1}\right) \times \cdots \times \left(\mathfrak{m}_{\widetilde{Y},y}/\mathfrak{m}_{\widetilde{Y},y}^{\ell+1}\right)$$

which we denote by $p_1 \times \cdots \times p_\gamma$, and where $d \geq d_\ell$.

Let $\mathcal{K}_1 = \ker(p_1)$. By the surjectivity of p_1 from Sublemma 2.8 we have

$$\text{codim}\, \mathcal{K}_1 \ = \ \binom{p + \gamma + \ell}{\ell} - 1$$

We now fix $f \in \mathcal{M}_{d,y}$ with $\widetilde{f} \stackrel{\text{def}}{=} p_1(f) \neq 0$. Set $\mathcal{A}_{2,f} \stackrel{\text{def}}{=} p_1^{-1}(\widetilde{f}) = f + \mathcal{K}_1$. (This is an affine space which we consider to be a vector space with f as its origin.) Set $\widetilde{Y}_2 = \widetilde{Y} \cap \{f = 0\}$ and consider the composition

$$\mathcal{A}_{2,f} \ \stackrel{p_2}{\longrightarrow} \ \mathfrak{m}_{\widetilde{Y},y}/\mathfrak{m}_{\widetilde{Y},y}^{\ell+1} \ \stackrel{\pi}{\longrightarrow} \ \mathfrak{m}_{\widetilde{Y}_2,y}/\mathfrak{m}_{\widetilde{Y}_2,y}^{\ell+1}$$

where π is the natural projection. Denote this composition by $p_{2,f}$. Now by 2.9, this map **depends only on the class** $\widetilde{f}$. By 2.8, the map p_2 restricted to $\mathcal{K}_1$ is surjective, and hence the linear map $p_{2,f}$ is also **surjective**.

Let $\mathcal{K}_{2,f} = \ker(p_{2,f})$. By the surjectivity of $p_{2,f}$ we have

$$\text{codim}\, \mathcal{K}_{2,f} \text{ in } \mathcal{A}_{2,f} \ \geq \ \binom{p + \gamma - 1 + \ell}{\ell} - 1$$

If the germ $\widetilde{Y}_2$ at y is reducible, we can apply the above construction to each of its irreducible components $\widetilde{Y}_{2,\alpha}$. This will give us a finite number of "bad" linear subspaces

$\mathcal{K}_{2,\alpha,f}$, each satisfying the codimension condition above. To keep the exposition simpler, we proceed as though the varieties were irreducible at each stage. Redoing the argument by looking at these distinct components at each stage does not change the proof qualitatively and the details are left to the reader.

We now fix $f \in \mathcal{M}_{d,y}$ with $\widetilde{f}_1 \stackrel{\text{def}}{=} p_1(f) \neq 0$ and $\widetilde{f}_2 \stackrel{\text{def}}{=} p_{2,f}(f) \neq 0$. Set $\mathcal{A}_{3,f} \stackrel{\text{def}}{=} (p_1 \times p_{2,f})^{-1}(\widetilde{f}_1, \widetilde{f}_2) = f + \mathcal{K}_{2,f}$. (This is an affine space which we consider to be a vector space with f as its origin.) Set $\widetilde{Y}_3 = \widetilde{Y}_2 \cap \{g_2^* f = 0\} = \widetilde{Y} \cap \{f = 0\} \cap \{g_2^* f = 0\}$, and consider the composition

$$\mathcal{A}_{3,f} \xrightarrow{p_3} \mathfrak{m}_{\widetilde{Y},y}/\mathfrak{m}_{\widetilde{Y},y}^{\ell+1} \xrightarrow{\pi} \mathfrak{m}_{\widetilde{Y}_3,y}/\mathfrak{m}_{\widetilde{Y}_3,y}^{\ell+1}$$

where π is the natural projection. Denote this composition by $p_{3,f}$. Now by 2.9, this map **depends only on the classes $\widetilde{f}_1$ and $\widetilde{f}_2$**. By 2.8, the map p_2 restricted to the dernel of (p_1, p_2) is surjective. Hence the linear map $p_{3,f}$ is also **surjective**.

Let $\mathcal{K}_{3,f} = \ker(p_{3,f})$. By the surjectivity of $p_{3,f}$ we have

$$\operatorname{codim} \mathcal{K}_{3,f} \text{ in } \mathcal{A}_{3,f} \;\geq\; \binom{p+\gamma-2+\ell}{\ell} - 1$$

We now consider $f \in \mathcal{M}_{d,y}$ with $\widetilde{f}_1 = p_1(f) \neq 0$, $\widetilde{f}_2 = p_{2,f}(f) \neq 0$ and $\widetilde{f}_3 = p_{3,f}(f) \neq 0$, and proceed as above. We repeat this process inductively up to the γ^{th} stage. where for $f \in \mathcal{M}_{d,y}$ having $\widetilde{f}_1 \neq 0, \ldots, \widetilde{f}_{\gamma-1} \neq 0$ we construct the spaces $\mathcal{K}_{\gamma,f} \subset \mathcal{A}_{\gamma,f}$ with

$$\operatorname{codim} \mathcal{K}_{\gamma,f} \text{ in } \mathcal{A}_{\gamma,f} \;\geq\; \binom{p+1+\ell}{\ell} - 1$$

The main observation now is the following. Set

$$\mathcal{B}_y \stackrel{\text{def}}{=} \{f \in \mathcal{M}_{d,y} \;:\; \mathbb{D} \text{ meets } \widetilde{Y} \text{ improperly at } y, \text{ where } D = \operatorname{Div}(f)\}.$$

Then we have

$$\mathcal{B}_y \;\subset\; \mathcal{K}_1 \cup \bigcup_{f \notin \mathcal{K}_1} \mathcal{K}_{2,f} \cup \bigcup_{f \notin \mathcal{K}_1 \cup \mathcal{K}_{2,f}} \mathcal{K}_{3,f} \cup \ldots$$

and the codimension estimates above show that

$$\operatorname{codim}(\mathcal{B}_y) \text{ in } \mathcal{M}_{d,y} \;\geq\; \binom{p+1+\ell}{\ell} - 1 \tag{2.9.1}$$

Suppose now that we have a divisor $D \in \mathcal{B}_Y$. By definition (2.6.3) this means that $\dim \widetilde{Y} \cap |\mathbb{D}| \geq p+1$. Therefore, by our assumption that $p \geq e(V)$, there exists a point $y \in \widetilde{Y} \cap |\mathbb{D}|$ such that $G_y = \{1\}$ (i.e., the orbit of y must have γ distinct points). Then we must have $D \in \mathcal{B}_y$. It follows that

$$\mathcal{B}_Y \;\subset \bigcup_{\mathbb{P}(V \oplus V_0')_{\text{free}} \cap \widetilde{Y}} \mathcal{B}_y$$

and so

$$\operatorname{codim}(\mathcal{B}_Y) \;\geq\; \binom{p+1+\ell}{\ell} - 1 - \dim \widetilde{Y} = \binom{p+1+\ell}{\ell} - p - \gamma - 1,$$

for $d \geq d_\ell$. In particular, given any $M > 0$, one can find ℓ such that $\binom{p+1+\ell}{\ell} - p - \gamma - 1 > M$ and hence, for $d \geq d_\ell$ one has $f(d) = \operatorname{codim}(\mathcal{B}_Y) > M$, proving Lemma 2.7. ■

We now conclude the proof of Proposition 2.4. Given a cycle $c = \sum_i n_i Y_i \in \mathcal{C}_{p+\gamma}$ we set

$$\mathcal{B}_c \overset{\text{def}}{=} \bigcup_i \mathcal{B}_{Y_i}$$

and observe that Lemma 2.7 holds with $\mathcal{B}_Y$ replaced by $\mathcal{B}_c$. Now in Proposition 2.4 it will suffice to let X be a finite union of components of $\mathcal{C}_{p+\gamma}$. For such X

$$\mathcal{B}_X \overset{\text{def}}{=} \bigcup_{c \in X} \mathcal{B}_c$$

is a constructible subset of Div_d with $\operatorname{codim}(\mathcal{B}_X) \;\geq\; f(d) - \dim X$. Hence,

(2.9.2) $$\operatorname{codim}(\mathcal{B}_X) >> 0$$

for all d sufficiently large. For any such d consider the family of lines $\mathcal{L}_\Delta$ in the projective space Div_d which pass through the point Δ (cf. (2.5.1)). For amost all lines $\ell \in \mathcal{L}_\Delta$, the intersection $\ell \cap \mathcal{B}_X$ is finite (in fact, consists only of $\{\Delta\}$) by (2.9.2). For any such ℓ, there exists $D \in \ell$ such that $D_t \equiv (1-t)\Delta + tD \notin \mathcal{B}_X$ for all $0 < t \leq 1$. The family $\Psi_t = \Psi_{D_t}$ has the desired property. This proves Proposition 2.4 ■

To prove Assertion 2 we employ the fact that an equivariant map $\Psi : A \to B$ between G-CW-complexes is a G- homotopy equivalence if Ψ restricts to an ordinary homotopy equivalence of fixed-point sets

$$\Psi \,:\, A^H \;\longrightarrow\; B^H$$

for all subgroups $H < G$. (See [tD, p. 107]).

Fix $H < G$. Let us abbreviate notation by letting $\mathcal{C}_{p+\gamma,d}$ denote the cycles of degree d in $\mathcal{C}_{p+\gamma}$, and setting $\mathcal{Z}_{p+\gamma} = \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0 \oplus V_0)$ and $\mathcal{T}\mathcal{Z}_{p+\gamma} = \mathcal{T}\mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0 \oplus V_0)$, etc..

We want to prove that the homomorphism

$$j_* \,:\, (\mathcal{T}\mathcal{Z}_{p+\gamma})^H \;\longrightarrow\; (\mathcal{Z}_{p+\gamma})^H$$

induced by the inclusion j is an isomorphism for all $k \geq 0$.

Fix k and suppose that $f : S^k \longrightarrow (\mathcal{Z}_{p+\gamma})^H$ is a continuous map. Then, there is a $D > 0$ such that $f(S^k) \subset \mathcal{Z}^H_{p+\gamma,\leq D}$, where $\mathcal{Z}_{p+\gamma,\leq D}$ is the image of the compact set $\coprod_{r+s\leq D} \mathcal{C}_{p+\gamma,r} \times \mathcal{C}_{p+\gamma,s}$ under the canonical projection $\pi : \mathcal{C}_{p+\gamma} \times \mathcal{C}_{p+\gamma} \to \mathcal{Z}_{p+\gamma}$. Denote $X = \coprod_{r\leq D} \mathcal{C}_{p+\gamma,r}$ and let $\tilde{\Psi}_t : \mathcal{Z}_{p+\gamma} \to \mathcal{Z}_{p+\gamma}$ be the extension of the monoid morphism

$\Psi_t : \mathcal{C}_{p+\gamma} \to \mathcal{C}_{p+\gamma}$ from Proposition 2.4 to a G-equivariant group homomorphism. Then the composition $\tilde{\Psi}_t \circ f$ gives a homotopy $F_t : S^k \to (\mathcal{Z}_{p+\gamma})^H$ satisfying

$$F_0 = d^\gamma \cdot f \qquad \text{and} \qquad F_t(S^k) \subset (\mathcal{T}\mathcal{Z}_{p+\gamma})^H \quad \forall\, t > 0,$$

where d is as in Proposition 2.4.

This proves that for any element $\alpha \in \pi_k\left(\mathcal{Z}_{p+\gamma}\right)^H$, we have $d^\gamma \alpha \in \text{Image}(j_*)$ for all d sufficiently large. If we choose d_1, d_2 relatively prime, then d_1^γ, d_2^γ are relatively prime, and this proves that $\alpha \in \text{Image}(j_*)$. Thus j_* is surjective.

Consider now a continuous map $f : D^{k+1} \longrightarrow (\mathcal{Z}_{p+\gamma})^H$ such that $f(\partial D^{k+1}) \subset (\mathcal{T}\mathcal{Z}_{p+\gamma})^H$. Using Proposition 2.4 and the same argument as above one finds d_0 such that for each $d \geq d_0$ there is a homotopy $F_t : D^{k+1} \longrightarrow (\mathcal{Z}_{p+\gamma})^H$ with

$$F_0 = d^\gamma \cdot f \qquad \text{and} \qquad F_t(D^{k+1}) \subset (\mathcal{T}\mathcal{Z}_{p+\gamma})^H \quad \forall t > 0.$$

This proves that for any element $\beta \in \pi_k\left(\mathcal{T}\mathcal{Z}_{p+\gamma}\right)^H$ with $j_*\beta = 0$, there exists d_0 such that $d^\gamma \beta = 0$ for all $d \geq d_0$. Choosing d_1, d_2 relatively prime as above shows that $\beta = 0$. Thus j_* is injective. This completes the proof of Assertion 2. Assertions 1 and 2 clearly imply the theorem. ■

§**3. Some immediate consequences.** Let G be a finite group of order γ and let V be a finite dimensional G-module. Denote by V_0 the regular representation of G. Then beginning with any $p < \dim V - 1$ we have a sequence of G-equivariant homomorphisms

$$\mathcal{Z}_p(\mathbb{P}(V)) \xrightarrow{\Sigma^{V_0}} \mathcal{Z}_{p+\gamma}(\mathbb{P}(V \oplus V_0)) \xrightarrow{\Sigma^{V_0}} \mathcal{Z}_{p+2\gamma}(\mathbb{P}(V \oplus 2V_0)) \xrightarrow{\Sigma^{V_0}} \dots \tag{3.0.1}$$

given by algebraic suspension to V_0. We shall see in the next section that these homomorphisms are definitely not G-equivalences in general. However, Lemma 2.2 (3) and the Theorem 2.3 immediately imply the following.

Theorem 3.1. (The Stability Theorem) *Let q be the smallest prime divisor of $\gamma = |G|$. Then the homomorphism*

$$\Sigma^{V_0} : \mathcal{Z}_{p+m\gamma}(\mathbb{P}(V \oplus mV_0)) \longrightarrow \mathcal{Z}_{p+(m+1)\gamma}(\mathbb{P}(V \oplus (m+1)V_0))$$

is a G-homotopy equivalence for all

$$m \geq \tfrac{q}{\gamma(q-1)} \left\{ \dim V - (p+1) + \tfrac{\gamma}{q} \right\}.$$

For groups of prime order Lemma 2.2(2) and Theorem 2.3 give a more refined result.

Theorem 3.2. *Let $G = \mathbb{Z}/q$ where q is prime. Then*

$$\Sigma^{V_0} : \mathcal{Z}_p(\mathbb{P}(V)) \longrightarrow \mathcal{Z}_{p+q}(\mathbb{P}(V \oplus V_0))$$

is a G-homotopy equivalence for all $p \geq \max_{\rho \in \widehat{G}} \dim V_\rho$. In particular, Σ^{V_0} gives the series of G-homotopy equivalences

$$\mathcal{Z}_p(\mathbb{P}(kV_0)) \xrightarrow{\cong} \mathcal{Z}_{p+q}(\mathbb{P}((k+1)V_0)) \xrightarrow{\cong} \mathcal{Z}_{p+2q}(\mathbb{P}((k+2)V_0)) \xrightarrow{\cong} \dots$$

whenever $p \geq k$.

§4. **Important examples.** At first glance the hypothesis that $p > e(V)$ in Theorem 2.3 seems awkward and technical since the function $e(V)$ is not easy to compute in general. However there is very good evidence to indicate that this hypothesis is sharp.

Certainly without the hypothesis the proof fails. When $p \leq e(V)$, cycles can be "trapped" in fixed-point sets of subgroups of G so that no equivariant move will displace them to be "transversal". Furthermore, calculations show that often when this hypothesis fails, such cycles cannot in any way be equivariantly displaced. In such cases Σ^{V_0} is **not** a G-equivalence, and these cycles are contributing to new elements in $\pi_0 \mathcal{Z}^G$. We summarize some of these calculations here.

Let G be a abelian of order γ and for $m \geq 0$, $k > 0$, set $\mathcal{Z}(m,k) \stackrel{\text{def}}{=} \mathcal{Z}_{m\gamma}((\mathbb{P}(k+m)V_0))$. Consider the sequence of G-equivariant suspension maps

$$\mathcal{Z}(0,k) \xrightarrow{\Sigma_{V_0}} \mathcal{Z}(1,k) \xrightarrow{\Sigma_{V_0}} \mathcal{Z}(2,k) \xrightarrow{\Sigma_{V_0}} \mathcal{Z}(3,k) \xrightarrow{\Sigma_{V_0}} \dots$$

By Theorem 2.3 these maps become G-equivalences starting with the first $\mathcal{Z}(m,k)$ where $m(\gamma - 1) > k$. However, these maps are certainly not G-equivalences at the outset. This is seen already at the level of π_0

In [LLM$_1$] direct calculations of $\pi_0(\mathcal{Z}(m,k)^G)$ were carried out, and in particular, the following was established.

Fact 1. The inclusions $\mathbb{P}(kV_0) \subset \mathbb{P}((k+1)V_0)$ induce isomorphisms

$$P_G \stackrel{\text{def}}{=} \pi_0(\mathcal{Z}(0,1)^G) \xrightarrow{\cong} \pi_0(\mathcal{Z}(0,2)^G) \xrightarrow{\cong} \pi_0(\mathcal{Z}(0,3)^G) \xrightarrow{\cong} \dots$$

and $P_G \cong \mathbb{Z} \oplus \widetilde{P}_G$ where $\widetilde{P}_G$ is a torsion group of rank $< \gamma$. In particular, for prime integers q,

$$\widetilde{P}_{(\mathbb{Z}/q\mathbb{Z})} \cong (\mathbb{Z}/q\mathbb{Z})^{q-1}.$$

Fact 2. The inclusion-suspension maps

$$\dots \hookrightarrow \pi_0(\mathcal{Z}(k,2k)^G) \hookrightarrow \pi_0(\mathcal{Z}(k+1,2(k+1))^G) \hookrightarrow \pi_0(\mathcal{Z}(k+2,2(k+2))^G) \hookrightarrow \dots$$

are injective and not surjective for all k. Set $Q_G \stackrel{\text{def}}{=} \lim_{k\to\infty} \pi_0\big(\mathcal{Z}(k,2k)^G\big)$. Then Q_G can be written as the localization of a graded ring

$$Q_G \;=\; \mathbb{H}^*(G)_{\mathcal{M}}$$

where $\mathbb{H}^*$ is a functor on abelian groups with the following properites:

(1) $\mathbb{H}^0 \;=\; \mathbb{Z}$ and $\mathbb{H}^1 \;\cong\; G$.

(2) $\mathbb{H}^*(G_1 \oplus G_2) \;\cong\; \mathbb{H}^*(G_1) \otimes \mathbb{H}^*(G_2)$

(3) If G is cyclic of prime order q, then there is a natural isomorphism

$$\mathbb{H}^*(G) \;\cong\; H^{2*}(G;\, \mathbb{Z})$$

with cohomology with coefficients in the trivial G-module $\mathbb{Z}$.

and where $\mathcal{M}$ is the multiplicative system generated by $1+x$ for $x \in \mathbb{H}^1(G) \cong G$.

Note that Q_G embeds in the completion $\widetilde{\mathbb{H}}^*(G)$ of $\mathbb{H}^*(G)$ with repsect to the aumentation ideal.

Clearly P_G and Q_G are radically different. P_G is a finitely generated abelian group and Q_G is a ring which is infinitely generated additively. When $G = \mathbb{Z}/q\mathbb{Z}$ for q prime, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow Q_G \longrightarrow \widetilde{Q}_G \longrightarrow 0$$

where

$$\widetilde{Q}_G \;\cong\; (\mathbb{Z}/q\mathbb{Z})[x]_{(1+x)}.$$

At intermediate stages there is a short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_0\big(\mathcal{Z}(k,2k)^G\big) \longrightarrow \widetilde{Q}_G^{(k)} \longrightarrow 0$$

and an additive isomorphism

$$\widetilde{Q}_G^{(k)} \;\cong\; (\mathbb{Z}/q\mathbb{Z})[x,\bar{x}]/\big\langle (1+x)^{k+1}, (1+\bar{x})^{k+1}, (1+x)(1+\bar{x})-1\big\rangle$$

Another set of examples comes from of divisors. Suppose G is abelian and let V be a complex G-module of dimension $n+1$. Then it can be shown that $\mathcal{Z}_{n-1}(\mathbb{P}(V))_o = \varinjlim_d \mathrm{Div}_d = \varinjlim_d \mathbb{P}(S^dV)$ where S^dV is the d^{th} symmetric tensor power of V and $\mathcal{Z}_{n-1}(\mathbb{P}(V))_o$ denote the divisors of degree zero in $\mathbb{P}(V)$. As in the proof of 2.2 we see that $\#(\pi_0\mathbb{P}(S^dV)^G)$ is exactly the number of distinct characters of G which appear in the representation S^dV.

If $G = \mathbb{Z}/q\mathbb{Z}$ and V is non-trivial, then all characters occur in S^dV for $d > 2q$. To see this note that if S^dV has characters a and b, then S^dV has all characters $na + mb$ for non-negative integers n, m with $n + m = d$. One can solve the system

$$\begin{aligned} na + mb &\equiv c (\mathrm{mod}\ q) \\ n + m &\equiv d (\mathrm{mod}\ q) \end{aligned}$$

for all c, and for d large we can solve the second equation in positive integers. Therefore if V has at least two distinct characters, then $\pi_0\mathcal{Z}_{n-1}(\mathbb{P}(V))^G \cong \mathbb{Z}/q\mathbb{Z}$. Thus if V is the trivial 2-dimensional module, we have

$$\{0\} = \pi_0\mathcal{Z}_0(\mathbb{P}(V))^G \xrightarrow{\Sigma^{V_0}} \pi_0\mathcal{Z}_q(\mathbb{P}(V\oplus V_0))^G = \mathbb{Z}/q\mathbb{Z}.$$

More generally one has $\pi_0\mathcal{Z}_{n+\gamma-1}(\mathbb{P}(V\oplus V_0))^G \cong G$, but $\pi_0\mathcal{Z}_{n+\gamma-1}(\mathbb{P}(V))^G \neq G$ if say V is a representation of a non-trivial quotient. These examples show again that our hypothesis $p \geq e(V)$ in Theorem 2.3 is sharp.

§**5. Equivariant suspension for varieties.** The proof of Theorem 2.3 goes through without change if one replaces $\mathbb{P}(V)$ with a G-invariant subvariety of $\mathbb{P}(V)$.

Theorem 5.1. *Let G be a finite group and V a finite-dimensional complex G-module. Suppose that $X \subset \mathbb{P}(V)$ is a G-invariant algebraic subvariety, and set*

$$e(X) \stackrel{\text{def}}{=} \max_{\substack{g\in G \\ g\neq\{1\}}} \dim \left(\Sigma^{V_0}X\right)^g$$

Let V_0 denote the regular representation of V_0 and $\gamma = \dim V_0 = |G|$. Then for any $p \geq e(X)$, the suspension homomorphism

$$\Sigma^{V_0} : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p+\gamma}(\Sigma^{V_0}X)$$

is a G-homotopy equivalence.

Note 5.2. In Theorem 5.1 we can take $\dim\emptyset = -\infty$. That is, if G acts freely on X, then the equivariant suspension theorem holds for all $p \geq 0$. One needs only to check the case $p = 0$. The proof goes through here because the action of G on $\mathbb{P}(V \oplus V_0')$ is free outside $\mathbb{P}(V_0')$ and the subvarietes $\widetilde{Y} \cap \mathbb{D}$ never meet $\mathbb{P}(V_0')$ because D is chosen so that $\mathbb{D} \cap \mathbb{P}(V_0') = \emptyset$.

Proof of Theorem B. The Stability Theorem B stated in the introduction is proved as follows. There are two G-equivariant projections

$$\pi : \Sigma^{V_0}X - \mathbb{P}(V_0) \to X \qquad \text{and} \qquad \pi' : \Sigma^{V_0}X - X \to \mathbb{P}(V_0).$$

If $v \in \Sigma^{V_0}X$ is fixed by $g \in G$ and if $v \notin \mathbb{P}(V_0) \cup X$, then both $\pi(v)$ and $\pi'(v)$ are also g-fixed and belong to the same g-character spaces of $V \oplus V_0$. We estimate $e(X)$ by observing that

$$\dim \left(\Sigma^{V_0}X\right)^g \leq \max_\rho \dim \Sigma^{V_\rho}(X^g) = \dim X^g + \tfrac{\gamma}{q}$$

where V_ρ denotes the subspace of V_0 with g-character ρ. Therefore,

$$\dim \left(\Sigma^{(m+1)V_0}X\right)^g \leq \dim X^g + \tfrac{(m+1)\gamma}{q},$$

from which it follows that

$$e(\not\Sigma^{mV_0} X) \ \le \ e_0(X) + \tfrac{(m+1)\gamma}{q}.$$

Theorem B now follows from Theorem A, which is equivalent to Theorem 5.1 above.

§**6. The quaternionic suspension theorem.** The proof of Theorem 2.3 given above carries over with no essential change to prove a "quaternionic suspension theorem". The algebraic subvarieties if $\mathbb{P}_{\mathbb{C}}(\mathbb{H}^n)$ which are invariant under the involution induced by multiplication by j, have interesting geometric structure and arise naturally for example in certain twistor constructions. Families of such j-invariant cycles also play an interesting role in topology. For example, using them one can make quaternionic analogues of the constructions in [BLLMM] concerning the total Chern and Stiefel- Whitney classes. A key to computing the topological structure of these spaces are Theorems 6.3 and 6.5 below. These applications are carried out in [LLM_2].

Let $\mathbb{H}$ denote the quaternions with standard basis $1, i, j, k$, and let $\mathbb{C}^2 \xrightarrow{\cong} \mathbb{H}$ be the canonical isomorphism which associates to a pair (u, v) the quaternion $u + vj$. For any n this gives a canonical complex isomorphism

$$\mathbb{C}^{2n} \ \xrightarrow{\cong} \ \mathbb{H}^n.$$

Under this identification left scalar multiplication by j in $\mathbb{H}^n$ becomes the complex-antilinear map

$$j : \mathbb{C}^{2n} \ \longrightarrow \ \mathbb{C}^{2n}$$

with $j^2 = -\text{Id}$, given in coordinates $(u, v) = (u_1, ..., u_n, v_1, ..., v_n)$ by

$$j(u, v) \ = \ (-\overline{v}, \overline{u}).$$

This induces a **fixed-point free, antiholomorphic** map

(6.0.1) $$\overline{j} : \mathbb{P}_{\mathbb{C}}^{2n-1} \ \longrightarrow \ \mathbb{P}_{\mathbb{C}}^{2n-1}$$

with $\overline{j}^2$ =Id. There is a natural fibration $\mathbb{P}_{\mathbb{C}}^{2n-1} \longrightarrow \mathbb{P}_{\mathbb{H}}^{n-1}$ whose fibres are $\overline{j}$-invariant projective lines, and on these lines $\overline{j}$ is linearly equivalent to the antipodal map.

The involution $\overline{j}$ carries algebraic subvarieties of $\mathbb{P}_{\mathbb{C}}^{2n-1}$ to themselves and induces a continuous, antiholomorphic involution

$$\overline{j}_* : \mathcal{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^{2n-1}) \ \longrightarrow \ \mathcal{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^{2n-1})$$

for all p and d. This involution is additive on the monoid $\mathcal{C}_p(\mathbb{P}_{\mathbb{C}}^{2n-1}) = \coprod_d \mathcal{C}_{p,d}(\mathbb{P}_{\mathbb{C}}^{2n-1})$ and extends to a continuous involutive automorphism

$$\overline{j}_* : \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n-1}) \ \longrightarrow \ \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n-1}).$$

There is a $\overline{j}$-equivariant algebraic suspension homomorphism

$$\Sigma^{\mathbb{H}} : \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n-1}) \longrightarrow \mathcal{Z}_{p+2}(\mathbb{P}_{\mathbb{C}}^{2n+1}) \tag{6.0.2}$$

defined as follows. Write $\mathbb{H}^{n+1} = \mathbb{H}^n \oplus \mathbb{H}$ and set $\mathbb{P}_{\mathbb{C}}^{2n-1} = \mathbb{P}_{\mathbb{C}}(\mathbb{H}^n \oplus \{0\})$, $\mathbb{P}_{\mathbb{C}}(\mathbb{H}) = \mathbb{P}(\{0\} \oplus \mathbb{H})$. Then given a p-cycle c on $\mathbb{P}_{\mathbb{C}}^{2n-1}$, we set

$$\Sigma^{\mathbb{H}}(c) = c\#\mathbb{P}_{\mathbb{C}}(\mathbb{H}).$$

If $V \subset \mathbb{P}_{\mathbb{C}}^{2n-1}$ is an irreducible subvariety with homogeneous cone $\widetilde{V} \stackrel{\text{def}}{=} \bigcup\{\ell : \ell \in V\} \subset \mathbb{H}^n$, then the homogeneous cone of $\Sigma^{\mathbb{H}}(V)$ in $\mathbb{H}^{n+1}$ is just

$$\widetilde{\Sigma^{\mathbb{H}}(V)} = \widetilde{V} \oplus \mathbb{H} \subset \mathbb{H}^n \oplus \mathbb{H}.$$

From this one sees that $\Sigma^{\mathbb{H}}$ commutes with $\overline{j}_*$. Our first main result is the following

Theorem 6.1. *For all p and n with $0 \le p < 2n-1$, the algebraic suspension map*

$$\Sigma^{\mathbb{H}} : \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n-1}) \longrightarrow \mathcal{Z}_{p+2}(\mathbb{P}_{\mathbb{C}}^{2n+1}),$$

as a map of $\mathbb{Z}/2\mathbb{Z}$-spaces under the action of $\overline{j}_*$ is a $\mathbb{Z}/2\mathbb{Z}$-homotopy equivalence.

Corollary 6.2. *Let $\mathcal{Z}_p^{\mathbb{H}}(n) \subset \mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n-1})$ denote the $\overline{j}_*$-fixed point set, i.e., the group of $\overline{j}_*$-invariant algebraic p-cycles. Then the algebraic suspension homomorphism*

$$\Sigma^{\mathbb{H}} : \mathcal{Z}_p^{\mathbb{H}}(n) \longrightarrow \mathcal{Z}_{p+2}^{\mathbb{H}}(n+1)$$

is a homotopy equivalence.

Proof. The proof of 6.1 follows exactly the arguments given for the proof of Theorem 2.3. Certain homomorphisms of local rings which were complex linear are complex antilinear in this case. However, all dimension estimates go through without change. Since the action of $G = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{P}_{\mathbb{C}}^{2n-1}$ is free, there are no restrictions on the dimension p. Details are left to the reader. ∎

In analogy with [Lam] and [LM$_2$] we now consider the "Galois quotient"

$$\widetilde{\mathcal{Z}}_p^{\mathbb{H}}(n) \stackrel{\text{def}}{=} \mathcal{Z}_p^{\mathbb{H}}(n)/\mathcal{A}_p(n)$$

of the $\overline{j}_*$-fixed cycles by the subgroup $\mathcal{A}_p(n) \stackrel{\text{def}}{=} (1+\overline{j}_*)\mathcal{Z}_p(\mathbb{P}_{\mathbb{C}}^{2n+1})$ of $\overline{j}_*$- averaged ones. Note that this is the (topological) 2-torsion group generated by the irreducible quaternionic subvarieties of dimension p. For $p = 1$ these subvarieties can be thought of as the "non-orientable algebraic curves".

Corollary 6.3. *For all p, n with $0 \le p < 2n-1$ the algebraic suspension homomorphism*

$$\Sigma\!\!\!/^{\mathbb{H}} : \widetilde{\mathcal{Z}}_p^{\mathbb{H}}(n) \longrightarrow \widetilde{\mathcal{Z}}_{p+2}^{\mathbb{H}}(n+1)$$

is a homotopy equivalence.

Proof. This is a direct consequence of the following two statements

(1) $\Sigma\!\!\!/^{\mathbb{H}} : \mathcal{A}_p(n) \longrightarrow \mathcal{A}_{p+2}(n+1)$ is a homotopy equivalence.

(2) $\mathcal{A}_p(n) \longrightarrow \mathcal{Z}_p^{\mathbb{H}}(n) \longrightarrow \widetilde{\mathcal{Z}}_p^{\mathbb{H}}(n)$ is a quasi-fibration sequence.

To establish (1), one checks that the arguments for Theorem 2.3 can be carried through for the subgroup $\mathcal{A}_p(n)$. Statement (2) follows from [LLM$_1$,§3.4] and [M], once one identifies $\widetilde{\mathcal{Z}}_p^{\mathbb{H}}(n)$ with the homotopy quotient $B(\mathcal{Z}_p^{\mathbb{H}}(n), \mathcal{A}_p(n), *)$. ∎

It is useful, in light of Theorem 6.1, to introduce the following notation. Let $\mathcal{Z}_{\mathbb{H}}^q(n)$ denote the $\overline{j}_*$-fixed cycles of complex *codimension* q in $\mathbb{P}_{\mathbb{C}}^{2n-1}$, and let $\widetilde{\mathcal{Z}}_{\mathbb{H}}^q(n)$ denote its analogue for the quotient. (Thus $\mathcal{Z}_{\mathbb{H}}^q(n) = \mathcal{Z}_{2n-1-q}^{\mathbb{H}}(n)$.) The corollaries above assert that the homotopy type of these spaces depends on q but not on n, so we abreviate the notation to $\mathcal{Z}_{\mathbb{H}}^q$ and $\widetilde{\mathcal{Z}}_{\mathbb{H}}^q$.

Theorem 6.4. *For all integers $q \ge 0$ there are a homotopy equivalences:*

$$\mathcal{Z}_{\mathbb{H}}^{2q+1} \cong \prod_{i=0}^{q} K(\mathbb{Z}, 4i) \times \prod_{i=0}^{q} K(\mathbb{Z}/2\mathbb{Z}, 4i+1) \qquad \textit{and}$$

$$\widetilde{\mathcal{Z}}_{\mathbb{H}}^{2q+1} \cong \mathrm{pt}$$

Proof. By Corollary 6.2 there are homotopy equivalences

$$\mathcal{Z}_{\mathbb{H}}^{2q+1} \cong \mathcal{Z}_0^{\mathbb{H}}(\mathbb{P}_{\mathbb{C}}^{2q+1}) \qquad \text{and} \qquad \widetilde{\mathcal{Z}}_{\mathbb{H}}^{2q+1} \cong \mathcal{Z}_0^{\mathbb{H}}(\mathbb{P}_{\mathbb{C}}^{2q+1})/(1+\overline{j}_*)\mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^{2q+1}).$$

Now $\overline{j}$ acts freely on $\mathbb{P}_{\mathbb{C}}^{2q+1}$. Hence the group of $\overline{j}_*$-fixed 0-cycles coincides with the group of averaged 0-cycles and is homeomorphic to $\mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^{2q+1}/\mathbb{Z}_2)$. By [DT] we have that

$$\pi_* \mathcal{Z}_0(\mathbb{P}_{\mathbb{C}}^{2q+1}/\mathbb{Z}_2) \cong H_*(\mathbb{P}_{\mathbb{C}}^{2q+1}/\mathbb{Z}_2, \mathbb{Z})$$

and these latter groups are easily computed from the Serre spectral sequence. ∎

§**7. Quaternionic suspension for varieties.** As noted in §5, the arguments given for cycles in projective space carry over to cycles in any invariant subvariety. This gives us a the following general version of Theorem 6.1.

Theorem 7.1. *Let $X \subset \mathbb{P}^{2n-1}_{\mathbb{C}}$ be an algebraic subvariety which is invariant under the involution $\overline{j}$. (Such subvarieties can be thought of as "quaternionic".) Then for all p with $0 \leq p \leq \dim X$, the quaternionic algebraic suspension homomorphism*

$$\Sigma^{\mathbb{H}} : \mathcal{Z}_p(X) \longrightarrow \mathcal{Z}_{p+2}(\Sigma^{\mathbb{H}} X),$$

as a map of $\mathbb{Z}/2\mathbb{Z}$-spaces under the action of $\overline{j}_*$ is a $\mathbb{Z}/2\mathbb{Z}$-homotopy equivalence.

References

[BLLMM] Boyer, C.P., H.B. Lawson, Jr, P. Lima-Filho, B. Mann, and M.-L. Michelson, *Algebraic cycles and infinite loop spaces,* Inventiones Math. , **113** (1993), 373-388.

[tD] tom Dieck, T., *Transformation Groups,* de Gruyter Studies in Mathematics **8**, Walter de Gruyter, New York, 1987.

[DT] Dold, A. and R. Thom, *Quasifaserungen und unendliche symmetrische produkte,* Ann. of Math. (2) **67** (1956), 230-281.

[FL_1] Friedlander, E. and H.B. Lawson, Jr., *A theory of algebraic cocycles*, Annals of Math., **136** (1992), 361-428.

[FL_2] Friedlander, E. and H.B. Lawson, Jr., *Moving algebraic cycles of bounded degree*, preprint, 1994.

[Fu] Fulton, W., *Intersection theory*, Springer-Verlag, New York, 1984.

[H] Harris, J., *Algebraic Geometry*, Springer-Verlag, New York, 1992.

[Lam] Lam T.-K., *Spaces of Real Algebraic Cycles and Homotopy Theory*, Ph.D. thesis, SUNY, Stony Brook, 1990.

[L_1] Lawson, H.B. Jr, *Algebraic cycles and homotopy theory*, Ann. of Math. **129** (1989), 253-291.

[L_2] Lawson, H.B. Jr, *On spaces of algebraic cycles*, Surveys in Differential Geometry, A.M.S., Providence, 1995.

[LLM_1] Lawson, H.B. Jr, P.C. Lima-Filho and M.-L. Michelsohn, *Finite groups, algebraic cycles and equivariant cohomology theories*, to appear in the Proceedings of the London Mathematical Society.

[LLM_2] Lawson, H.B. Jr, P.C. Lima-Filho and M.-L. Michelsohn, *Quaternionic algebraic cycles*, (to appear).

[LM_1] Lawson, H.B. Jr. and M.-L. Michelsohn, *Algebraic cycles, Bott periodicity, and the Chern characteristic map*, in *The Mathematical Heritage of Hermann Weyl,* A.M.S., Providence, 1988, pp. 241-264.

[LM_2] Lawson, H.B. Jr and M.-L. Michelsohn, *Algebraic cycles and group actions* in *Differential Geometry*, Longman Press, 1991, pp. 261-278.

[Li_1] Lima-Filho, P.C. *Lawson homology for quasiprojective varieties,* Compositio Math **84** (1992), 1-23.

[Li_2] Lima-Filho, P.C., *Completions and fibrations for topological monoids,* Trans. Amer. Math. Soc. **340** no. 1 (1993), 127-146.

[Li_3] Lima-Filho, P.C., *On the generalized cycle map,* J. Diff. Geom. **38** (1993), 105-130.

[Li_4] Lima-Filho, P.C., *On the equivariant homotopy of free abelian groups on G- spaces and G-spectra*, Preprint, 1994.

[Li_5] Lima-Filho, P.C., *On the topological group structure of algebraic cycles,* Duke Math. J. **75** (1994), no. 2,467–491.

[M] May, J. P., *Classifying Spaces and Fibrations,* Mem. Amer. Math. Soc. **155**, 1975.

[Mo] Mostovoy, J., *Quaternion flavored cycle spaces,* I.C.M.S. Preprint, Edinburgh University, 1975.