Poincaré Duality via Forms and Currents

Lemma 1. *Let*

$$A \xrightarrow{\ d\ } B \xrightarrow{\ \delta\ } C$$

be continuous linear maps of Frêchet spaces with $\delta \circ d = 0$. Suppose d and δ have closed range. Consider the dual sequence

$$A' \xleftarrow{\ d'\ } B' \xleftarrow{\ \delta'\ } C'.$$

(These adjoint maps automatically have closed range by a general theorem.) Then

$$H(B') \cong H(B)'$$

i.e.,

$$\frac{\ker d'}{\mathrm{Im}\delta'} \cong \left\{\frac{\ker\delta}{\mathrm{Im} d}\right\}'$$

Proof. An $L' \in B'$ with $d' \circ L \equiv L \circ d = 0$ determines

$$\widetilde{L} : \frac{B}{dA} \to \mathbf{R}.$$

Suppose $L_1 = L + \delta' M \equiv M \circ \delta$ for some $M \in C'$. Then $L_1 - L = M \circ \delta = 0$ on $\ker(\delta)$. Hence we have a map

$$\frac{\ker d'}{\mathrm{Im}\delta'} \to \left\{\frac{\ker\delta}{\mathrm{Im} d}\right\}' \tag{1}$$

by restriction and quotient.

Surjectivity. Let

$$\widetilde{\lambda} : \frac{\ker\delta}{\mathrm{Im} d} \to \mathbf{R}$$

be a given continuous linear map, and lift $\widetilde{\lambda}$ to

$$\lambda : \ker\delta \to \mathbf{R}.$$

Since $\ker\delta$ is closed, there exists an extension of λ

$$L : B \to \mathbf{R}$$

by the Hahn-Banach Theorem. Hence (1) is surjective.

Injectivity. Suppose we are given $L_0, L_1 \in B'$ with $L_0 \circ d = L_1 \circ d = 0$ and

$$L_0\big|_{\ker\delta} \;=\; L_0\big|_{\ker\delta}.$$

We want to show that

$$L_1 - L_0 \;=\; \delta' M \;\equiv\; M \circ \delta$$

for some $M \in C'$. Now δ gives an isomorphism

$$\delta : \frac{B}{\ker \delta} \;\to\; \mathrm{Im}\,\delta \;\subset\; C,$$

and by assumption $\mathrm{Im}\,\delta \subset C$ is **closed**. Again by Hahn-Banach the continuous linear map

$$M_0 \;\equiv\; (L_1 - L_0) \circ \delta^{-1} : \mathrm{Im}\,\delta \;\to\; \mathbf{R}$$

extends to a continuous linear map $M : C \to \mathbf{R}$. qed.

Suppose now that X is an oriented n-dimensional manifold. We have the two complexes of forms and their dual complexes currents:

$$\mathcal{E}^0(X) \;\to\; \mathcal{E}^1(X) \;\to\; \ldots \to \mathcal{E}^n(X)$$

$$\mathcal{E}'_0(X) \;\leftarrow\; \mathcal{E}'_1(X) \;\leftarrow\; \ldots \leftarrow \mathcal{E}'_n(X)$$

and

$$\mathcal{D}^0(X) \;\to\; \mathcal{D}^1(X) \;\to\; \ldots \to \mathcal{D}^n(X)$$

$$\mathcal{D}'_0(X) \;\leftarrow\; \mathcal{D}'_1(X) \;\leftarrow\; \ldots \leftarrow \mathcal{D}'_n(X)$$

Now it is a general (and surprising) fact that the range of d is always closed. When the cohomology of X is finite dimensional (e.g., when X is compact), this follows from the general theorem that: *when the image of a continuous linear map has finite codimension, it is closed.* This gives the following reult.

$$H_p(\mathcal{E}'_*(X), d) \;=\; H^p(\mathcal{E}^*(X), d)'$$

$$H_p(\mathcal{D}'_*(X), d) \;=\; H^p(\mathcal{D}^*(X), d)'$$

We know that with

$$\mathcal{E}'_p(X) \;\equiv\; \mathcal{E}'^{\,n-p}(X) \quad \text{and} \quad \mathcal{D}'_p(X) \;\equiv\; \mathcal{D}'^{\,n-p}(X)$$

we have that

$$0 \;\to\; \mathbf{R} \;\to\; \mathcal{E}^0 \;\to\; \mathcal{E}^1 \;\cdots\; \to\; \mathcal{E}^n \;\to\; 0$$

and

$$0 \;\to\; \mathbf{R} \;\to\; \mathcal{D}'^0 \;\to\; \mathcal{D}'^1 \;\cdots\; \to\; \mathcal{D}'^n \;\to\; 0$$

are resolutions of the constant sheaf $\mathbf{R}$. Hence $H^k(X;\mathbf{R})$ (and $H^k_{\mathrm{cpt}}(X;\mathbf{R})$) are computed by taking the cohomology of the complex of global sections (or global sections with compact support respectively). When X is compact, things simplify, since $\mathcal{D}^*(X) = \mathcal{E}^*(X)$, and we conclude the following.

Theorem 2. (Poincaré Duality). *If X is compact and oriented, then*

$$H^{n-p}(X;\mathbf{R}) \;\cong\; H^p(X;\mathbf{R})$$

More generally we have

Theorem 3. (Poincaré Duality). *If X is oriented (but not necessarily compact), then*

$$H^{n-p}_{\mathrm{cpt}}(X;\mathbf{R}) \;\cong\; H^p(X;\mathbf{R})$$