THE DIRICHLET PROBLEM
WITH PRESCRIBED INTERIOR SINGULARITIES

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ABSTRACT

In this paper we solve the nonlinear Dirichlet problem (uniquely) for functions with prescribed asymptotic singularities at a finite number of points, and with arbitrary continuous boundary data, on a domain in $\mathbf{R}^n$. The main results apply, in particular, to subequations with a Riesz characteristic $p \geq 2$. It is shown that, without requiring uniform ellipticity, the Dirichlet problem can be solved uniquely for arbitrary continuous boundary data with singularities asymptotic to the Riesz kernel $\Theta_j K_p(x - x_j)$ where

$$K_p(x) \;=\; \begin{cases} -\frac{1}{|x|^{p-2}} & \text{for } 2 < p < \infty, \\ \log|x| & \text{if } p = 2. \end{cases}$$

at any prescribed finite set of points $\{x_1, ..., x_k\}$ in the domain and any finite set of positive real numbers $\Theta_1, ..., \Theta_k$. This sharpens a previous result of the authors concerning the discreteness of high-density sets of subsolutions.

Uniqueness and existence results are also established for finite-type singularities such as $\Theta_j |x - x_j|^{2-p}$ for $1 \leq p < 2$.

The main results apply similarly with prescribed singularities asymptotic to the fundamental solutions of Armstrong-Sirakov-Smart (in the uniformly elliptic case).

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1. Introduction and Statement of Some Main Results.

The aim of this paper is to study the Dirichlet problem for functions with prescribed asymptotic singularities on a domain in $\mathbf{R}^n$. We shall adopt the notation and definitions in our previous work [HL$_{1,3,6,10}$].

Throughout the paper $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ will denote a closed set which satisfies the weakest possible ellipticity condition:

$\qquad$ (F1). $\ A \in F$ and $P \geq 0 \ \ \Rightarrow \ \ A + P \in F$,

so that solutions can be taken in the viscosity sense (cf. [C], [CIL], [CC]). Said differently, F is a constant coefficient, pure second-order subequation in $\mathbf{R}^n$.

In addition we will always assume that:

$\qquad$ (F2). $\ F$ is a cone with vertex at the origin (i.e., $tF = F$ for $t > 0$).

We will refer to an F satisfying (F1) and (F2) succinctly as a **cone subequation**.

For certain existence results we shall also make the mild requirement that

$\qquad$ (F3). $\ -P_e \notin F$ for all unit vectors $e \in \mathbf{R}^n$

where P_e denotes orthogonal projection onto the line $\mathbf{R} \cdot e$. Said differently, quadratic functions such as $u(x) = -x_1^2$ are not allowed to be F-subharmonic. This property (F3) is equivalent to the following condition on the dual subequation $\widetilde{F} \equiv \sim (-F)$:

$\qquad$ (B3). $\ $ All smooth boundaries are strictly $\widetilde{F}$-convex.

See Proposition 1.7 below for the proof of this equivalence and further discussion.

Our asymptotic singularities will be prescribed by functions of the following type.

Definition 1.1. A downward-pointing singular F-subharmonic at a point $x_0 \in \mathbf{R}^n$ is a continuous $[-\infty, \infty)$-valued F-subharmonic function ψ defined on a neighborhood U of x_0 with $\psi(x) > \psi(x_0)$ for $x \in U - \{x_0\}$, such that either:

$\qquad$ **(Polar Case):** $\quad \psi(x_0) = -\infty, \quad$ or

$\qquad$ **(Finite Case):** $\quad \psi(x_0) > -\infty$, and ψ has no test functions at x_0.

If in addition ψ is F-harmonic on $U - \{x_0\}$, then ψ will be referred to as a **downward-pointing singular F-harmonic** at x_0, and will usually be denoted by h instead of ψ.

Note that in the polar case ψ also has no test functions at x_0.

The problem we want to address is the following, which we will refer to as the **(DPPS)**.

THE DIRICHLET PROBLEM WITH PRESCRIBED SINGULARITIES.

Let Ω be a bounded domain in $\mathbf{R}^n$, and fix a finite number of points $x_1, ..., x_k \in \Omega$.

Boundary Data: This consists of a function $\varphi \in C(\partial\Omega)$ along with a downward-pointing singular F-harmonic function h_j at each point x_j.

A function H is a **solution** of the corresponding (DPPS) if

(1) $H \in C(\overline{\Omega} - \{x_1, ..., x_k\})$

(2) H is F-harmonic on $\Omega - \{x_1, ..., x_k\}$,

(3) $H\big|_{\partial\Omega} = \varphi$,

(4) H is **asymptotically equivalent to** h_j **at** x_j. By definition this means that:

    (4a) (In the Polar Case). There exists a constant $C > 0$ such that for each $j = 1, ..., k$

$$h_j(x) - C \ \leq \ H(x) \ \leq \ h_j(x) + C \qquad \text{near } x_j.$$

    (4b) (In the Finite Case). For each $j = 1, ..., k$

$$\lim_{x \to x_j} \frac{H(x) - H(x_j)}{h_j(x) - h_j(x_j)} \ = \ 1.$$

Remark 1.2. One easily verifies that asymptotic equivalence is indeed an equivalence relation (in fact, on the larger space of upper semi-continuous functions defined in a neighborhood of the point in question). Each equivalence class is invariant under the change of functions by an additive constant. We will denote *asymptotic equivalence*

      by $H \approx h$ in the polar case, and by $H \sim h$ in the finite case.

The following result is an immediate consequence of the comparison Theorems 5.2 and 5.4 proved in Section 5.

THEOREM 1.3. (Uniqueness). *For any cone subequation F there is at most one function H with properties (1) through (4).*

An existence construction is presented in Section 6 and then completed in the polar case by proving the following theorem. Our existence results in the finite case are stated and proved in Section 7.

THEOREM 1.4. (Existence in the Polar Case). *Suppose that F is a cone subequation satisfying Condition (F3) = (B3), and that the boundary of Ω is smooth and strictly F-convex. Assume that each prescribed singularity h_j at x_j is a downward-pointing singular F-harmonic of polar type. Finally assume*

Hypothesis (H): *There exists a continuous F-subharmonic function ψ on $\overline{\Omega}$, finite except at $x_1, ..., x_k \in \Omega$, with $\psi \approx h_j$ at x_j for each j.*

Then the Dirichlet Problem with Prescribed Singularities described above has a solution. Moreover, it is uniquely determined as the Perron function

$$H(x) \;=\; \sup_{v \in \mathcal{F}} v(x)$$

for the family

$$\mathcal{F} \equiv \{v \in \mathrm{USC}(\overline{\Omega}) : v \text{ is } F\text{-subharmonic on } \Omega,\ v \leq \varphi \text{ on } \partial\Omega,$$
$$\text{and } v - h_j \text{ is bounded above near each } x_j\}.$$

By Remark 11.13 in [HL$_3$] this Theorem extends to domains which are finite intersections of domains with smooth strictly F-convex boundaries.

Typically the functions h_j arise from a single global F-subharmonic function h on $\mathbf{R}^n$ which is F-harmonic outside the origin and has a downward-pointing singularity at 0. Such an h will be called a *generalized fundamental solution* for the subequation F. It determines the data in the (DPPS) by taking $h_j(x) = \Theta_j h(x - x_j)$ with constants $\Theta_j > 0$ for $j = 1, ..., k$. For Hypothesis (H) consider the function

$$\psi(x) \;=\; \sum_{j=1}^{k} \Theta_j h(x - x_j). \tag{1.1}$$

In the polar case (where $h(0) = -\infty$) the condition $\psi \approx h_j$ at x_j is automatic since h is continuous outside the origin. Thus the hypothesis (H) reduces to

Hypothesis (H)$'$: $\psi(x) \;=\; \sum_j \Theta_j h(x - x_j)$ is F-subharmonic on $\mathbf{R}^n$.

There are two cases where this hypothesis is easily satisfied.

A Single Point Singularity at $x_0 \in \Omega$: Then $\psi(x) \equiv \Theta h(x - x_0)$ $(\Theta > 0)$ is F-subharmonic on $\mathbf{R}^n$ – in fact, F-harmonic on $\mathbf{R}^n - \{x_0\}$.

The subequation F is convex: Then ψ is F-subharmonic on $\mathbf{R}^n$ because sums and positive multiples of F-subharmonic functions are also F-subharmonic.

This yields two special cases where the (DPPS) is uniquely solvable.

Corollary 1.5. *Let F and Ω be as in Theorem 1.4, and suppose h is a generalized fundamental solution for F.*

(a) For each $x_0 \in \Omega$ and $\Theta > 0$ there exists a unique solution to the (DPPS) having boundary values φ on $\partial\Omega$ and asymptotic to $\Theta h(x - x_0)$ at x_0.

(b) If, in addition, F is convex, then the multi-pole (DPPS) with ψ as in (1.1), has a unique solution.

In the case of just one point singularity with $\Theta = 1$ and the outer boundary function $\varphi \equiv 0$, the solution provided by Corollary 1.5(a) will be denoted $G_\Omega(x; x_0, h)$ and referred

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to as the **nonlinear Green's function** for the subequation F on the domain Ω with asymptotic singularity determined by the generalized fundamental solution h.

In the multi-pole case where F is required to be convex, we again take $\varphi \equiv 0$. Then the solution to the (DPPS) given by Corollary 1.5(b) will be called the **multi-pole nonlinear Green's function** and denoted by $G_\Omega(x; x_1, ..., x_k; \Theta_1, ..., \Theta_k; h)$.

These functions extend the classical pluri-complex Green's function with logarithmic singularities (cf. [Lem], [K$_*$], [Le], [Z]) where $h(x) = \log|x|$ on $\mathbf{R}^{2n} = \mathbf{C}^n$. (See Theorem 3.5 ff. for further discussion.)

In Section 7 we establish existence results in the finite case. The strongest result, Theorem 7.4, applies when there is just one singularity. Here the asymptotic type of the singularity can be prescribed along with the outer boundary values. One begins with a function $h \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_0\}$ and has a downward-pointing singularity of finite type at x_0. One then says that a function $H \in C(\Omega)$ has h-**density** $\Theta \geq 0$ if

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{h(x) - h(x_0)} = \Theta.$$

This is equivalent to saying that $H \sim \Theta h$ at x_0. Theorem 7.4 asserts that under certain assumptions on F, the Dirichlet problem can be solved for a harmonic with any prescribed h-density $\Theta \geq 0$ at x_0 and $\varphi \in C(\Omega)$. It is then shown that the hypotheses in Theorem 7.4 are satisfied for two large and important classes of subequations. The first is the class of O(n)-invariant subequations whose Riesz-characteristic p satisfies $1 \leq p < 2$. Here $h(x) = |x - x_0|^{2-p}$. The second class consists of the uniformly elliptic subequations where h is taken to be the downward-pointing fundamental solution of Armstrong, Sirakov and Smart.

There is another existence question which is meaningful only in the finite case. As before we fix a boundary function $\varphi \in C(\partial\Omega)$ and points $\{x_1, ..., x_k\} \subset \Omega$. However, rather than prescribing densities, we instead prescribe the values v_j for H at each point x_j. This is the Dirichlet Problem on the Punctured Domain $\Omega - \{x_1, ..., x_k\}$, where we look for a function $H \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_1, ..., x_k\}$ with $H\big|_{\partial\Omega} = \varphi$ and $H(x_j) = v_j$ for each j. By comparison on $\Omega - \{x_1, ..., x_k\}$, if a solution H exists, it is unique (see for example, [HL$_9$, Thm. 6.2]). This leaves the important problem of exactly determining the set $\mathcal{V}al$ of values $v = (v_1, ..., v_k)$ for which a solution H exists. In general $\mathcal{V}al \subset \mathbf{R}^k$ is a proper subset which depends on the given data. In Section 8 we establish the existence of a large, and explicitly described, subset $\mathcal{V} \subset \mathcal{V}al$. For the case where $\varphi = 0$, it is shown that this set $\mathcal{V}$ is a convex cone with vertex at the origin and non-empty interior contained in the negative "octant" $\mathbf{R}^k_-$. When $\varphi = 0$ and there is only one point ($k = 1$), the "value problem", namely, that of determining $\mathcal{V}al$, is solved completely: $\mathcal{V}al = \{v \leq 0\} = \mathcal{V}$ for any choice of point $x_1 \in \Omega$. (See Proposition 8.5.)

The main theorem in Section 8 (Theorem 8.1) actually enables one to prescribe not only certain values of H at the points x_j but also to prescribe the tangent to H at x_j up to a positive multiple ≥ 1. The key hypothesis in Theorem 8.1 is that there must exist on $\overline{\Omega}$ an F-subharmonic function h which is $\leq \varphi$ on $\partial\Omega$ and asymptotic to the given tangent at each x_j. For convex subequations this is often easily done and one obtains the large subset $\mathcal{V} \subset \mathcal{V}al$ discussed above.

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Remark 1.6. (The Boundary Convexity Hypothesis). The hypothesis that $\partial\Omega$ is strictly F-convex in Theorem 1.4 is necessary for existence in the finite cases as well as the polar case. Of course there are many domains with smooth boundary where this is true. For example, if $\partial\Omega$ is strictly convex, then $\partial\Omega$ is also strictly F-convex for any cone subequation F because $\mathcal{P} \subset F$. On the other hand, there are many cone subequations F with the property that every smooth boundary is strictly F-convex, allowing $\partial\Omega$ to be arbitrary. This is true if and only if $P_e \in \mathrm{Int}F$ for all $|e| = 1$ by the following Proposition, applied to $\widetilde{F}$ instead of F.

Proposition 1.7. (Concerning Condition (F3)). *For a cone subequation F the conditions:*

(F3) $-P_e \notin F$ for any unit vector e, and

(B3) All boundaries are strictly $\widetilde{F}$-convex

are equivalent. Furthermore, if F is a convex subequation, then (F3) and (B3) are also equivalent to:

(F3)′ The subequation F is complete, i.e., it cannot be defined using the variables in a proper linear subspace of $\mathbf{R}^n$.

Finally, if F is an invariant cone subequation as in Lemma 3.4, then these conditions are equivalent to

(F3)″ F has finite Riesz characteristic p (see Definition 3.2).

We shall use condition (B3), while condition (F3) provides the simplest test for this assumption. Condition (F3)′ is automatic unless the subequation is so degenerate that it does not involve all the variables in $\mathbf{R}^n$.

Proof. For cone subequations the equivalence of (F3) and (B3) was established in Section 5 of the earlier paper [HL₁], but was not stated explicitly. The argument goes as follows. First, by Lemma 5.3 (ii)′ in [HL₁] condition (B3) is equivalent to

(F3)‴   For all $B \in \mathrm{Sym}^2(\mathbf{R}^n)$ and all unit vectors e

$$B + tP_e \in \mathrm{Int}\widetilde{F} \text{ for all } t \geq \text{ some } t_0.$$

(This can be considered to be the definition of strict $\widetilde{F}$-convexity.) Second, by the Elementary Property (5) in [HL₁, §3] with $B' \equiv 0$, this is equivalent to

(F3)$^{(iv)}$ $P_e \in \mathrm{Int}\widetilde{F}$ for all unit vectors e.

Finally, by definition we have $-\mathrm{Int}\widetilde{F} = \sim F$, so that (F3)$^{(iv)} \iff$ (F3).

The equivalence of the two structural conditions (F3) and (F3)′, for F convex, was established in Proposition 3.6 of [HL₂].

For the equivalence of (F3) and (F3)″ see Lemma 3.4. ∎

2. Examples of Downward-Pointing F-Harmonics.

Example 2.1. (Riesz Kernels). Perhaps the most important such examples are the classical Reisz kernels K_p (with $p \geq 1$) which are defined and discussed in the next section. Each has a downward-pointing singularity. They play a fundamental role in standard potential theory (cf. [L]). Moreover, each K_p is actually a punctured harmonic for a large family of subequations – those of *Riesz characteristic p*. For such subequations F the Riesz kernels are central to the study of tangents and densities in the associated F-potential theories [HL$_{10}$], [HL$_{11}$]. In particular, they often arise as the unique tangent to any F-subharmonic function.

A large and important class of subequations with characteristic-p, which are convex cones but not uniformly elliptic, come from:

Geometrically Defined Subequations: These are the convex cone subequations $\mathcal{P}(\mathbb{G})$ determined by a closed subset $\mathbb{G} \subset G(p, \mathbf{R}^n)$, of the Grassmannian of p-dimensional subspaces of $\mathbf{R}^n$, by the requirement that

$$A \in F \qquad \Longleftrightarrow \qquad \text{tr}\left(A\big|_W\right) \geq 0 \qquad \forall\, W \in \mathbb{G}.$$

The following **Fullness Condition on $\mathbb{G}$**:

$$\text{Each unit vector } \ e \in \mathbf{R}^n \ \text{ is contained in a subspace } \ W \in \mathbb{G}, \tag{2.1}$$

is equivalent to the pth Riesz kernel being a punctured $\mathcal{P}(\mathbb{G})$-harmonic on $\mathbf{R}^n - \{0\}$. Moreover, if $\mathbb{G}$ satisfies (2.1), then Condition (F3) = (B3) also holds for $\mathcal{P}(\mathbb{G})$.

These examples contain the subequations naturally associated to many calibrations. They also include the Lagrangian subequations on $\mathbf{C}^n$. If $\mathbb{G} = G(p, \mathbf{R}^n)$, the resulting subequation $\mathcal{P}(\mathbb{G})$ (denoted $\mathcal{P}_p$ in Example 3.6(1)) is basic in geometry (cf. [Wu], [Sha] and [HL$_5$,] for example). See Section 4 of [HL$_{10}$] and Appendix A in [HL$_{11}$] for many more examples.

Perhaps it deserves mentioning here that the potential theory associated with the subequation $\mathcal{P}(\mathbb{G})$ is more appropriately called the $\mathbb{G}$-*pluripotential theory* because of the fact that: u is $\mathcal{P}(\mathbb{G})$-subharmonic $\Leftrightarrow$ $u\big|_W$ is Δ-subharmonic for every affine $W \in \mathbb{G}$ (proved in [HL$_4$]).

Example 2.2. (Homogeneous Singularities). Suppose that the function ψ in Definition 1.1 is homogeneous. This means (assuming $x_0 = 0$ for simplicity) that

$$\psi(x) = |x|^\alpha \psi\left(\frac{x}{|x|}\right) \qquad \text{for some} \ \ \alpha \neq 0$$

or, in the case "$\alpha = 0$", $\psi(x) = \psi(\frac{x}{|x|}) + \Theta \log|x|$ where $\sup_{|x|=1} \psi = 0$ and $\Theta > 0$. Note that the Riesz kernel K_p has homogeneity $\alpha = 2 - p$.

Suppose that ψ is a homogeneous F-subharmonic on $\mathbf{R}^n - \{0\}$. Then ψ is downward-pointing (Definition 1.1) as follows.

Case: $0 < \alpha \leq 1$. Then ψ has a strict minimum at $x_0 = 0$ if and only if $\psi(x) > 0$ for $x \neq 0$, in which case $\psi(0) = 0$ and $c|x|^\alpha \leq \psi(x)$ with $c = \inf_{\partial B} \psi$. The condition of having

no test functions at 0 follows easily from this inequality since $0 < \alpha \leq 1$. Thus ψ has a downward-pointing singularity at $0 \iff \psi(x) > 0$ on $|x| = 1$.

Case: $\alpha < 0$. Then ψ has a strict minimum at $x_0 = 0$ if and only if $\psi(x) < 0$ for $x \neq 0$, in which case $\psi(0) = -\infty$. Thus ψ has a downward-pointing singularity at $0 \iff \psi(x) < 0$ on $|x| = 1$.

Case: $\alpha = 0$. For any $\psi(\frac{x}{|x|})$ continuous we have $\psi(0) = -\infty$, while $\psi(x)$ is finite for $x \neq 0$. Thus ψ always has a downward-pointing singularity at 0.

Example 2.3. (The Special Cases $\mathcal{P}$, $\mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$). These are the geometrically defined subequations obtained by taking $\mathbb{G}$ to be $G(1, \mathbf{R}^n)$, $G^{\mathbf{C}}(1, \mathbf{C}^n) \subset G(2, \mathbf{R}^{2n})$ and $G^{\mathbf{H}}(1, \mathbf{H}^n) \subset G(4, \mathbf{R}^{4n})$ respectively. Here there are huge families of downward-pointing F-harmonic functions h. Those which are homogeneous can be classified as follows. (The differential inequalities are in the viscosity sense.)

The $\mathcal{P}$ Case:

$$h(x) = |x| g\left(\frac{x}{|x|}\right) \qquad \text{with } g \in C(S^{n-1}) \qquad \text{satisfying}$$

$$\mathrm{Hess}_{S^{n-1}} g + gI \geq 0, \quad \text{and} \quad \inf_{S^{n-1}} g > 0$$

The $\mathcal{P}^{\mathbf{C}}$ Case:

$$h(x) = g([x]) + \Theta \log|x| \qquad \text{with } g \in C(\mathbf{P}(\mathbf{C}^n)) \text{ and } \Theta > 0 \quad \text{satisfying}$$

$$i\partial\bar{\partial}g + \Theta\omega \geq 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g = 0$$

$$(\text{equivalently} \quad \mathrm{Hess}^{\mathbf{C}}_{\mathbf{P}(\mathbf{C}^n)} g + \Theta I \geq 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g = 0).$$

The $\mathcal{P}^{\mathbf{H}}$ Case:

$$h(x) = \frac{1}{|x|^2} g([x]) \qquad \text{with } g \in C(\mathbf{P}(\mathbf{H}^n)) \qquad \text{satisfying}$$

$$\mathrm{Hess}^{\mathbf{H}}_{\mathbf{P}(\mathbf{H}^n)} g - 2gI \geq 0, \quad \text{and} \quad \sup_{\mathbf{P}(\mathbf{C}^n)} g < 0$$

See Section 5 in [HL$_{11}$] for details and proofs.

Example 2.4. (Bedford-Taylor Singularities). Downward-pointing F-harmonics need not be homogeneous. The $\mathcal{P}^{\mathbf{C}}$-harmonic (i.e., maximal plurisubharmonic) functions on $\mathbf{C}^n - \{0\}$ given by

$$h(z, w) = \log(|z|^2 + |w|^4)$$

and each of its unitary rotates, define a family of distinct asymptotic equivalence classes at the origin, which are different from the basic punctured harmonic $\log(|z|^2 + |w|^2)$. There

are also the $\mathcal{P}^{\mathbf{C}}$-harmonic functions $\log(|z|^\alpha + |w|^\beta)$ for $\alpha, \beta > 0$. These examples go back to Bedford and Taylor [BT].

Example 2.5. (Armstrong-Sirakov-Smart Singularities). In the very interesting paper [AS$_1$] the authors consider pure second-order cone subequations exactly as in this paper but with the additional hypothesis of uniform ellipticity (where (F3) is automatic). Under this hypothesis they establish the existence of a canonical *fundamental solution* Φ which is F-harmonic on $\mathbf{R}^n - \{0\}$, has the homogeneity property $\Phi(tx) = t^{2-p}\Phi(x), \forall t > 0$ for some $1 < p < \infty$, and has a downward-pointing singularity at 0. In stark contrast to the three special cases $\mathcal{P}, \mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$ in Example 2.3 above, in this uniformly elliptic case there is precisely one asymptotic equivalence class of downward-pointing F-harmonic, namely Φ, up to a positive scale (see [AS$_1$]).

Now start with any cone subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ and consider the family $F(\delta), \delta > 0$ of elliptic regularizations consisting of uniformly elliptic cone subequations converging to F as $\delta \to 0$ (See Appendix A in [HL$_{11}$]). The paper [AS$_1$] applies to each $F(\delta)$ presenting us with a large set of functions $h_\delta \equiv \Phi$, other than the Riesz kernels, which satisfy the conditions of Definition 1.1.

Now suppose that F is convex, so that for small δ, $F(\delta)$ is also convex. The smallest possible subequation F_δ for which the function $h_\delta = \Phi$ is an entire *sub*harmonic is constructed by taking the closed convex cone on $\{D_x^2\Phi : |x| = 1\} + \mathcal{P}$. (See Example 3.6 where this is carried out in the cases $F = \mathcal{P}, \mathcal{P}^{\mathbf{C}}$ and $\mathcal{P}^{\mathbf{H}}$.) If Φ is also a harmonic for this smallest F_δ (outside the origin), then Φ is a downward-pointing harmonic for all cone subequations F such that $F_\delta \subset F \subset F(\delta)$. For each such F, Theorems 1.3 and 1.4 apply, as well as Theorem 7.1 in the finite case.

Other examples of this phenomenon can be constructed by applying a non-orthogonal linear transformation to any of the many convex subequations studied in [HL$_{10}$] and [HL$_{11}$].

3. Riesz Kernels and the Riesz Characteristic.

Of particular importance for the study of isolated singularities of subsolutions are the classical *Riesz kernels*:

$$K_p(x) \;=\; \begin{cases} -\dfrac{1}{p-2}\dfrac{1}{|x|^{p-2}} & \text{for } 1 \le p < \infty, \ p \ne 2 \\[2mm] \log|x| & \text{if } p = 2. \end{cases}$$

One sees easily that

$$D_x K_p \;=\; \frac{x}{|x|^p} \qquad \text{and} \qquad D_x^2 K_p \;=\; \frac{1}{|x|^p}\left(P_{x^\perp} - (p-1)P_x\right)$$

where $P_{x^\perp}$ and P_x denote orthogonal projection onto the hyperplane perpendicular to x and the line through x respectively. From this one sees the following.

Proposition 3.1. *Given a cone subequation F, the pth Riesz kernel*

$$K_p \ \text{ is } \ F \text{ harmonic on } \ \mathbf{R}^n - \{0\} \qquad \Longleftrightarrow \qquad P_{e^\perp} - (p-1)P_e \in \partial F \quad \forall\, |e| = 1. \quad (3.1)$$

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Definition 3.2. A (not necessarily convex) cone subequation $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ is said to have a **finite Riesz characteristic** $p_F = p$ if

$$P_{e^\perp} - (p-1)P_e \ \in \ \partial F \qquad \text{for all unit vectors} \ \ e \in \mathbf{R}^n.$$

If F has finite Riesz characteristic p, then $1 \leq p$ since $\mathrm{Int}\mathcal{P} \subset \mathrm{Int}F$. In addition, the Riesz kernel K_p and each of its translates is F**-harmonic** outside its singularity. Thus

$$h(x) \equiv K_p(x) \ \text{ is a downward pointing, singular } F \text{ harmonic on } \mathbf{R}^n - \{0\} \qquad (3.2)$$

for any cone subequation with Riesz characteristic p. This is perhaps the most important example satisfying the conditions in Definition 1.1. It is discussed in detail after a few additional comments on the polar case $p \geq 2$.

Remark 3.3. It is natural to extend the definition of Riesz characteristic. We say that F **has Riesz characteristic** $p_F = \infty$ if

$$-P_e \in \partial F, \ \text{ or equivalently } \ -P_e \in F, \quad \forall \, |e| = 1. \qquad (3.3)$$

The equivalence is because $-P_e \notin \mathrm{Int}F$ for a cone subequation F unless $F = \mathrm{Sym}^2(\mathbf{R}^n)$.

Of course, there are cone subequations which do not have a Riesz characteristic. However, as seen in the next lemma, for "invariant" cone subequations the Riesz characteristic is very easy to compute, and condition (F3) holds $\iff$ p is finite.

Lemma 3.4. *Suppose that F is invariant under a subgroup $H \subset \mathrm{O}(n)$ which acts transitively on the sphere $S^{n-1} \subset \mathbf{R}^n$. Then*

$$p_F \ = \ \sup\{q : P_{e^\perp} - (q-1)P_e \ \in \ F\} \qquad (3.4)$$

for some (and therefore any) unit vector $e \in \mathbf{R}^n$. In particular, $p_F = \infty \iff$ (F3) fails for all $|e| = 1$.

The proof is straightforward. Note that in appropriate coordinates $P_{e^\perp} - (q-1)P_e$ is diagonal matrix with one eigenvalue $-(q-1)$ and the remaining entries all equal to 1.

The principal Corollary 1.5 has as its most important special case the following.

THEOREM 3.5. (The Polar Case). *Suppose F is a cone subequation with finite Riesz characteristic $p \geq 2$ and property (F3), and let Ω be a domain with a smooth boundary which is strictly F-convex.*

(a) For each $\varphi \in C(\partial\Omega)$, $x_0 \in \Omega$, and $\Theta > 0$, there exists a unique solution H to the (DPPS) having boundary values φ on $\partial\Omega$ and asymptotic singularity $H(x) \approx \Theta K_p(x - x_0)$ at x_0.

(b) If, in addition, F is convex, then the multi-pole (DPPS) with asymptotics $\Theta_j K_p(x - x_j)$ has a unique solution H.

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Moreover, in either of these cases, if $\varphi \equiv 0$, this provides the existence and uniqueness of a nonlinear Green's function G_Ω with

$$G_\Omega(x) \;\approx\; \Theta K_p(x - x_0) \qquad \text{at} \ \ x_0$$

in the case of a single pole, and, provided that F is convex,

$$G_\Omega(x) \;\approx\; \Theta_j K_p(x - x_j) \qquad \text{at each} \ \ x_j$$

in the multi-pole case.

The existence and uniqueness of the multi-pole Green's function for the subequation $F = \mathcal{P}^{\mathbf{C}}$ was proved by Lelong in 1989 ([Le]). This built on previous work for single-point Green's functions (Lempert [Lem] and Klimek [K$_*$]). An even more general version was established by A. Zeriahi [Z].

Theorem 3.5 includes all invariant cone subequations whose Riesz characteristic is finite (by Lemma 3.4).

There are many more subequations F which have a finite Riesz characteristic than one might at first imagine. We start by mentioning four extreme examples of characteristic p subequations. The first explains our choice of normalization in the definitions of K_p and characteristic p. When p is an integer, this example coincides with the geometric subequation $\mathcal{P}(G(p, \mathbf{R}^n))$ discussed in Section 2.

Examples. 3.6.

(1) $\mathcal{P}_p = \{A : \lambda_1(A) + \cdots + \lambda_{[p]}(A) + (p - [p])\lambda_{[p]+1}(A) \geq 0\}$ where $\lambda_1(A) \leq \lambda_2(A) \leq \cdots$ are the ordered eigenvalues of A and $1 \leq p \leq n$.

(2) $\mathcal{P}(\delta_p) = \left\{A : A + \frac{\delta_p}{n}\text{tr}(A)I \;\geq\; 0\right\}$, where $\delta_p = \frac{n(p-1)}{n-p}$.

(3) $\mathcal{P}_p^{\min/\max} = \{A : \lambda_{\min}(A) + (p - 1)\lambda_{\max}(A) \geq 0\}$.

(4) $\mathcal{P}_p^{\min/2} = \{A : \lambda_{\min}(A) + (p - 1)\lambda_2(A) \geq 0\}$.

Note that $\mathcal{P}_p^{\min/2} \subset \mathcal{P}_p \subset \mathcal{P}(\delta_p) \subset \mathcal{P}_p^{\min/\max}$ and that each subequation F with $\mathcal{P}_p^{\min/2} \subset F \subset \mathcal{P}_p^{\min/\max}$ has finite Riesz characteristic p. It is somewhat surprising that, under a mild restriction, there exist both a "largest" and a "smallest" characteristic p subequation. More precisely, *every invariant (as in Lemma 3.4) cone subequation F with finite Riesz characteristic p satisfies*

$$\mathcal{P}_p^{\min/2} \;\subset\; F \;\subset\; \mathcal{P}_p^{\min/\max} \tag{3.5}$$

and if F is $O(n)$-invariant and convex, then

$$\mathcal{P}_p \;\subset\; F \;\subset\; \mathcal{P}(\delta_p). \tag{3.6}$$

This is proved in Appendix A of [HL$_{11}$] where many more examples of characteristic p subequations are given. These include subequations of Monge-Ampère type arising from

Gårding operators. Among these is the following *Hessian equation*, which has been studied by Trudinger-Wang [TW$_*$], Labutin [La$_*$] and others. We have drawn heavily from [La$_3$] in this paper.

(5) $\Sigma_k = \{A : \sigma_1(A) \geq 0, ..., \sigma_k(A) \geq 0\}$, where $p_F = \frac{n}{k}$.

One also has

(6) $F = \{A : \text{tr}(A^q) \geq 0\}$ where $p_F = 1 + (n-1)^{\frac{1}{q}}$ for $q \in \mathbf{Z}$ odd. (This F is not convex.)

Suppose F is an O(n)-invariant subequation with $p_F = p$, and let $F(\mathbf{C}), F(\mathbf{H})$ be the complex and quaternionic analogues given by the same conditions on the eigenvalues of their hermitian symmetric components. Then $p_{F(\mathbf{C})} = 2p_F$ and $p_{F(\mathbf{H})} = 4p_F$. These examples contain the complex and quaternionic Monge-Ampère equations [BT], [A$_*$], [AV], as well as the complex and quaternionic hessian equations. There are also the complex (as well as quaternionic) analogues:

$$\mathcal{P}_p^{\min/2}(\mathbf{C}) \subset \mathcal{P}_p(\mathbf{C}) \subset \mathcal{P}(\delta_p)(\mathbf{C}) \subset \mathcal{P}_p^{\min/\max}(\mathbf{C}). \tag{3.7}$$

with

$$\mathcal{P}_p^{\min/2}(\mathbf{C}) \subset F(\mathbf{C}) \subset \mathcal{P}_p^{\min/\max}(\mathbf{C}) \qquad \text{and} \qquad \mathcal{P}_p^{\min/2}(\mathbf{C}) \subset F(\mathbf{C}) \subset \mathcal{P}_p^{\min/\max}(\mathbf{C})$$

as in (3.5) and (3.6).

4. The Dirichlet Problem with Prescribed Densities.

Assume, as in the previous section, that F is a cone subequation with a finite Riesz characteristic p. For an arbitrary F-subharmonic function u, the density of u at a point x_0 in its domain, is the limit

$$
\begin{aligned}
\Theta(u, x_0) &\equiv \lim_{r \to 0} \frac{\sup_{B_r(x_0)} u}{K_p(x - x_0)} & \text{for } p \geq 2 \quad \text{and} \\
\Theta(u, x_0) &\equiv \lim_{r \to 0} \frac{\sup_{B_r(x_0)} u - u(x_0)}{K_p(x - x_0)} & \text{for } 1 \leq p < 2,
\end{aligned}
\tag{4.1}
$$

which always exists by [HL$_{10}$].

The **Dirichlet Problem with Prescribed Densities**, denoted by (DPPD), is the same as the (DPPS) except that the asymptotic requirement

$$H \approx \Theta_j K_p(x - x_j) \quad \text{for } p \geq 2 \qquad \text{or} \qquad H \sim \Theta_j K_p(x - x_j) \quad \text{for } 1 \leq p < 2 \tag{4.2}$$

is replaced by prescribing the density

$$\Theta(H, x_j) = \Theta_j \tag{4.3}$$

at each point x_j.

Lemma 4.1. *For any F-subharmonic function u, the condition (4.2) implies (4.3).*

Proof. Assume $x_j = 0$. The notion $u \sim \Theta K_p$ at the origin is defined for $p \geq 2$ by requiring that

$$\lim_{x \to 0} \frac{u(x)}{K_p(x)} = \Theta$$

(see (A.1) and also (4a) in (DPPS)). By Proposition A.3 the condition $u \approx \Theta K_p$ at 0 implies that $u \sim \Theta K_p$ at 0, so even when $p \geq 2$, we can assume $u \sim \Theta K_p$. Finally, the limits in (4.1) equal Θ since the limits in (A.1) as $x \to 0$ equal Θ. ∎

Corollary 4.2. *If existence holds for the (DPPS), then existence holds for the (DPPD). Moreover, uniqueness holds for the (DPPD) if and only if for any downward-pointing singular F-harmonic H with density $\Theta(H, 0) = \Theta$ we have*

$$H \approx \Theta K_p \text{ at } 0 \text{ if } 2 \leq p \qquad \text{or} \qquad H \sim \Theta K_p \text{ at } 0 \text{ if } 1 \leq p < 2 \qquad (4.4)$$

The Polar Case $(p \geq 2)$

Existence for the (DPPS) provided by Theorem 3.5 implies existence for the (DPPD) because of Corollary 4.2.

THEOREM 4.3. (Polar Case $p \geq 2$) *Let F, Ω and φ be as in Theorem 3.5. Then there exits a solution H to the (DPPD) with prescribed positive density Θ at any given point $x_0 \in \Omega$. Moreover, if F is convex, there exists a solution H with prescribed positive densities $\Theta_1, ..., \Theta_k$ at an arbitrary collection of distinct points $x_1, ..., x_k \in \Omega$.*

Uniqueness of the non-linear Green's function implies uniqueness in the general (DPPD).

THEOREM 4.4. (Polar Case $p \geq 2$). *Suppose that the Riesz kernel K_p is the only solution to the (DPPD) on a ball about the origin with the same boundary values and asymptotic behavior as K_p. Then uniqueness holds for the general Dirichlet problem with prescribed densities (DPPD) in Theorem 4.3.*

Proof. Suppose H is a solution to the (DPPD) in Theorem 4.3. By the definition of the (DPPD) we know that H has density Θ_j at x_j. It suffices to prove (4.4), i.e., to show that $H \approx \Theta_j K_p(x - x_j)$ because then the uniqueness part of Theorem 3.5 applies. As in [La$_3$, Thm. 3.6] here is how (4.4) can be proved for H. Normalize so that $x_j = 0$, $\Theta_j = 1$ and $B \subset\subset \Omega$ is a ball about the origin. By the existence part of Theorem 4.3, we obtain $h \in C(\overline{B} - \{0\})$, which is F-harmonic on $B - \{0\}$, equal to the constant K_p on ∂B, and satisfies $h \approx H$ at 0. Now $h \approx H$ at 0 implies that h also has density 1 at 0. By the hypothesis, this proves $h = K_p$. ∎

Corollary 4.5. (Polar Case $p \geq 2$). *Suppose F is an orthogonally invariant subequation of finite Riesz characteristic p. Then both existence and uniqueness hold for the (DPPD) in Theorem 4.3.*

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Proof. The classical moving plane argument shows that if h is F-harmonic on B, constant on ∂B and $h(0) = -\infty$, then h is a radial function (see [La$_3$], [GLN]). Hence, $h(x) = \Theta K_p(x) + k$ by [HL$_{10}$, Prop. 3.5]. ■

Final Note. For orthogonally invariant subequations F (and many others as well) it has been shown ([HL$_{10}$], [HL$_{11}$]) that for any F-**sub**harmonic function u on $\Omega \subset \mathbf{R}^n$ and $c > 0$,

$$\text{the set} \ \ E_c(u) \ \equiv \ \{x \in \Omega : \Theta(u, x) \geq c\} \ \ \text{is discrete}.$$

The results here show that any finite set $E \subset \Omega$ can occur as $E_c(H)$ for an F-harmonic function H on Ω.

5. Comparison.

In this section we prove the Uniqueness Theorem 1.3 for the (DPPS) by establishing a comparison theorem in the setting of prescribed singularities. The key idea is contained in a local result. First we consider the polar case.

Lemma 5.1. (The Polar Case). *Suppose v is F-subhamronic and w is $\widetilde{F}$-subharmonic in a deleted neighborhood of a point $x_0 \in \mathbf{R}^n$, and that h is a downward-pointing singular F-harmonic at x_0. Assume that for some constants c and k*

$$v \ \leq \ h + c \quad \text{and} \quad w \ \leq \ -h + k \quad \text{near} \ x_0. \tag{5.1}$$

Then, with $u \equiv v + w$ extended to x_0 by setting

$$u(x_0) \ \equiv \ \overline{\lim_{x \to x_0}} \, (v(x) + w(x)) \tag{5.2}$$

we have that

$$u \ \text{is subaffine on a neighborhood of} \ \ x_0. \tag{5.3}$$

Proof. We recall the notion of subaffine functions [HL$_1$]. These are the functions $u \in \text{USC}(\Omega)$ such that for any affine function a and $K \subset\subset \Omega$, $u \leq a$ on $\partial K \ \Rightarrow \ u \leq a$ on K. It turns out that this is a local condition on the function, namely that it satisfy the subequation $\widetilde{\mathcal{P}}$ dual to $\mathcal{P}$ (see [HL$_1$]). Comparison follows from (5.3) since subaffine functions clearly satisfy the maximum principle.

Consider the sum $u = v + w$ which defines an upper semi-continuous $[-\infty, \infty)$-valued function in a deleted neighborhood of x_0. This function has the following two properties:

$$u \ \text{is bounded above across} \ \ x_0, \tag{5.4}$$

$$u \ \text{is subaffine on a deleted neighborhood of} \ \ x_0. \tag{5.5}$$

Obviously (5.1) implies (5.4). For (5.5) recall (see [HL$_1$] or [HL$_9$, Thm. 6.2]) that for any constant coefficient, pure second-order subequation F, the sum of an F-subharmonic function and an $\widetilde{F}$-subharmonic function is $\widetilde{\mathcal{P}}$-subharmonic. The condition (5.4) implies

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that the extension of u defined by (5.2) is upper semi-continuous, with values in $[-\infty, \infty)$, in a neighborhood of x_0. It remains to show that u is $\widetilde{\mathcal{P}}$-subharmonic. (Note that the function $-|x - x_0|$ satisfies (5.4) and (5.5) and, even though it is continuous across x_0, it is not $\widetilde{\mathcal{P}}$-subharmonic across x_0. Said differently, Lemma 5.1 is not simply a removable singularity theorem for $\widetilde{\mathcal{P}}$.)

To prove that u is subaffine across x_0 we approximate u by

$$u_\epsilon \; = \; u + \epsilon v \; = \; (1 + \epsilon)v + w. \tag{5.6}$$

Since u is bounded above and $v(x_0) = -\infty$, if we define $u_\epsilon(x_0) = -\infty$, then u_ϵ is upper semi-continuous on a neighborhood of x_0. Note that u_ϵ has no test functions at x_0, so to prove that u_ϵ is subaffine on a neighborhood V of x_0, we need only prove that u_ϵ is subaffine on $V - \{x_0\}$. However, u_ϵ is the sum of the F-subharmonic function $(1 + \epsilon)v$ and the $\widetilde{F}$-subharmonic function w. This implies, as above, that u_ϵ is $\widetilde{\mathcal{P}}$-subharmonic on $V - \{x_0\}$ as desired. (Here we have used that F is a cone.)

Now on a neighborhood of x_0 the subaffine functions u_ϵ increase pointwise to u (since $v < 0$). By the "families bounded above" property (see [HL$_1$]) and the fact that u is upper semi-continuous with values in $[-\infty, \infty)$, this proves that u is $\widetilde{\mathcal{P}}$-subharmonic. ∎

Comparison can be stated as follows.

THEOREM 5.2. (Comparison in the Polar Case). *Suppose Ω is a domain and $h_1, ..., h_k$ are downward-pointing singular F-harmonics at $x_1, ..., x_k$ respectively. Given $v, w \in \mathrm{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$ with v F-subharmonic and w $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$, suppose that near each x_j, $j = 1, ..., k$ we have*

$$v \; \leq \; h_j + c_j \qquad \text{and} \qquad w \; \leq \; -h_j + k_j \qquad \text{for some constants } c_j \text{ and } k_j. \tag{5.7}$$

Then comparison holds on Ω, that is,

$$\text{If} \quad v + w \; \leq \; 0 \quad \text{on} \quad \partial\Omega, \quad \text{then} \quad v + w \; \leq \; 0 \quad \text{on} \quad \Omega - \{x_1, ..., x_k\}. \tag{5.8}$$

Proof. By Lemma 5.1 the function $u \equiv v + w$, defined on $\overline{\Omega} - \{x_1, ..., x_k\}$, extends to an upper semi-continuous $[-\infty, \infty)$-valued function u on $\overline{\Omega}$ which is subaffine on Ω. Hence, $\sup_{\overline{\Omega}} u \leq \sup_{\partial\Omega} u$ by the maximum principle. ∎

Proof of the Uniqueness Theorem 1.3 in the Polar Case. It suffices to prove that

$$\text{If} \quad H \quad \text{is a solution to the (DPPS), then} \quad H(x) \; = \; \sup_{v \in \mathcal{F}} v(x). \tag{5.9}$$

where $\mathcal{F}$ is the family defined in Theorem 1.4. We can apply the Comparison Theorem 5.2 to $v \in \mathcal{F}$ and $w \equiv -H$. On $\partial\Omega$, we have $v \leq \varphi$ and $w = -\varphi$, and so $v + w \leq 0$ (or $v \leq H$) on $\overline{\Omega}$ by (5.8). Since $H \in \mathcal{F}$, this proves (5.9). ∎

Now we turn to the finite case.

Lemma 5.3. (The Finite Case). *Suppose v is F-subharmonic and w is $\widetilde{F}$-subharmonic on a deleted neighborhood of x_0, and that h is a downward-pointing singular F-harmonic at x_0. Assume that*

$$\liminf_{x \to x_0} \frac{v(x) - v(x_0)}{h(x) - h(x_0)} \geq 1 \qquad \text{and} \qquad \liminf_{x \to x_0} \frac{w(x) - w(x_0)}{h(x) - h(x_0)} \geq -1 \qquad (5.10)$$

Then with $u \equiv v + w$ extended to x_0 by setting

$$u(x_0) \equiv \limsup_{x \to x_0} u(x) \qquad (5.11)$$

we have that

$$u \ \text{is subaffine on a neighborhood of} \ \ x_0. \qquad (5.12)$$

Proof. To prove that u is subaffine, we approximate $\overline{u} \equiv u - v(x_0) + w(x_0)$ by

$$\overline{u}_\epsilon \ = \ \overline{u} + \epsilon(v(x) - v(x_0)) \ = \ (1 + \epsilon)(v(x) - v(x_0)) + (w(x) - w(x_0)) \qquad (5.13)$$

and prove that $\overline{u}$ is subaffine. Note that as in Lemma 5.1, $\overline{u}_\epsilon$ is subaffine on a deleted neighborhood of x_0 since F is a cone. To show that $\overline{u}_\epsilon$ is subaffine on a neighborhood of x_0, we need only show that at x_0, $\overline{u}_\epsilon$ has no test functions.

The hypothesis (5.10) on v implies that for $1 < \alpha < 1 + \epsilon$, there exists a neighborhood of x_0 with $v(x) - v(x_0) \geq \frac{\alpha}{1+\epsilon}(h(x) - h(x_0))$.

The hypothesis (5.10) on w implies that for $1 < \beta < \alpha$, we have $w(x) - w(x_0) \geq (\alpha - \beta)(h(x) - h(x_0))$ near x_0. Hence, $\overline{u}_\epsilon \geq (\alpha - \beta)(h(x) - h(x_0))$, which proves that $\overline{u}_\epsilon$ has no test functions at x_0, since h has no test functions at x_0 (note that $\overline{u}_\epsilon(x_0) = 0$.) This proves that each $\overline{u}_\epsilon$ is subaffine in a neighborhood of x_0.

Finally the fact that $v(x) - v(x_0)$ is bounded below by a positive multiple of $h(x) - h(x_0)$ implies that $\overline{u}_\epsilon$ is decreasing pointwise as $\epsilon \to 0$ in a neighborhood of x_0. However, outside x_0, $\overline{u}$ is the sum of the F-subharmonic function $v(x) - v(x_0)$ and the $\widetilde{F}$-subharmonic function $w(x) - w(x_0)$, which implies that $\overline{u}$ is subaffine. ∎

THEOREM 5.4. (Comparison in the Finite Case). *Suppose that Ω is a domain and $h_1, ..., h_k$ are downward-pointing singular F-harmonics at $x_1, ..., x_k \in \Omega$ respectively. Given $v, w \in \text{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$ with v F-subharmonic and w $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$, suppose that for $j = 1, ..., k$ we have*

$$\liminf_{x \to x_j} \frac{v(x) - v(x_j)}{h(x) - h(x_j)} \geq 1 \qquad \text{and} \qquad \liminf_{x \to x_j} \frac{w(x) - w(x_j)}{h(x) - h(x_j)} \geq -1. \qquad (5.14)$$

Then (with $u \equiv v + w$ defined at x_j by $(v + w)(x_j) \equiv \limsup_{x \to x_j}(v + w)(x)$),

$$\text{If} \ \ v + w \ \leq \ 0 \ \ \text{on} \ \ \partial\Omega, \ \ \text{then} \ \ v + w \ \leq \ 0 \ \ \text{on} \ \ \overline{\Omega}. \qquad (5.15)$$

Proof. This follows from Lemma 5.3 as in the proof of Theorem 5.2. ∎

Corollary 5.5. (Uniqueness in the Finite Case). *Let $\mathcal{F}$ denote the family of $v \in$ USC$(\overline{\Omega})$ satisfying: v is F-subharmonic on Ω, $v \leq \varphi$ on $\partial\Omega$, and for $j = i, ..., k$*

$$\liminf_{x \to x_j} \frac{v(x) - v(x_j)}{h(x) - h(x_j)} \geq 1.$$

If H is a solution to the (DPPS), then

$$H(x) = \sup_{v \in \mathcal{F}} v(x). \tag{5.16}$$

Proof. Note that if $v \in \mathcal{F}$ and H is a solution, then comparison implies that $v \leq H$. Since $H \in \mathcal{F}$, this proves (5.16). ∎

6. A Basic Construction and the Proof of Existence in the Polar Case.

In this section we describe the basic construction of the solution to the (DPPS) in some generality, and then complete the existence proof in the polar case. The finite case is finished in Section 7. Our starting point is the standard (DP) on the domain with a neighborhood of the singular points removed.

The Dirichlet Problem on the Perforated Domain.

We fix $r_0 > 0$ so that the closed balls $\overline{B}_{r_0}(x_j) = \{|x - x_j| \leq r_0\}$, $j = 1, ..., k$, are mutually disjoint and contained in Ω. Let

$$D_r \equiv \bigcup_{j=1}^{k} \overline{B}_r(x_j) \qquad \text{for } 0 < r \leq r_0.$$

Consider the perforated domain

$$\Omega_r \equiv \Omega - D_r$$

whose oriented boundary is the sum of the *outer boundary* $\partial\Omega$ and the *inner boundary* $-\partial D_r$.

Consider the Dirichlet Problem for a subequation $F \subset \text{Sym}^2(\mathbf{R}^n)$ on Ω_r with given boundary functions $\underline{\varphi} \in C(\partial D_r)$ and $\varphi \in C(\partial\Omega)$. As discussed in the proof of Lemma 5.1 comparison and hence uniqueness holds on any domain. For existence, consider the **Perron family** $\mathcal{F}$ consisting of all $u \in \text{USC}(\overline{\Omega_r})$ such that

$$u\big|_{\Omega_r} \in F(\Omega_r), \qquad u\big|_{\partial D_r} \leq \underline{\varphi}, \qquad \text{and} \qquad u\big|_{\partial\Omega} \leq \varphi.$$

along with its **Perron function**

$$H_r(x) \equiv \sup_{u \in \mathcal{F}} u(x).$$

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By [HL$_3$, §12] the Perron function will solve the Dirichlet problem for the given bound-ary values provided that, at each point of $\partial\Omega_r = \partial\Omega - \partial D_r$, one can construct barriers as in Propositions F and $\widetilde{F}$ in [HL$_3$] on page 453.

We list our assumptions.

Assumption (B1): The outer boundary $\partial\Omega$ is strictly F-convex.

Note that $\partial\Omega$ is strictly $\widetilde{F}$-convex by (F3) = (B3). Thus barriers, as in Propositions F and $\widetilde{F}$ exist for each point $x_0 \in \partial\Omega$.

Note similarly that the inner boundary $-\partial D_r$ is also strictly $\widetilde{F}$-convex by (F3) = (B3). This provides a barrier as in Proposition $\widetilde{F}$ at points $x_0 \in \partial D_r$.

To obtain an F-barrier at a point $x_0 \in \partial D_r$ we assume that the inner boundary function $\underline{\varphi}$ is of a special nature, namely,

Assumption (B2): $\underline{\varphi} \equiv \psi\big|_{\partial D_r}$ where ψ is F-subharmonic on $\overline{\Omega}$, and either

> **Polar Case:** $\psi : \overline{\Omega} \to [-\infty, \infty)$ is continuous and $= -\infty$ precisely at the points $x_1, ..., x_k$, or

> **Finite Case:** $\psi \in C(\overline{\Omega})$ with each x_j, $j = 1, ..., k$ a strict local minimum point with no test functions.

Assumption (B4): $\psi\big|_{\partial\Omega} \leq \varphi$.

Since ψ and φ satisfy (B2) and (B4), we can use the F-subharmonic function ψ to construct barriers at points $x_0 \in \partial D_r$ as in Proposition F. This is done by setting $\underline{u}(x) \equiv \psi(x) - \delta + \epsilon|x - x_0|^2$ for $\epsilon > 0$ sufficiently small.

This establishes the following existence result.

THEOREM 6.1. *Assume that F is a cone subequation which satisfies Condition (F3) = (B3). Suppose $\varphi \in C(\partial\Omega)$. Suppose that $\partial\Omega$ is strictly F-convex (B1), and that ψ satisfies (B2) and (B4). Then the Perron function H_r solves the Dirichlet Problem:*

(a) $H_r \in C(\overline{\Omega}_r)$

(b) H_r is F-harmonic on Ω_r

(c) $H_r\big|_{\partial D_r} = \psi\big|_{\partial D_r}$ and $H_r\big|_{\partial\Omega} = \varphi$.

The Candidate for the Solution to the (DPPS)

We continue with the same notation and hypotheses.

The proposed solution to the (DPPS) is constructed as a pointwise increasing limit

$$H(x) \equiv \lim_{r \to 0} \overline{H}_r(x) \quad \text{on } \overline{\Omega} \tag{6.1}$$

of functions $\overline{H}_r$ which extend the functions H_r to $\overline{\Omega}$.

Lemma 6.2. *The continuous functions*

$$\overline{H}_r \;\equiv\; \begin{cases} H_r & \text{on } \overline{\Omega}_r \\ \psi & \text{on } D_r \end{cases} \tag{6.2}$$

are F-subharmonic on Ω and pointwise increasing on $\overline{\Omega}$ as $r \to 0$.

Proof. The function ψ is in the Perron family on Ω_r since $\psi|_{\partial D_r} = \underline{\varphi}$ and $\psi|_{\partial\Omega} \leq \varphi$. This proves that

$$\psi \leq H_r \quad \text{on } \overline{\Omega}_r. \tag{6.3}$$

Consequently,

$$\overline{H}_r(x) \;\equiv\; \begin{cases} \max\{\psi(x), H_r(x)\} & \text{on } \overline{\Omega}_r \\ \psi(x) & \text{on } D_r. \end{cases} \tag{6.4}$$

To see that

$$\overline{H}_r \quad \text{is } F\text{ subharmonic on } \Omega, \tag{6.5}$$

note that $\overline{H}_r^{\epsilon} = \max\{\psi + \epsilon, H_r\}$ defines an F-subharmonic function on Ω since $\psi + \epsilon > H_r$ on ∂D_r, and then note that $\overline{H}_r^{\epsilon}$ decreases pointwise as $\epsilon \to 0$ to $\overline{H}_r$ on $\overline{\Omega}$.

Finally we show that for all ρ with $0 < \rho < r \leq r_0$, one has

$$\overline{H}_r \;\leq\; \overline{H}_\rho \qquad \text{on } \overline{\Omega}. \tag{6.6}$$

On D_r we have $\overline{H}_r = \overline{H}_\rho = \psi$. In particular, on ∂D_ρ, $\overline{H}_r = \psi$, while on the outer boundary $\partial\Omega$, $\overline{H}_r = \varphi$. Thus $\overline{H}_r$ is in the Perron family for H_ρ on this larger domain Ω_ρ, proving that $\overline{H}_r \leq H_\rho$ on $\overline{\Omega}_\rho$, which establishes (6.6). ∎

Let H^{DP} denote the solution to the standard Dirichlet Problem (DP) on Ω. That is, $H^{\mathrm{DP}} \in C(\Omega)$, H^{DP} is F-harmonic on Ω, and $H^{\mathrm{DP}}|_{\partial\Omega} = \varphi$. This function exists since the outer boundary is assumed to be strictly F-convex, and by (F3) = (B3) it is also strictly $\widetilde{F}$-convex.

Proposition 6.3. *The proposed solution H to the (DPPS) defined by (6.1) satisfies:*

(1) (a) $H^* \in \mathrm{USC}(\overline{\Omega})$ *and*   (b) $-H = (-H)^* \in \mathrm{USC}(\overline{\Omega} - \{x_1, ..., x_k\})$,

(2) (a) H^* *is F-subharmonic on Ω,*   (b) $-H$ *is $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$*

(3) $H^*|_{\partial\Omega} = H|_{\partial\Omega} = \varphi$

(4) $\psi \leq H \leq H^{\mathrm{DP}}$ *on $\overline{\Omega}$.*

Proof. Assertion (1) is immediate while (2a) follows from the "families bounded above property" and (2b) by the "decreasing limit property". To prove (4) first note that (6.3) implies that $\psi \leq H$. Now the function H^{DP} is the Perron function for the standard Dirichlet Problem on Ω with boundary values φ. Therefore, H^{DP} is larger than ψ by assumption (B4). Thus $H_r \leq H^{\mathrm{DP}}$ on $\partial\Omega_r$, and this remains true on $\overline{\Omega}_r$. Therefore, $H \leq H^{\mathrm{DP}}$, which implies that

$$H^* \;\leq\; H^{\mathrm{DP}}. \tag{6.7}$$

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In turn this implies (3). ∎

Existence in the Polar Case – The Proof of Theorem 1.4

First note that in the polar case the notion of asymptotic equivalence is preserved by subtracting a constant. This implies that the hypothesis (B4) in Theorem 6.1 and Proposition 6.3 can always be satisfied.

Now (4) in Proposition 6.3 yields the left hand inequality in Part (4a) of the (DPPS) stated in the introduction. For the right hand inequality in (4a) we prove the following.

Lemma 6.4. *Near each x_j we have $H \leq h_j + c_j$ for some constant c_j.*

Proof. For $0 < \rho \leq r$ we have that

$$H_\rho - h_j \quad \text{is} \quad \widetilde{\mathcal{P}} \text{ subharmonic on} \quad A_{\rho,r} \; = \; B_r - \overline{B}_\rho.$$

On the inner boundary ∂B_ρ we have, since $\psi \approx h_j$ implies $\psi - h_j \leq C$, that

$$H_\rho - h_j \; = \; \psi - h_j \; \leq \; C,$$

and on the outer boundary ∂B_r we have

$$H_\rho - h_j \; \leq \; U - h_j \; \leq \; C(r)$$

independent of ρ. Hence,

$$H - h_j \; \leq \; \max\{C, C(r)\} \; \equiv \; C' \qquad \text{on} \quad B_r(x_j). \qquad ∎$$

Now we apply the Comparison Theorem 5.2 to $v \equiv H^*$ and $w \equiv -H$. By (1a) H^* takes values in $[-\infty, \infty)$ on Ω, while by (4) w takes values in $(-\infty, \infty)$ except at $x_1, ..., x_k$ where w equals $+\infty$. The inequality

$$v \; \equiv \; H^* \; \leq \; h_j + c_j \qquad \text{near} \; x_j$$

is immediate from Lemma 6.4. Observe now that by combining (4) $\psi \leq H$ ($\equiv -w$) from Proposition 6.3 with the inequality $h_j - k_j \leq \psi$ near x_j, which is part of Hypothesis (H) in Theorem 1.4, gives

$$w \; \leq \; -h_j + k_j \qquad \text{near} \; x_j.$$

This establishes the hypothesis (5.7) in Theorem 5.2. Finally, by (3) in Proposition 6.3, $v = \varphi$ and $w = -\varphi$ on $\partial\Omega$, so that $v + w = 0$ on $\partial\Omega$. We can now apply Theorem 5.2 to conclude that $v + w \leq 0$ on $\overline{\Omega} - \{x_1, ..., x_k\}$, i.e., $H^* \leq H$ on $\overline{\Omega} - \{x_1, ..., x_k\}$. This proves that $H^* = H$ on $\overline{\Omega} - \{x_1, ..., x_k\}$. Since we already have $H = H_*$ (because it is an increasing limit of continuous functions), we conclude that H is continuous on $\overline{\Omega} - \{x_1, ..., x_k\}$, and Conditions (1) and (2) are proved. This completes the proof of Theorem 1.4. ∎

The (DPPS) with Singularities on a Compact Polar Set

The arguments given above adapt to prove a version of Theorem 1.4 with $\{x_1, ..., x_k\}$ replaced by a compact polar set.

THEOREM 6.5. *Let F be a cone subequation satisfying Condition (F3). Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with smooth boundary $\partial\Omega$ which is strictly F-convex, and let $\Sigma \subset \Omega$ be a compact subset. Suppose there exists a continuous function $h : \overline{\Omega} \to [-\infty, \infty)$ such that $\Sigma = h^{-1}(-\infty)$ and h is F-harmonic on $\Omega - \Sigma$.*
Then for any $\varphi \in C(\partial\Omega)$ we have the following.

Existence. *There exists $H \in C(\overline{\Omega} - \Sigma)$ such that:*

(1) H *is F-harmonic on $\Omega - \Sigma$,*

(2) $H\big|_{\partial\Omega} = \varphi,$

(3) H *is asymptotically equivalent to h, i.e., there exists $c, C \in \mathbf{R}$ such that*

$$h(x) + c \;\; \leq \;\; H(x) \;\; \leq \;\; h(x) + C \qquad \text{on} \;\; \Omega - \Sigma$$

Uniqueness. *There is at most one function $H \in C(\overline{\Omega} - \Sigma)$ satisfying (1), (2) and (3). (Here the F-convexity of $\partial\Omega$ is not required.)*

Proof. The uniqueness is proved as in Section 5. For existence we choose a sequence of regular values $\{r_j\}$ of h with $r_j \downarrow -\infty$. Again by adjusting h with an additive constant we can assume that (B4) is satisfied. For each j define

$$D_j \;\; \equiv \;\; \{x \in \Omega : h(x) > r_j\} \qquad \text{and} \qquad \Omega_j \;\; \equiv \;\; \Omega - \overline{D}_j.$$

Let H_j be the solution to the Dirichlet Problem on Ω_j for F-harmonic functions with boundary values

$$H_j\big|_{\partial\Omega} \;\; = \;\; \varphi \qquad \text{and} \qquad H_j\big|_{\partial D_j} \;\; = \;\; h\big|_{\partial D_j}$$

on the outer and inner boundaries respectively. As before this solution exists due to the assumption of F-convexity on $\partial\Omega$, the hypothesis (F3) = (B3), the fact that h is in the Perron family by (B4), and the F-harmonicity of h outside Σ.

Now define $\overline{H}_j(x)$ as in Lemma 6.2 and note that the assertions of Lemma 6.3 hold by exactly the same arguments.

It follows that the increasing limit

$$H \;\; \equiv \;\; \lim_{j \to \infty} \overline{H}_j$$

has the properties that H^* is F-subharmonic and $-H$ is $\widetilde{F}$-subharmonic on $\Omega - \Sigma$. Using the solution H^{DP} to the Dirichlet Problem on Ω with boundary values φ the same arguments as above show that

$$H^*\big|_{\partial\Omega} = \varphi. \tag{6.8}$$

We now prove (3). The left-hand inequality in (3) holds since h is in the Perron family for H_j on Ω_j for all j. On the other hand, for $C > 0$ sufficiently large we will have $h + C > \varphi$ on $\partial\Omega$. Since $h + C$ is F-harmonic on $\Omega - \Sigma$ and greater than H_j on $\partial\Omega_j$, we have $h + C > H_j$ on Ω_j by comparison. This establishes the right-hand inequality in (3).

As noted above we have

$$H^* \in F(\Omega - \Sigma) \qquad \text{and} \qquad -H \in \widetilde{F}(\Omega - \Sigma).$$

As before this implies that $H^* - H \in \widetilde{\mathcal{P}}_p(\Omega - \Sigma)$. Now Condition (3) implies Condition (3) for H^* and therefore $H^* - H \leq C - c$ on $\Omega - \Sigma$. We now apply the removable singularity argument in [HL$_7$] using the polar function h (as before) to conclude that $H^* - H$ is F-subharmonic on Ω. By (6.8) $H^* - H$ is ≤ 0 on $\partial\Omega$. Hence, we have $H^* - H \leq 0$ on Ω. We have proved that $H^* = H$ on Ω. This proves $H \in C(\overline{\Omega} - \Sigma)$ and condition (1), and we are done. ∎

7. Existence in the Finite Case.

We now take up the proof of existence in the finite case. It proceeds in two stages. The first (Theorem 7.1) is a construction which provides a family of F harmonics which only satisfy a weakened form of asymptotic equivalence in a range. The second stage shows that one member of the family has the desired asymptotic singularity at the given point.

Throughout this section we assume that F is a cone subequation with the property that all boundaries are strictly $\widetilde{F}$-convex, and that Ω is a domain with smooth strictly F-convex boundary. We limit the discussion to the case of a single singular point $x_0 \in \Omega$. (However, some of the arguments extend to multiple singular points as in the polar case.) The following assumption replaces the hypothesis (H) in Theorem 1.4.

Hypothesis (H1) (Finite Case). We are given a function $h \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_0\}$ and has a downward-pointing singularity at x_0 with $h(x_0) < \inf_{\partial\Omega} h$.

Applying the maximum principle to $-h$ on $\Omega - B_r(x_0)$, for r small, proves that:

$$\text{the point } \ x_0 \ \text{ is a strict global minimum for } \ h \ \text{ on } \ \overline{\Omega}. \tag{7.1a}$$

Moreover, for convenience, we assume by rescaling that

$$h(x_0) \ = \ 0, \qquad \sup_{\partial\Omega} h \ = \ 1 \qquad \text{and} \qquad h(x) \ > \ 0 \ \text{ for } \ x \neq x_0 \ \text{ in } \ \overline{\Omega}. \tag{7.1b}$$

The Construction

It is similar to the construction in the polar case. However, for each $t \geq 0$ we construct a function H^t which is a candidate for the solution with $H^t \sim th$ at x_0. When $t = 0$, the construction will yield the solution to the standard Dirichlet Problem on Ω, namely:

$$H^0 \in C(\overline{\Omega}), \ \ H^0 \text{ is } F \text{ harmonic on } \Omega, \ \text{ and } \ H^0\big|_{\partial\Omega} \ = \ \varphi. \tag{7.2}$$

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The construction of H^t is based on using the function

$$\underline{\psi}_t(x) \;\equiv\; H^0(x) + t(h(x) - 1).\tag{7.3}$$

to prescribe the boundary values on the inner boundary $\partial B_r(x_0)$. An upper bound for H^t is provided by the function

$$\overline{h}_t(x) \;\equiv\; \lambda h(x) + H^0(x_0) + t(h(x) - 1)\tag{7.4}$$

for λ sufficiently large. Note that $\overline{h}_t$ is F-harmonic on $\Omega - \{x_0\}$ since h is F-harmonic there. We need the following "monotonicity" hypothesis to ensure that $\underline{\psi}_t$ is F-subharmonic.

Hypothesis (H2). The function $H^0 + th$ is F-subharmonic on Ω, and therefore, so is each $\underline{\psi}_t$. (Of course (H2) is satisfied if F is a convex subequation.)

Let H_r^t denote the solution, given by Theorem 6.1, to the Dirichlet Problem on $\Omega_r = \Omega - \overline{B}_r(x_0)$ with boundary values φ on $\partial\Omega$ and boundary values $\underline{\psi}_t$ on $\partial B_r(x_0)$. As in Section 6, we will show that these functions H_r^t are pointwise increasing as $r \to 0$. We define H^t by

$$H^t(x) \;\equiv\; \lim_{r\to 0} H_r^t(x) \ \text{ on } \ \overline{\Omega} - \{x_0\} \qquad \text{and} \qquad H^t(x_0) \;\equiv\; H^0(x_0) - t.\tag{7.5}$$

This is our candidate for a "solution" to the (DPPS) with singularity prescribed by th at x_0. Next we show that H^t satisfies all the required conditions with the exception of the asymptotic equivalence $H^t \sim th$ at x_0.

THEOREM 7.1. *For each $t \geq 0$,*
   (1) $H^t \in C(\overline{\Omega})$,
   (2) H^t *is F-harmonic on $\Omega - \{x_0\}$,*
   (3) $H^t\big|_{\partial\Omega} = \varphi$.
In addition,
   (4a) $\underline{\psi}_t \leq H^t$ *with equality at x_0,*
   (4b) $H^t \leq \overline{h}_t$ *(for λ large) with equality at x_0.*

Proof. First note that $\underline{\psi}_t \in C(\overline{\Omega})$ has the properties

$$\underline{\psi}_t \ \text{ is } \ F\text{ subharmonic on } \Omega \qquad \text{and} \qquad \underline{\psi}_t \leq \varphi\tag{7.6}$$

because of (H2) and $h\big|_{\partial\Omega} \leq 1$ respectively. (We also have $\underline{\psi}_t(x_0) = H^0(x_0) - t$ since $h(x_0) = 0$.) Therefore $\underline{\psi}_t$ is in the Perron family for H_r^t, which proves that

$$\underline{\psi}_t \;\leq\; H_r^t \quad \text{on} \ \ \Omega_r.\tag{7.7}$$

Define

$$\overline{H}_r^t \;\equiv\; \begin{cases} \max\{\underline{\psi}_t, H_r^t\} = H_r^t & \text{on } \ \overline{\Omega}_r \\ \underline{\psi}_t & \text{on } \ \overline{B}_r. \end{cases}\tag{7.8}$$

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Then exactly as in Section 6 one proves that

$$\overline{H}_r^t \ \text{ is } F \text{ subharmonic on } \Omega \text{ and increasing in } r \text{ as } r \to 0. \tag{7.9}$$

Define

$$H^t(x) \ \equiv \ \lim_{r \to 0} \overline{H}_r^t(x) \qquad \text{for all } \ x \in \overline{\Omega}. \tag{7.10}$$

Now Property (3) is immediate since $H_r^t\big|_{\partial\Omega} = \varphi$ independent of r. Next note that $\underline{\psi}_t \leq \overline{H}_r^t \leq H^t$ by (7.7). Together with the equality $\underline{\psi}_t(x_0) = \overline{H}_r^t(x_0) = H^t(x_0) - t$ at x_0, this proves (4a).

We begin the proof of (1) and (2). The family $\{\overline{H}_r^t\}_{r>0}$ of functions on $\overline{\Omega}$ is bounded. The upper bound $\overline{H}_r^t \leq H^0$ implies that $H^t \leq (H^t)^* \leq H^0$, and hence (3) can be strengthened to

(3)′ $\ \ H^t\big|_{\partial\Omega} \ = \ (H^t)^*\big|_{\partial\Omega} \ = \ \varphi.$

Also, since H^t is an increasing limit of continuous functions, it is lower semi-continuous. Thus,

$$(H^t)^* \in \mathrm{USC}(\overline{\Omega}) \qquad \text{and} \qquad H^t = (H^t)_* \in \mathrm{LSC}(\overline{\Omega}) \tag{7.11}$$

are finite-valued. Moreover, by the "families bounded above" property and the "decreasing limit" property, we have

$(a)\ \ (H^t)^* \text{ is } F \text{ subharmonic on } \Omega, \quad (b)\ \ -H^t \text{ is } \widetilde{F} \text{ subharmonic on } \Omega - \{x_0\}. \tag{7.12}$

Hence $u \equiv (H^t)^* - H^t \in \mathrm{USC}(\overline{\Omega})$ satisfies $u \geq 0$ on $\overline{\Omega}$, and $u\big|_{\partial\Omega} = 0$ by (3)′. As discussed in the proof of Lemma 5.1, u is subaffine on $\Omega - \{x_0\}$. Therefore, by the Maximum Principle for subaffine functions on the domain $\Omega - \{x_0\}$ we have

$$(H^t)^*(x) - H^t(x) \ \leq \ (H^t)^*(x_0) - H^t(x_0) \ = \ u(x_0) \quad \text{on } \ \overline{\Omega}. \tag{7.13}$$

It remains to show that $u(x) \equiv (H^t)^*(x) - H^t(x)$ equals zero at x_0, i.e.,

$$(H^t)^*(x_0) \ = \ H^0(x_0) - t. \tag{7.14}$$

Once this is established, we will have $(H^t)^* = H^t$, which implies both (1) and (2). Next note that (7.14) follows immediately from the upper bound (4b) since $\overline{h}_t(x_0) = H^0(x_0) - t$ and $\overline{h}_t \in C(\overline{\Omega})$.

Thus it remains to prove (4b). As noted, equality in (4b) holds at x_0. Hence it suffices to prove that $H^t \leq \overline{h}_t$ on $\overline{\Omega} - \{x_0\}$, or that

$$H_r^t \ \leq \ \overline{h}_t \qquad \text{on } \ \Omega_r \ \text{ for small } \ r. \tag{7.15}$$

Since $\overline{h}_t$ is F-harmonic on Ω_r by (H1), it suffices, by comparison, to show that

$$H_r^t \ \leq \ \overline{h}_t \qquad \text{on } \ \partial\Omega_r. \tag{7.16}$$

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That is, we must show that

$$\varphi \;\leq\; \overline{h}_t \qquad \text{on the outer boundary } \partial\Omega, \text{ and} \qquad (7.16a)$$

$$\underline{\psi}_t \;\leq\; \overline{h}_t \qquad \text{on the inner boundary } \partial B_r. \qquad (7.16b)$$

Since $\inf_{\partial\Omega} h > 0$ and $\operatorname{osc}_{\partial\Omega}(h) = 1 - \inf_{\partial\Omega} h$, it is straightforward to see that

$$\lambda \;\geq\; \frac{\sup \varphi - H^0(x_0) + t \operatorname{osc}_{\partial\Omega}(h)}{\inf_{\partial\Omega} h} \qquad \Longleftrightarrow \qquad \sup \varphi \;\leq\; \inf_{\partial\Omega} \overline{h}_t \qquad (7.17)$$

which implies (7.16a). We might as well take $\lambda \equiv \lambda(t)$ to be the affine function of t defined by equality in (7.17).

For (7.16b) first note that $\underline{\psi}_t \leq H^0$ implies $\underline{\psi}_t\big|_{\partial\Omega} \leq \varphi$, and hence by (7.16a), that $\underline{\psi}_t\big|_{\partial\Omega} \leq \overline{h}_t\big|_{\partial\Omega}$. We also have $\underline{\psi}_t(x_0) = \overline{h}_t(x_0)$ since $h(x_0) = 0$. Since $u \equiv \underline{\psi}_t - \overline{h}_t \in \widetilde{\mathcal{P}}(\Omega - \{x_0\})$, we can apply the maximum principle to u on $\Omega - \{x_0\}$ to conclude that $u \leq 0$ on $\overline{\Omega}$. In particular, $u \leq 0$ on $\partial B_r(x_0)$, which is (7.16b). This completes the proof of (4b) and, therefore, of Theorem 7.1. ∎

Prescribing the Density at a Point

We make two additional assumptions on h.

Hypothesis (H3). For each H which is F-subharmonic near $x_0 \in \Omega$ and F-harmonic on a deleted neighborhood of x_0,

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{h(x)} \;\equiv\; \Theta \qquad \text{exists and } \Theta \geq 0, \qquad (7.18)$$

that is, $H \sim \Theta h$ at x_0 for some $\Theta \geq 0$.

There are many examples where this is true. They will be discussed later in this section.

Definition 7.2. Under the hypothesis (H3) the h-**density of H at** x_0, denoted $\Theta_{x_0}(H)$ is defined to be the limit in (7.18).

The second additional assumption on h is that F-harmonics have vanishing densities.

Hypothesis (H4). If H is F-harmonic in a neighborhood of x_0, then $\Theta_{x_0}(H) = 0$, i.e.,

$$\lim_{x \to x_0} \frac{H(x) - H(x_0)}{h(x)} \;=\; 0.$$

.

Definition 7.3. The **Dirichlet Problem with a Prescribed density**, abbreviated (DPPD), is said to be **uniquely solvable for** F if for all $\varphi \in C(\partial\Omega)$ and $\Theta \geq 0$, there exists a unique $H \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_0\}$ and satisfies

$$H\big|_{\partial\Omega} \;=\; \varphi \qquad \text{and} \qquad \Theta_{x_0}(H) \;=\; \Theta \quad (\text{i.e. } H \sim \Theta h).$$

As in Theorem 7.1, we assume (H1). We shall assume that F is convex, so (H2) is unnecessary. In addition we assume (H3) and (H4).

THEOREM 7.4. (Existence for a Prescribed Density). *Assume that F is a convex cone subequation. Then for each $\Theta \geq 0$ and $\varphi \in C(\partial\Omega)$ the (DPPD) is uniquely solvable.*

For the terminology used in the next corollary see the discussion following Corollary 1.5.

Corollary 7.5. (The Nonlinear Green's Function). *There exists a unique nonlinear Green's function $G(x) = G_\Omega(x; x_0, h)$ for the subequation F on the domain Ω with asymptotic singularity determined by the generalized fundamental solution h. Moreover, x_0 is a strict global minimum point for G on Ω.*

Proof. By definition G is the unique solution on Ω with h-density 1 at x_0 and boundary values $\varphi \equiv 0$. Since G has no test functions at x_0, G is F-subharmonic on $\overline{\Omega}$. Hence, by the maximum principle, $G \leq \sup_{\partial\Omega} G = 0$ on $\overline{\Omega}$. Now $-G$ is $\widetilde{F}$-subharmonic on $\Omega - \{x_0\}$. Therefore, by the maximum principle applied to $-G$ on $\Omega - \{x_0\}$, we first get $G(x_0) < 0$ (because $G(x_0) = 0$ implies $-G \leq 0$, and so $G \equiv 0$ on $\overline{\Omega}$ contradicting $\Theta_{x_0}(G) = 1$). Then exactly as in the proof of (7.1a) we get that x_0 is a strict global minimum for G on $\overline{\Omega}$. ∎

Proof of Theorem 7.4. Let H^t, $t \geq 0$ denote the family of downward-pointing F-harmonics at x_0 on Ω with $H^t\big|_{\partial\Omega} = \varphi$, constructed in Theorem 7.1. Let

$$f(t) \equiv \Theta(H^t).$$

To prove Theorem 7.4 it is enough to show that $f([0,\infty)) = [0,\infty)$, and then choose $t \in f^{-1}(\Theta)$ and $H = H^t$. The next lemma supplies this needed fact.

Lemma 7.6. *The function $f(t)$ satisfies:*

(A) $f(0) = 0$,

(B) $f(s) + (t - s) \leq f(t)$ *for* $0 \leq s \leq t$,

(C) $f(t)$ *is concave.*

Proof. Part (A) is immediate from the assumption (H4) since H^0 is F-harmonic on Ω. To prove (B) we show

(B)′ $H^s(x) - H^s(x_0) + (t - s)h(x) \leq H^t(x) - H^t(x_0)$.

The functions

$$u \equiv H_r^s(x) + (t - s)(h(x) - 1) \qquad \text{and} \qquad v \equiv H_r^t(x)$$

have the same boundary values on ∂B_r since $\underline{\psi}_s + (t - s)(h - 1) = \underline{\psi}_t$, while on $\partial\Omega$ we have $u = \varphi + (t - s)(h(x) - 1) \leq \varphi = v$. By the convexity assumption on F, u is F-subharmonic on $\Omega - B_r$. Since v is F-harmonic on $\Omega - B_r$, comparison implies that $u \leq v$ on $\Omega - B_r$. Taking $r \downarrow 0$ gives

$$H^s(x) + (t - s)(h(x) - 1) \leq H^t(x)$$

which implies (B)′ since $H^s(x_0) = H^0(x_0) - s$, $H^t(x_0) = H^0(x_0) - t$, and $h(x_0) = 0$.

For (C) we show that for $0 \leq \sigma \leq 1$ and $0 < s < t$,

$$(C)' \quad \sigma \left(H^s(x) - H^s(x_0) \right) + (1-\sigma) \left(H^t(x) - H^t(x_0) \right) \leq H^{\sigma s + (1-\sigma)t}(x) - H^{\sigma s + (1-\sigma)t}(x_0)$$

on $\overline{\Omega}$, which implies (C) by (H3). Since $H^s(x_0) = H^0(x_0) - s$ for all s, the inequality $(C)'$ is equivalent to

$$(C)'' \qquad \qquad \sigma H^s(x) + (1-\sigma) H^t(x) \leq H^{\sigma s + (1-\sigma)t}(x) \quad \text{on } \overline{\Omega}.$$

This follows from

$$\sigma H_r^s(x) + (1-\sigma) H_r^t(x) \leq H_r^{\sigma s + (1-\sigma)t}(x) \quad \text{on } \overline{\Omega} - B_r, \qquad (7.19)$$

which is true because the LHS and the RHS have the same boundary values on $\partial(\Omega - \overline{B}_r)$ and, by the hypothesis that F is a convex cone, the sum of the two F-harmonics on the LHS is F-subharmonic and therefore in the Perron family for $H_r^{\sigma s + (1-\sigma)t}$. ∎

Remark 7.7. Part (4b) of Theorem 7.1 implies $f(t) \leq t + \lambda(t)$ where $\lambda(t)$ is the affine function of t used in the definition of $\overline{h}_t$. By Lemma 7.6 (B) we have $t \leq f(t)$. We ask the question: When does $f(t) = t$?

Applications of Theorem 7.4.

There are many interesting examples of subequations F and singularities h for which the hypotheses (H3) and (H4) hold.

Case 7.8. (Strong Uniqueness of Tangents). Here we assume that the subequation F is invariant under $O(n)$. Then F has a well-defined Riesz characteristic p which we assume to satisfy $1 \leq p < 2$. The function $h(x) = |x|^{2-p}$ provides the asymptotic singularity Every F-subharmonic function u defined near a point $x_0 \in \mathbf{R}^n$ has a well-defined density $\Theta = \Theta(u, x_0)$ at x_0 defined to be $\lim_{r \to 0} \sup_{B_r(x_0)} u/r^{2-p}$. We say that F satisfies **Strong Uniqueness of Tangents** if, for each u, every tangent to u at x_0 is $\Theta(u, x_0)|x - x_0|^{2-p}$. This Strong Uniqueness holds for every F with the exception of $F = \mathcal{P}$, which is the only possibility when $p = 1$ (see [HL$_{10}$], [HL$_{11}$]).

Proposition 12.6 in [HL$_{10}$] states that for any F-subharmonic (not just a punctured F-harmonic) function u, strong uniqueness of the tangent to u holds if and only if (H3) holds, i.e., $u \sim \Theta K_p$.

This leads to a result where the several hypotheses are automatic

THEOREM 7.9. *Suppose F is $O(n)$-invariant convex cone subequation with Riesz characteristic $1 < p < 2$ Take $h(x) = |x - x_0|^{2-p}$. Then the (DPPD) is uniquely solvable for F.*

Proof. The hypotheses (H3) and (H4) are true for h so that Theorem 7.4 applies. ∎

Case 7.10. (Armstrong-Sirakov-Smart Fundamental Solutions) Now we assume instead that the subequation F in Theorem 7.1 is uniformly elliptic, and let $h = \Phi$ be the fundamental solution discussed in Example 2.5.

THEOREM 7.11. (Armstrong-Sirakov-Smart [AS₁]). *Every F-harmonic function H on $B_r(x_0) - \{x_0\}$, which has a downward-pointing singularity at x_0 is asymptotically equivalent to $\Theta\Phi(x - x_0)$ for some $\Theta \geq 0$.*

This has the following corollary.

THEOREM 7.12. *Suppose the subequation F is uniformly elliptic, and that the downward-pointing fundamental solution Φ of Armstrong, Sirakov and Smart is of finite type (i.e.. of homogeneity > 0). Then with $h = \Phi$ the hypotheses (H3) and (H4) are satisfied and Theorem 7.4 applies.*

8. Prescribing Values at Singularities in the Finite Case.

In the finite case one can also consider the Dirichlet Problem with multiple prescribed singularities where the value of the solution, instead of the density, is given at each singular point. The h_j-density at each singular point x_j is not precise but is replaced by two-sided bounds on the difference quotient. The general result is the following.

THEOREM 8.1. *Let F be a cone subequation satisfying (F3). Fix a function $\varphi \in C(\partial\Omega)$ and points $x_1, ..., x_k \in \Omega$. Suppose that for each $j = 1, ...k$ we are given a function*

$$h_j \in C(\overline{\Omega}) \text{ which is } F\text{-harmonic on } \Omega - \{x_j\}$$

with a downward-pointing singularity of finite type at x_j. Furthermore, assume that

(i)   the boundary of Ω is smooth and strictly F-convex,

(ii)   each h_j has a strict global minimum at x_j.

(iii) there exists a function $h \in C(\overline{\Omega})$ which is F-subharmonic on $\Omega - \{x_1, ..., x_k\}$ with $h \sim h_j$ at x_j for each j and with $h\big|_{\partial\Omega} \leq \varphi$.

Then there exists a function H such that

(1) $H \in C(\overline{\Omega})$,

(2) H is F-harmonic on $\Omega - \{x_1, ..., x_k\}$,

(3) $H\big|_{\partial\Omega} = \varphi$ and $H(x_j) = h(x_j)$ for $j = 1, ..., k$,

(4)   There exists a constant $c > 1$ such that for any $\epsilon > 0$

$$1 - \epsilon \; \leq \; \frac{H(x) - h(x_j)}{h_j(x) - h_j(x_j)} \; \leq \; c \qquad \text{for } x \text{ sufficiently near } x_j, \quad j = 1, ..., k.$$

We postpone the proof to the end of this section, and first examine some special cases. For example, this theorem can be applied to subequations F as in Case 7.8 above with the additional hypothesis of convexity.

Corollary 8.2. *Suppose that F is a convex, $O(n)$-invariant subequation whose Riesz characteristic p satisfies $1 < p < 2$. Let $\Omega \subset\subset \mathbf{R}^n$ be a domain with a smooth strictly*

F-convex boundary. Then given points $x_1, ..., x_k \in \Omega$, positive numbers $\gamma_1, ..., \gamma_k > 0$, and a function $\varphi \in C(\Omega)$, the following holds. For every constant C such that the restriction

$$\left\{ h \;\equiv\; h_C \;\equiv\; \sum_{j=1}^{k} \gamma_j |x - x_j|^{2-p} + C \right\}\Bigg|_{\partial\Omega} \;\leq\; \varphi, \tag{8.1}$$

there exists a function H with properties (1) – (4) above.

Proof. Since F is convex, the function h is F-subharmonic. It satisfies the boundary hypothesis in (iii) by (8.1). Finally, one has that $h(x) \sim \gamma_j |x - x_j|^{2-p}$ at x_j for each j. (To see this note that if $h(x) = \Theta|x|^\alpha + g(x)$ where g is smooth and $0 < \alpha < 1$, then $(h(x) - h(0))/|x|^\alpha \to \Theta$ as $x \to 0$.) Hence, Theorem 8.1 applies. ∎

The simplest subequation of this type is $\mathcal{P}_p$.

We note that in the case where $p = 1$ (i.e., $F = \mathcal{P}$), multiple singularities cannot exist, because a convex function cannot have more than one strict local minimum. Thus Theorem 8.1 does not apply since the global function h does not exist when $k > 1$.

Corollary 8.2 allows us to prescribe certain values for the function H at the singular points. This can be thought of as the Dirichlet problem, if one considers $x_1, ..., x_k$ as additional boundary points. Uniqueness holds for this problem by comparison (see [HL$_9$, Thm. 6.2]) , but in general the values at these interior points cannot be given arbitrarily (see Proposition 8.5 below). However, Corollary 8.2 does provides a set $\mathcal{V} \subset \mathbf{R}^k$ of values which can be prescribed. More precisely, let F, Ω, φ and $x_1, ..., x_k$ be as in Corollary 8.2. Given $\gamma_1, ..., \gamma_k > 0$ and a constant C, let

$$h_{\gamma, C}(x) \equiv \sum_j \gamma_j |x - x_j|^{2-p} + C.$$

Define the set $\mathcal{V} \subset \mathbf{R}^k$ by the condition that $= (v_1, ..., v_k) \in \mathcal{V}$ iff

$$v_j \;=\; h_{\gamma, C}(x_j) \quad \text{for some} \;\; \gamma, C \;\; \text{such that} \;\; h_{\gamma, C}\big|_{\partial\Omega} \leq \varphi. \tag{8.2}$$

Corollary 8.3. ($\mathcal{V} \subset \mathcal{V}al$). *For every $v \in \mathcal{V}$ there exists a solution H to the "value problem". That is, there exists $H \in C(\overline{\Omega})$ which is F-harmonic on $\Omega - \{x_1, ..., x_k\}$, and takes on the boundary values*

$$H\big|_{\partial\Omega} \;=\; \varphi \qquad \text{and} \qquad H(x_j) \;=\; v_j \quad \text{for} \;\; j = 1, ..., k.$$

In addition, H satisfies (4) above with $h_j(x) = \gamma_j |x - x_j|^{2-p}$. Moreover the set $\mathcal{V}$ has non-empty interior and satisfies $\mathcal{V} + (c, ..., c) \subset \mathcal{V}$ for all $c \leq 0$.

Proof. If v is given as in (8.2) with $h_{\gamma, C}\big|_{\partial\Omega} < \varphi$, then $h_{\gamma', C'}\big|_{\partial\Omega} < \varphi$ for (γ', C') in a neighborhood of (γ, C) and the resulting values v' fill out a neighborhood of v in $\mathbf{R}^k$. To see this consider the symmetric $k \times k$ matrix $A = ((a_{ij}))$ where $a_{ij} = |x_i - x_j|^{p-2}$. Then

$\det(A) \neq 0$ and so the mapping $\gamma \mapsto A \cdot \gamma$ is open. This proves that $\mathcal{V}$ has non-empty interior. The remaining assertions are clear. ∎

Note 8.4. (The Nonlinear Green's Functions). In the case of a single point $x_0 \in \Omega$ and boundary values $\varphi = 0$, Corollary 8.3 produces the family of Green's functions $G(x) = G_\Omega(x; x_0, \Theta|x - x_0|^{2-p})$ in Corollary 7.5. To see this, note that the function $H(x)$ given in Corollary 8.3 is ≤ 0 on $\overline{\Omega}$ since by (4) it has no test functions at x_0 and is therefore F-subharmonic on Ω. If $H(x_0) = 0$, then $H \equiv 0$ on $\overline{\Omega}$ by the maximum principle on $\Omega - \{x_0\}$ applied to the $\widetilde{F}$-subharmonic function $-H$. Thus $H(x_0) < 0$ and by uniqueness $H(x) = G(x; x_0, \Theta|x - x_0|^{2-p})$ where $\Theta \geq \gamma > 0$ is the density of H at x_0. It is now obvious by rescaling (since $\varphi = 0$) that the set $\mathcal{V}$ defined by (8.2) at x_0 is exactly $\{v \leq 0\}$.

There remains the question of describing $\mathcal{V}al$, i.e., when, if at all, can one prescribe $v = H(x_0) > 0$.

Proposition 8.5. *Suppose $H \in C(\overline{\Omega})$ is F harmonic on $\Omega - \{x_0\}$ and satisfies $H\big|_{\partial\Omega} = 0$. Then $H(x_0) \leq 0$. Thus, $\mathcal{V} = \{v \leq 0\}$ is exactly the set of values $\mathcal{V}al$ that can be prescribed for the one-point Dirichlet problem with $\varphi = 0$.*

Proof. Assume $x_0 = 0$ for simplicity, and suppose that $H(0) > 0$. Choose $R > 0$ so that $\overline{\Omega} \subset\subset B_R(0)$, and for $q > 2$ and $r > 0$ consider the function

$$v_r(x) \equiv r \left\{ \frac{1}{|x|^{q-2}} - \frac{1}{R^{q-2}} \right\}.$$

The ordered eigenvalues of $D_x^2(v_r)$, up to the positive factor $\frac{r(q-2)}{|x|^q}$, are

$$-1, \ ... \ , -1, (q-1)$$

and so the eigenvalues of $D_x^2(v_r)$ satisfy

$$\lambda_{\min} + (p-1)\lambda_{\max} \cong -1 + (p-1)(q-1) \ < \ 0 \tag{8.3}$$

if we choose

$$q \ < \ 1 + \frac{1}{p-1}$$

which is possible since $0 < p - 1 < 1$.

Now for $r > 0$ sufficiently large we have $v_r > H$ on $\overline{\Omega}$. (In fact by the maximum principle on $\Omega - \{0\}$ one has that $H(0)$ is a global maximum of H on $\overline{\Omega}$.) Let

$$r_0 \ \equiv \ \inf\{r : v_r > H \ \text{ on } \ \overline{\Omega}\} \ \geq \ 0.$$

Now r_0 cannot be 0 since we are assuming that H is continuous with $H(0) > 0$. Thus, there is a point $y \in \overline{\Omega}$ where $H(y) = v_{r_0}(y)$. Note that $y \notin \partial\Omega$ since $H = 0$ there, and $y \neq 0$ since $v_{r_0}(0) = \infty$.

We conclude that $y \in \Omega - \{0\}$ and v_{r_0} is a test function for H at y. In particular, $D_y^2(v_{r_0}) \in F$. However, since F is $O(n)$-invariant and of characteristic p, we have by

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Proposition 3.13 in [HL$_{10}$] that $F \subset \mathcal{P}_p^{\text{min/max}} \equiv \{\lambda_{\min} + (p-1)\lambda_{\max} \geq 0\}$. Thus we have $D_y^2(v_{r_0}) \in \mathcal{P}_p^{\text{min/max}}$ contradicting (8.3). ∎

For multiple singular points things are much more complicated.

Corollary 8.6. (Prescribing Values at Multiple Singularities). *Let F and Ω be as above, and fix points $x_1, ..., x_k \in \Omega$. Take boundary values $\varphi = 0$ and choose $\gamma \in \mathbf{R}_+^k$. Then for each C satisfying $h_{\gamma,C}\big|_{\partial\Omega} \leq 0$, one obtains from Corollary 8.3 the existence of a* **multi-pole Green's function** *$G(x) = G_\Omega(x; x_1, ..., x_k)$ with a singularity of type $|x - x_j|^{2-p}$ at x_j but with an unknown density $\Theta_j \geq \gamma_j$ at x_j. As in (8.2) let $\mathcal{V} \subset \mathbf{R}^k$ be the set of all values $v = (G(x_1), ..., G(x_k))$ at the singular points, obtained by varying γ. Then $\mathcal{V}$ is a convex cone in the negative "octant" $\mathbf{R}_-^k \subset \mathbf{R}^k$.*

Proof. The set $\mathcal{V}$ given in Corollary 8.3 is described as follows. For each $\gamma = (\gamma_1, ..., \gamma_k) \in \mathbf{R}_+^k$ set

$$F_\gamma(x) = \sum_{j=1}^k \gamma_j |x - x_j|^{2-p} \qquad \text{and} \qquad C(\gamma) = -\sup_{x \in \partial\Omega} F_\gamma(x)$$

Then $v = (v_1, ..., v_k) \in \mathcal{V}$ if and only if there exist γ and C such that

$$v_j = F_\gamma(x_j) - C \quad \text{for} \quad j = 1, ..., k \qquad \text{and} \qquad C \leq C(\Omega). \tag{8.5}$$

It is clear that $C(t\Omega) = tC(\Omega)$ and $F_{t\gamma} = tF_\gamma$ for all $t \geq 0$, so $\mathcal{V}$ is a cone with vertex the origin. It remains to show that $\mathcal{V} + \mathcal{V} \subset \mathcal{V}$. We begin by observing that

$$C(\gamma + \gamma') \geq C(\gamma) + C(\gamma') \tag{8.6}$$

(since the sup of the sum is $\leq$ the sum of the sups). Consider the vectors $v(\gamma)$ obtained by taking $C = C(\Omega)$ in (8.5). Then we have

$$
\begin{aligned}
v_j(\gamma + \gamma') &= F_{\gamma+\gamma'}(x_j) + C(\gamma + \gamma') \\
&= F_\gamma(x_j) + F_{\gamma'}(x_j) + C(\gamma) + C(\gamma') - \kappa \\
&= v_j(\gamma) + v_j(\gamma') - \kappa
\end{aligned}
$$

where $\kappa \equiv C(\gamma + \gamma') - C(\gamma) + C(\gamma') \geq 0$ by (8.6). Thus by (8.5) we have $v(\gamma) + v(\gamma') \in \mathcal{V}$, and the assertion follows easily. ∎

For a slightly different perspective, note that $F_\gamma(x) = \gamma \cdot F(x)$ where

$$F(x) = (|x - x_1|^{2-p}, |x - x_2|^{2-p}, ..., |x - x_k|^{2-p}).$$

and consider the symmetric $k \times k$ matrix with positive entries:

$$A = \begin{pmatrix} F(x_1) \\ F(x_2) \\ \vdots \\ F(x_k) \end{pmatrix} = (F(x_1)^t, F(x_2)^t, ..., F(x_k)^t)$$

Then the set $\mathcal{V}$ is given by:

$$v \in \mathcal{V} \qquad \Longleftrightarrow \qquad v = A \cdot \gamma + C(1, ..., 1) \quad \text{for} \ \ \gamma \in \mathbf{R}_+^k \ \ \text{and} \ \ C \leq C(\Omega) \qquad (8.7)$$

Taking $\gamma = e_j = (0, ..., 0, 1, 0, ..., 0)$ (the jth coordinate vector in $\mathbf{R}^k$) for $j = 1, ..., k$ we have the following.

Lemma 8.7. *The set $\mathcal{V}$ contains the convex cone in the negative octant generated by the vectors*

$$V_j \ \equiv \ F(x_j) - \sup_{x \in \partial \Omega} |x - x_j|^{2-p}(1, 1, ..., 1), \qquad j = 1, ..., k.$$

Remark 8.8. It is important to note that the functions $G_\Omega(x; x_1, ..., x_k)$ constructed in Corollary 8.6 do not have h_j- density equal to γ_j at x_j (where $h_j(x) = |x - x_j|^{2-p}$). By Theorem 8.1 (4) the constant γ_j is a lower bound on the density and there exists a global upper bound depending on the data.

Question 8.9. Can one determine these densities, at least in geometrically simple cases?

Question 8.10. Can one determine the value set $\mathcal{V}$ defined by (8.2), at least in relatively simple cases?

Question 8.11. What is the full set $\mathcal{Val} \supset \mathcal{V}$ of possible values v (both depending on $\varphi \in C(\partial \Omega)$) for which there exists a solution H to the "value problem" (as defined in Corollary 8.3).

Note 8.4 and Proposition 8.5 answer this question in the case of a single point.

Proof of Theorem 8.1. We apply the existence construction given in Section 6. Consider $r > 0$ sufficiently small that the closed balls $\overline{B}_r(x_j), 1 \leq j \leq k$ are mutually disjoint and contained in Ω. Let H_r be the F-harmonic function on $\Omega_r \equiv \Omega - \bigcup_j \overline{B}_r(x_j)$ with boundary values φ on $\partial \Omega$ and h on each $\partial B_r(x_j)$. We extend H_r to $\overline{H}_r \in C(\overline{\Omega})$ by setting $\overline{H}_r = h$ on each $B_r(x_j)$. The arguments of Section 6 show that

$$\overline{H}_r \ \uparrow \ H \qquad \text{as} \ \ r \to 0.$$

As before H^* is F-subharmonic and $-H$ is $\widetilde{F}$-subharmonic on $\Omega - \{x_1, ..., x_k\}$. As before, let H^{DP} be the solution to the Dirichlet problem on Ω with boundary values φ. Then $\overline{H}_r \leq H^{\text{DP}}$ and therefore $H \leq H^{\text{DP}}$ on $\overline{\Omega}$. Thus $H^* \leq H^{\text{DP}}$ and in particular $H^*\big|_{\partial \Omega} = \varphi$. Note also that $h \leq H \leq H^*$.

Now for each j we choose a constant $c_j > 0$ sufficiently large that the punctured F-harmonic function

$$H^j(x) \ \equiv \ c_j(h_j(x) - h_j(x_j)) + h(x_j)$$

satisfies

$$H^j \ > \ h \ \ \text{on} \ \ \overline{\Omega} - \{x_j\} \qquad \text{and} \qquad H^j \ > \ \varphi \ \ \text{on} \ \ \partial \Omega.$$

This can be done since h_j has a strict global minimum at x_j. It follows that $\overline{H}_r \leq H^j$ on $\overline{\Omega}$ for all small r, and therefore $H \leq H^j$. In sum we have that

$$h \ \leq \ H \ \leq \ H^* \ \leq \ H^j \quad \text{for each} \ j = 1, ..., k,$$

and in particular H and H^* are continuous at each x_j with $H(x_j) = H^*(x_j) = h(x_j)$. Subtracting $h(x_j)$ and dividing by the positive function $h(x) - h(x_j)$ gives

$$1 \leq \frac{H(x) - h(x_j)}{h(x) - h(x_j)} \leq \frac{H^*(x) - h(x_j)}{h(x) - h(x_j)} \leq \frac{H^j(x) - h(x_j)}{h(x) - h(x_j)} = c_j \quad \text{near } x_j \tag{8.8}$$

We now consider the function $u \equiv H^* - H$ which is continuous on $\overline{\Omega}$, subaffine on $\Omega - \{x_1, ..., x_k\}$, and satisfies $u\big|_{\partial\Omega} = 0$ and $u(x_j) = 0$ for all j. It follows that $u \leq 0$ on $\overline{\Omega}$. Hence, $H = H^*$ and so H is F-harmonic on $\Omega - \{x_1, ..., x_k\}$.

The assertion (4) now follows from (8.8) and the fact that $h \sim h_j$ at x_j, i.e.,

$$\lim_{x \to x_j} \frac{h(x) - h(x_j)}{h_j(x) - h_j(x_j)} = 1. \qquad \blacksquare$$

Appendix A. Asymptotic Equivalences and Tangent Flows (for p $\neq$ 2).

The asymptotic equivalence classes can be related to the tangent flow very generally. Suppose u is an upper semi-continuous function defined in a neighborhood of the origin. Recall the tangent flow and the Riesz kernel:

$$u_r(x) \equiv \begin{cases} r^{p-2} u(rx) & \text{if } p \geq 2 \\ \frac{u(rx) - u(0)}{r^{2-p}} & \text{if } 1 \leq p < 2. \end{cases} \quad \text{and} \quad K(x) \equiv \begin{cases} -\frac{1}{|x|^{p-2}} & \text{if } p \geq 2 \\ |x|^{2-p} & \text{if } 1 \leq p < 2. \end{cases}$$

Note that $K_r = K$ for the full range $p \geq 1$, $p \neq 2$.

Fix $\Theta > 0$. We define the first notion of asymptotic equivalence at 0 as follows.

Definition A.1. $u \sim \Theta K$ at 0 if

$$\lim_{x \to 0} \frac{u(x)}{K(x)} = \Theta \qquad \text{when } p \geq 2$$
$$\lim_{x \to 0} \frac{u(x) - u(0)}{K(x)} = \Theta \qquad \text{when } 2 > p \geq 1 \tag{A.1}$$

We have a second alternative definition of asymptotic equivalence when $p > 2$.

Definition A.2. $u \approx \Theta K$ at 0 if

$$u - \Theta K \quad \text{is bounded in a neighborhood of the origin.} \tag{A.2}$$

This notion is stronger. (When $2 > p \geq 1$ it is weaker, in fact too weak to be useful.)

Proposition A.3. ($p > 2$). If $u \approx \Theta K$, then $u \sim \Theta K$. However, the converse is false.

Proof. Note that

$$u \approx \Theta K \qquad \Longleftrightarrow \qquad |u(x) - \Theta K(x)| \leq C \quad \text{near } x = 0$$
$$\Longleftrightarrow \qquad \left| \frac{u(x)}{K(x)} - \Theta \right| \leq \frac{C}{|K(x)|} \quad \text{near } x = 0$$

This implies that $\lim_{x \to 0} u(x)/K(x) = \Theta$ since $K(0) = -\infty$. See Example A.6 for a counterexample to the converse. ∎

These asymptotic equivalences are related to the tangent flows as follows.

Proposition A.4. ($p > 2$).

$$(a) \quad u \approx \Theta K \qquad \Rightarrow \qquad u_r \to \Theta K \ \text{ uniformly on } \ \overline{B}_R$$

$$(b) \quad u_r - \Theta K \ \text{ bounded near } \ 0 \ \text{ for some } r \qquad \Rightarrow \qquad u \approx \Theta K.$$

Proposition A.5.

(a) $(1 \le p < 2)$ $\quad u \sim \Theta K \quad \Rightarrow \quad u_r \to \Theta K \ \text{ uniformly on } \ \overline{B}_R$

(b) $(2 < p)$ $\quad u \sim \Theta K \quad \Rightarrow \quad u_r \to \Theta K \ \text{ uniformly on } \ \overline{A}_{s,R}$

(c) $(1 \le p < \infty)$ $\quad u_r \to \Theta K \ \text{ uniformly on some sphere } \partial B_R \quad \Rightarrow \quad u \sim \Theta K$

Proof of Proposition A.4(a). Assume $u \approx \Theta K$. The inequality

$$|u(y) - \Theta K(y)| \ \le \ C \tag{A.3}$$

can be rewritten with $y = rx$ as

$$|u_r(x) - \Theta K(x)| \ \le \ C r^{p-2} \tag{A.3$'$}$$

by applying the tangent flow to both sides. Thus if (A.3) holds for all $|y| \le \delta$, then (A.3)$'$ holds for all $|x| \le R$ and $r \le \delta/R$, which suffices to prove that u_r converges uniformly to ΘK on B_R.

Proof of Proposition A.5(a), (b). With y, r, x related by $y = rx$, the inequalities

$$(a) \quad \left| \frac{u(y)}{K(y)} - \Theta \right| \ \le \ \epsilon \qquad \text{and} \qquad (b) \quad |u_r(x) - \Theta K(x)| \ \le \ \epsilon |K(x)| \tag{A.4}$$

are equivalent. If (a) holds for $|y| \le \delta$, then (b) holds for $|x| \le R$ and $r \le \delta/R$.

Case: $1 \le p < 2$. Note that $|x| \le R \Rightarrow |K(x)| = K(x) \le K(R)$. This proves that (A.4a) for $|y| \le \delta$ implies $|u_r(x) - \Theta K(x)| \le \epsilon K(R)$ for $|x| \le R$ and $r < \delta/R$. Thus u_r converges uniformly to ΘK on $\overline{B}_R$.

Case: $2 < p$. Note that $s \le |x| \Rightarrow |K(x)| \le |K(s)|$. This proves that (A.4a) for $|y| \le \delta$ implies that $|u_r(x) - \Theta K(x)| \le \epsilon |K(s)|$ for $s \le |x| \le R$ and $r < \delta/R$, since $|y| = r|x| \le \frac{\delta}{R} R = \delta$. Thus u_r converges uniformly to ΘK on $A_{s,R}$.

Proof of Proposition A.4(b). With y, r, x related by $y = rx$, the inequalities

$$\begin{aligned} (a) \quad & |u_\delta(x) - \Theta K(x)| \ \le \ C \qquad \forall \, |x| \le R \qquad \text{and} \\ (b) \quad & \delta^{p-2} |u(y) - \Theta K(y)| \ \le \ C \qquad \forall \, |y| \le \delta R \end{aligned} \tag{A.5}$$

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are equivalent. If (a) holds for all $|x| \leq R$, then (b) holds for all $|y| \leq \delta R$. This proves that $u \approx \Theta K$ if (a) is true for some $\delta, C > 0$ and all $|x| \leq R$

Proof of Proposition A.5(c). Again if $y = rx$, then

$$(a) \quad |u_r(x) - \Theta K(x)| \leq \epsilon \qquad \text{and} \qquad (b) \quad \left| \frac{u(y)}{K(y)} - \Theta \right| \leq \frac{\epsilon}{|K(x)|} \qquad (A.6)$$

are equivalent. To see this divide both sides of (A.6a) by $|K(x)|$ and note that $\frac{u_r(x)}{K(x)} = \frac{u_r(x)}{K_r(x)} = \frac{u(rx)}{K(rx)}$. Suppose (A.6a) holds for all $|x| = R$ and $r \leq \delta$. Then (A.6b) holds for all $y = rx$ with $|x| = R$ and $r \leq \delta$, or for all $|y| \leq \delta R$. Since $K(x) = K(R)$ this is enough to prove that $\lim_{y \to 0} u(y)/K(y) = \Theta$.

Example A.6. (Counterexamples to: $u \sim \Theta K \Rightarrow u \approx \Theta K$). Take

$$F = \Delta, \qquad K(x) = -\frac{1}{|x|^{n-2}}, \qquad \text{and} \qquad u(x) \equiv -\frac{\Theta}{|x|^{n-2}} - \frac{1}{|x|^{p-2}}$$

with $p < n$. Then $\lim_{x \to 0} \frac{u(x)}{K(x)} = \lim_{x \to 0} (\Theta + |x|^{n-p}) = \Theta$ so that $u \sim K$. However, $u - \Theta K = -\frac{1}{|x|^{p-2}}$ is not bounded near the origin, i.e., $u \approx \Theta K$ is not true. Note that u is Δ-subharmonic.

In this example we could also replace n with any $q \leq n$ and take $p < q$. In this case the corresponding function u is $\mathcal{P}_q$-subharmonic since $\mathcal{P}_p \subset \mathcal{P}_q$.

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