Reflections on the Early Work of Eugenio Calabi

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I first met Gene[2] Calabi at Stanford in 1967 while I was still a graduate student. That encounter had special significance in my mathematical life. Most importantly it began a long friendship with a man whose mathematical originality and character I have always deeply admired. I can't remember a conversation with Gene in which I did not learn something fascinating. He has always been open and free with his ideas and wonderfully supportive of young mathematicians.

Much is made in this volume, and in the mathematical community in general, about Gene's role in the subjects of Calabi–Yau manifolds and extremal metrics. While his ideas and theorems about special metrics in complex geometry have certainly been revolutionary, his earlier work was also of transforming originality. For personal reasons I am particularly fond of that early work, and I would like to present it here and discuss its impact over the years.

I want to express my thanks and indebtedness to Gopal Prasad for the detailed comments below concerning the seminal impact of the work of Calabi and Vesentini on the subject of rigidity of lattices in Lie groups.

Note: Naturally I submitted a preliminary version of this article to Gene for his comments. Several of his responses are included at appropriate points in the text below. The reader should find them interesting.

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1. Calabi's Thesis: Isometric Holomorphic Embeddings

Gene wrote his Ph.D. thesis at Princeton under the direction of Salomon Bochner. It was concerned with the question:

> *Which complex hermitian manifolds (X, h) can be holomorphically and isometrically embedded into $\mathbf{C}^N$ (or more generally into a complex space form)?*

Calabi's answers to this problem constitute a gorgeous contribution to mathematics. It pre-dated by only a few years the work of Nash [21, 22] on the analogous riemannian embedding problem. As you shall see, these two cases are quite different in nature.

Note to begin that if (X, h) can be isometrically and holomorphically embedded into $\mathbf{C}^N$, then the metric h must be Kähler and real analytic. This holds also if $\mathbf{C}^N$ is replaced by any separable complex Hilbert space. Let $\mathbf{E}$ denote the indefinite complex Hilbert space of sequences $(z_1, z_{-1}, z_2, z_{-2}, \ldots)$ such that $\sum_k |z_k|^2 < \infty$, provided with the indefinite hermitian form $\mathrm{d}s^2 = \sum_k \operatorname{sgn}(k)|z_k|^2$.

Calabi's Universal Embedding Theorem. *Every complex manifold with an analytic Kähler metric can be locally embedded isometrically and holomorphically into* $\mathbf{E}$.

For embeddings into $\mathbf{C}^N$ (with positive definite metric) the story is quite different. The metrics that can be embedded are *highly restricted* and *intrinsically characterized*. Furthermore, when embeddings exist, they are unique up to ambient isometries. All this is true with $\mathbf{C}^N$ replaced by the complex space form $\mathbf{M}^N(c)$ of constant holomorphic sectional curvature c (complex projective, euclidean or hyperbolic spaces). Note that there are totally geodesic embeddings $\mathbf{M}^N(c) \subset \mathbf{M}^{N+N'}(c)$ unique up to isometries.

Calabi's Local Rigidity Theorem. *Suppose $f : X \to \mathbf{M}^N(c)$ and $f' : X \to \mathbf{M}^{N+N'}(c)$ are holomorphic isometric embeddings. Then $f = F \circ f'$ for some isometry F of $\mathbf{M}^{N+N'}(c)$.*

This is of course in stark contrast to the riemannian case. Let $\gamma : \mathbf{R} \hookrightarrow \mathbf{R}^2$ be a curve parameterized by arc-length. Then $(x, s) \mapsto (x, \gamma(s))$ is an isometric embedding of euclidean space $\mathbf{R}^n \hookrightarrow \mathbf{R}^{n+1}$.

Calabi's Intrinsic Characterization of Existence. *Let (X, h) be a connected Kähler manifold with a real analytic metric. Then there exists a holomorphic isometric embedding of a neighborhood of a point $p \in X$ into $\mathbf{M}^N(c)$ if and only if the metric satisfies an intrinsic, explicitly computable condition at p. Moreover, if this condition holds at one point, then it holds at all points of X.*

This intrinsic condition on the metric can be expressed in terms of Calabi's *diastatic function*, which we shall now examine. Recall that in holomorphic coordinates $z=(z_1,...,z_n)$ a Kahler metric $\mathrm{d}s^2=\sum_{i,j}h_{i\bar{j}}\mathrm{d}z_i\mathrm{d}\bar{z}_j$ can be written locally in terms of a real-valued potential function φ, that is,

$$h_{i\bar{j}}=\frac{\partial^2\varphi}{\partial z_i\partial\bar{z}_j}.$$

This potential is not unique, but any other such potential $\widetilde{\varphi}$ is of the form

$$\widetilde{\varphi}(z,\bar{z}) \;=\; \varphi(z,\bar{z})+f(z)+\overline{f(z)} \quad \text{where } f \text{ is holomorphic.} \tag{1.1}$$

Recall now that a real analytic function $\varphi(z,\bar{z})$ can be expressed as a convergent power series in z and $\bar{z}$ and therefore extends to a holomorphic function of $2n$ variables $\varphi(z,w)$. The expression in (1.1) becomes

$$\widetilde{\varphi}(z,w) \;=\; \varphi(z,w)+f(z)+\overline{f}(w) \quad \text{where } \overline{f}(w)=\overline{f(\bar{w})}. \tag{1.2}$$

From all of this one can deduce the following.

Calabi's Diastatic Function D**.** *Let $U\subset X$ be an open set with local holomorphic coordinates $z:U\xrightarrow{\cong}V\subset\mathbf{C}^n$ (V simply-connected). Then the* **diastatic function**

$$D:U\times U \;\longrightarrow\; \mathbf{R},$$

defined by

$$D(p,q) \;\equiv\; \varphi(z(p),\bar{z}(p))+\varphi(z(q),\bar{z}(q))-\varphi(z(p),\bar{z}(q))-\varphi(z(q),\bar{z}(p)),$$

is independent of the choice of potential and independent of the choice of local coordinates. Hence it is **intrinsically defined** *in terms of the Kähler metric on U. It is real-valued, $D(p,q)=D(q,p)$, and $D(p,p)=0$. Furthermore, if $d(p,q)$ is the intrinsic (riemannian) distance function on X, then $D(p,q)=d(p,q)^2+O(d(p,q)^4)$.*

This function can be placed in a global context. Given any complex manifold X, there is a complex conjugate manifold $\overline{X}$ obtained by taking the atlas $\{(U_\alpha,z_\alpha)\}_\alpha$ and replacing it with the atlas $\{(U_\alpha,\overline{z}_\alpha)\}_\alpha$ whose transition functions are also holomorphic. The assertions above show that for real analytic Kähler metrics, the diastatic function D is well-defined in a neighborhood of the diagonal in $X\times\overline{X}$.

The Fundamental Property of D**.** *Let Y be a Kähler manifold with real analytic metric and diastatic function D_Y. Suppose $f:X\to Y$ is a holomorphic isometric embedding. Then D_Y pulls back to be the diastatic function of X, i.e., $D_X(p,q)=D_Y(f(p),f(q))$.*

Thus if $f:X\to\mathbf{C}^N$ is a holomorphic isometric immersion, then

$$D(p,q) \;=\; |f(p)-f(q)|^2.$$

Therefore, for fixed q,

$$D_q(p) \equiv D(p,q) = \sum_{k=1}^{N} |f_k(p)|^2 \quad \text{where } f_1,\ldots,f_N \text{ are holomorphic functions.}$$

Taking power series expansions in local coordinates z we have

$$D_q = \sum_{\alpha,\beta} C_{\alpha\beta} z^\alpha \bar{z}^\beta \qquad \text{and} \qquad f_k = \sum_\alpha a_{k,\alpha} z^\alpha \qquad \text{for each } k.$$

Enumerating the multi-indices we get the following identity of $\infty \times \infty$-matrices

$$C = a_1 \cdot \overline{a}_1^t + \cdots + a_N \cdot \overline{a}_N^t,$$

where a_k is considered as an infinite column vector. Thus

$$C \text{ is a positive semidefinite matrix of rank } \leq N. \tag{1.3}$$

This is precisely the necessary and sufficient condition for the euclidean case in the Intrinsic Characterization Theorem above.

For $f : X \to \mathbf{M}^N(c)$ with $c \neq 0$, the function D_q takes the form $\frac{1}{c}\log(1 + c\sum_k |f_k|^2)$ in appropriate local coordinates, so one can exponentiate and apply the above.

These general results have the following very nice application.

Theorem 1.1. *Let $X \subset \mathbf{M}^N(c)$ be a (local) n-dimensional complex submanifold whose induced metric has constant holomorphic sectional curvature $\overline{c}$.*

(*i*) *If $c \leq 0$, then $\overline{c} = c$ and X is totally geodesic*

(*ii*) *If $c > 0$, then $\overline{c} = \frac{1}{k}c$ for some integer k with $\binom{n+1}{k} \leq N+1$, and (after normalizing to $c = 1$), X is congruent to a piece of the "round" Veronese embedding*

$$\Phi_k : \mathbf{P}^n \hookrightarrow \mathbf{P}^N$$

given in homogeneous coordinates $[Z] = [Z_0, \ldots, Z_n]$ by

$$[Z] \mapsto \left[Z_0^k, \ldots, \sqrt{\binom{n}{\alpha}} Z^\alpha, \ldots, Z_n^k, 0, \ldots, 0\right] \equiv [W],$$

so that $|W|^2 = |Z|^{2k}$.

2. The Special Case of Holomorphic Curves

Suppose now that $\dim_{\mathbf{C}}(X) = 1$, that is, we restrict attention to holomorphic isometric immersions

$$f : \Sigma \to \mathbf{M}^n(c)$$

of a metric Riemann surface Σ. This is the most transparent and beautiful case. It is also the basic one. The general results as stated above follow directly from the 1-dimensional versions.

In many ways it is intuitively correct to think of $f : \Sigma \to \mathbf{M}^n(c)$ as a complex curve – generalizing real curves in, say, euclidean space. Note, however, that by the Rigidity Theorem, *all the extrinsic invariants of this curve are intrinsically determined by the metric.* More specifically, for any such holomorphic curve, one can establish Frenet formulas in complete analogy with the real case. The normal bundle decomposes generically into a flag of osculating bundles with associated curvature functions $K_k > 0$, $k = 1, \ldots, n-1$. In the real case these curvature functions can be chosen freely. In the complex case, Calabi shows that each K_k can be computed explicitly in terms of the metric $\mathrm{d}s^2$ on Σ. Furthermore, points where the k^{th} osculating bundles fail to be defined directly (some $K_j = 0$ for $j \le k$) are isolated, and the bundle has a holomorphic extension across these points. These points determine a divisor on the Riemann surface Σ, and in the case of an algebraic curve $\Sigma \to \mathbf{P}^n$ one can deduce the classical Plücker formulas directly from the metric and these divisors.

To present these explicit calculations we work locally on the Riemann surface. Let

$$\mathrm{d}s^2 \;=\; 2F|\mathrm{d}z|^2$$

be a real analytic metric on the unit disk $\Delta = \{z : |z| < 1\}$. In [3, 4] (see also [8]) Calabi proves the following.

Theorem 2.1. *The metric* $\mathrm{d}s^2$ *is induced by a full homomorphic map*

$$f : \Delta \;\to\; \mathbf{M}^n(c)$$

if and only if the recursively defined functions

$$F_0 \;=\; 1, \qquad F_1 \;=\; F,$$

$$F_{k+1} \;=\; \frac{F_k^2}{F_{k-1}}\left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\log F_k + (k+1)c\right) \tag{2.1}$$

satisfy the conditions:

(1) *For $k \le n$ the function F_k is ≥ 0 and vanishes only at isolated points.*
(2) *The succeeding function F_{k+1} is defined by (2.1) away from those points, but extends to a real analytic function on all of Δ.*
(3) $F_{n+1} \;\equiv 0.$

(A “full” map is one whose image does not lie in any totally geodesic $\mathbf{M}^m(c) \subset \mathbf{M}^n(c)$ for $m < n$.)

If $(\Sigma, \mathrm{d}s^2)$ is a connected Riemann surface with an analytic metric, then the conditions in Theorem 2.1 hold in some local coordinate system on Σ if and only if they hold in every local coordinate system. In this case there exists a global holomorphic isometric immersion of the universal covering of Σ into $\mathbf{M}^n(c)$.

Associated to the F_k is a globally defined sequence of *curvature functions*

$$K_k \;\equiv\; \frac{F_{k+1}F_{k-1}}{FF_k} \;=\; \tfrac{1}{2}\left\{\Delta \log F_k + (k+1)c\right\}, \tag{2.2}$$

where $\Delta \equiv \frac{2}{F}\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}$. Note that each $K_k \geq 0$ and $K_1 = c - K$, where K is the Gauss curvature of the metric $\mathrm{d}s^2$.

To see the meaning of all this we first consider the euclidean case $c = 0$. Given a holomorphic map

$$f : \Delta \;\to\; \mathbf{C}^n$$

with induced metric $\mathrm{d}s^2 = 2F|\mathrm{d}z|^2$, we have

$$F \;=\; |\varphi|^2 \;=\; \sum_{j=1}^{n} |\varphi_j|^2,$$

where $\varphi(z) \equiv f'(z)$. Calculation shows that

$$\begin{aligned}
F_1 &= |\varphi|^2 \\
F_2 &= |\varphi \wedge \varphi'|^2 \\
F_3 &= |\varphi \wedge \varphi' \wedge \varphi''|^2 \\
&\;\vdots \\
F_k &= |\varphi \wedge \varphi' \wedge \varphi'' \wedge \cdots \wedge \varphi^{(k)}|^2 \\
&\;\vdots
\end{aligned}$$

from which the necessity of the conditions in Theorem 2.1 is clear, as is the relation of the functions F_k to the k^{th} osculating curve $\Phi_k \equiv [\varphi \wedge \varphi' \wedge \cdots \wedge \varphi^{(k-1)}] : \Delta \to \mathbf{P}^{\binom{n}{k}-1}$ of f.

The functions $K_1, K_2, K_3, \ldots$ are exactly the curvature functions which appear in the "Frenet formulas" for this given curve. (See [13] for details.)

The function K_k can also be expressed as the ratio $\mathrm{d}\sigma_k^2/\mathrm{d}s^2$ where $\mathrm{d}\sigma_k^2$ is the metric induced from $\mathbf{P}^{\binom{n}{k}-1}$ by the map Φ_k.

The projective case is similar. Let

$$f : \Delta \;\to\; \mathbf{P}^n$$

(with holomorphic sectional curvature 1) be represented in homogeneous coordinates by a holomorphic map

$$\varphi = (\varphi_0, \ldots, \varphi_n) : \Delta \;\to\; \mathbf{C}^{n+1}.$$

Then the functions in Theorem 2.1 are seen to be

$$F_k \;=\; \frac{|\varphi \wedge \varphi' \wedge \cdots \varphi^{(k)}|^2}{|\varphi|^{2(k+1)}}.$$

Again there is a well-defined osculating curve

$$\varPhi_k \;\equiv\; [\varphi \wedge \varphi' \wedge \cdots \wedge \varphi^{k)}] : \varSigma \;\to\; \mathbf{P}^{\binom{n+1}{k+1}-1}$$

and again K_k can also be expressed as the ratio $\mathrm{d}\sigma_k^2/\mathrm{d}s^2$ where $\mathrm{d}\sigma_k^2$ is the metric induced by the map $\varPhi_k$. When the Riemann surface $\varSigma$ is compact, the projective degree ν_k of the K^{th} osculating curve $\varPhi_k$ is given by

$$\nu_k \;=\; \frac{1}{2\pi}\int_{\varSigma} K_k \, dA.$$

For further details see [13].

Final Note. In the case of curves Calabi's Rigidity Theorem has an elementary proof. Suppose $f : \varDelta \to \mathbf{C}^n$ and $g : \varDelta \to \mathbf{C}^{n+m}$ are isometric holomorphic maps. We claim that $f = T \circ g$, where T is a holomorphic isometry of $\mathbf{C}^{n+m}$ (a unitary transformation followed by a translation). Setting $\varphi = f'$ and $\psi = g'$ we have $|\varphi|^2 \equiv |\psi|^2$ on $\varDelta$ since the induced metrics agree. Assume the first coordinate function $\varphi_1(0) \neq 0$ and set $\varPhi = \frac{1}{\varphi_1}\varphi$, $\varPsi = \frac{1}{\varphi_1}\psi$ on a neighborhood of zero. Then $1 + |\varPhi_2|^2 + |\varPhi_3|^2 + \cdots = |\varPsi|^2$. Taking $\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}$ gives $|\varPhi_2'|^2 + |\varPhi_3'|^2 + \cdots = |\varPsi'|^2$. By induction on n there exists a unitary transform U_0 with $\varPhi' = U_0\varPsi'$. Integrating and then multiplying through by φ_1 and checking constants of integration gives $\varphi = U\psi$ for a unitary matrix U.

Calabi's Comments on his Thesis: "The original idea came in the summer of 1949. I knew from recent lecture notes about Bieberbach's imbedding of the complex unit disc with the hyperbolic metric in the Hilbert space ℓ^2 (square-summable complex sequences), and was calculating, as an exercise, what happens when one extends real analytic curves in $\mathbf{R}^n$, with their Frenet–Serret formulas, to their corresponding extensions as holomorphic Riemann surfaces in $\mathbf{C}^n$. The thesis itself grew out of elaborating from the original drafts."

3. Minimal Surfaces in Euclidean Space

Calabi understood that his work on holomorphic curves had direct application to the theory of minimal surfaces. Every minimal surface in euclidean space can be realized locally by a conformal immersion

$$f : \varDelta \;\to\; \mathbf{R}^n$$

whose coordinate functions are harmonic. In particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}f \;=\; 0 \tag{3.1}$$

and conformality implies that

$$\left|\frac{\mathrm{d}f}{\mathrm{d}z}\right|^2 \;=\; \left|\frac{\mathrm{d}f}{\mathrm{d}\bar{z}}\right|^2 \;=\; F \qquad \text{and} \qquad \frac{\mathrm{d}f}{\mathrm{d}z}\cdot\frac{\mathrm{d}f}{\mathrm{d}z} \;=\; 0 \tag{3.2}$$

where the induced metric is $\mathrm{d}s^2 = 2F|\mathrm{d}z|^2$.

Let $\widetilde{f} : \Delta \to \mathbf{R}^n$ be the harmonic conjugate mapping. Then $\left|\frac{\mathrm{d}f}{\mathrm{d}z}\right|^2 = \left|\frac{\mathrm{d}\widetilde{f}}{\mathrm{d}z}\right|^2$, and we obtain a **holomorphic** mapping

$$\mathbf{f} \;\equiv \frac{1}{\sqrt{2}}\left(f + i\widetilde{f}\right) : \Delta \;\to\; \mathbf{C}^n \tag{3.3}$$

giving the **same** induced metric.

These remarks apply globally to the universal covering of any minimal surface in $\mathbf{R}^n$.

Calabi's results above can now be applied to:

(1) Intrinsically characterize the metrics which can appear on minimal surfaces in euclidean space.
(2) Classify the minimal surfaces up to congruence which are intrinsically isometric to a given one.

Part (1) is clear. These metrics are exactly the ones which can be realized on holomorphic curves in $\mathbf{C}^n$. For Part (2) the key observation is that by Calabi's Rigidity Theorem (in the simply-connected case)

> *Each class of isometric non-congruent minimal surfaces in euclidean space contains exactly one holomorphic curve.*

Starting with this holomorphic curve, all minimal surfaces isometric to it are obtained by applying an appropriate complex linear transformation and then taking the real part. This leads to the following theorem. Let Σ be a simply connected Riemann surface.

Calabi's Classification Theorem. *The space of non-conguent minimal immersions $f : \Sigma \to \mathbf{R}^m$, which are isometric to a given (full) holomorphic immersion $\mathbf{f} : \Sigma \to \mathbf{C}^m$, is naturally described as the set of all complex symmetric $m \times m$ matrices P such that*

(*a*) $1_m - P\overline{P} \;\geq\; 0$, *and*
(*b*) $(\mathbf{f})^t P \mathbf{f}' \;\equiv 0$,

that is, P annihilates all tangent vectors to the curve $\mathbf{f}$. Furthermore, let n be the dimension of the smallest affine subspace of $\mathbf{R}^m$ containing the minimal surface corresponding to P. Then

$$n - m \;=\; \mathrm{rank}(1_m - P\overline{P})$$

so that $m \le n \le 2m$*, and* $n = 2m \iff 1_m - P\overline{P} > 0$.

Examples. Let $f_\theta = \sqrt{2}\mathrm{Re}(e^{i\theta}\mathbf{f})$, where $\mathbf{f}$ is given as in (3.3). This is the family of *associate minimal surfaces* for f.

The classical case $m = 3$ is interesting. Here the intrinsic characterization goes back to Ricci–Curbastro. A metric $\mathrm{d}s^2 = 2F|\mathrm{d}z|^2$ appears on a minimal surface in $\mathbf{R}^3$ if and only if its Gauss curvature K is ≤ 0 with isolated analytic zeros, and the metric

$$\mathrm{d}\hat{s}^2 \;\equiv \sqrt{-K}\mathrm{d}s^2 \mathrm{isflat}.$$

All the other minimal surfaces in euclidean space which are isometric to this given one in $\mathbf{R}^3$ are either associate surfaces or they are linearly full in $\mathbf{R}^6$ ([14]).

4. Some Personal Reminiscences

My first encounter with Gene Calabi was an event that transformed my graduate – and indeed my mathematical – career. I had been working for my thesis on the problem of fully understanding the theorem of Ricci–Curbastro mentioned above. Doing this led me to the discovery of Theorem 1.1 and some of its applications. Checking with some experts led me to believe the result was new, and I was joyfully writing up the results when Calabi came to give the colloquium at Stanford. My advisor mentioned to Gene what I had been doing and learned the bad news, which he passed on to me in the early afternoon.

Later that day I went to Gene's colloquium, which turned out to be one of the most astonishing and beautiful mathematical lectures I had heard. It pulled me directly into the study of minimal submanifolds of spheres, which became the subject of my Ph.D. thesis.

For me it was quite a day.

What follows is a description of the results that Gene presented in that colloquium.

5. Minimal Surfaces in the Euclidean Sphere

By a minimal surface in $S^n = \{x \in \mathbf{R}^{n=1} : |x| = 1\}$ I mean an immersion of a smooth surface

$$f : \Sigma \;\to\; S^n$$

with mean curvature $\equiv 0$. Such immersions are real analytic. We take the induced metric on Σ which gives it the structure of a Riemann surface with an analytic metric.

Consider the case where Σ is compact (without boundary). In 1967 almost nothing was known about such surfaces, except for a theorem of Fred Almgren [1], which states that when $n=3$ and Σ has genus zero, the surface must be totally geodesic.

Calabi's Theorem on Minimal 2-spheres in S^n. *Let*

$$f : S^2 \;\to\; S^n \tag{5.1}$$

be a minimal 2-sphere which does not lie in any totally geodesic (equatorial) S^{n-1}. Then

(*i*) $n=2m$ *for some integer* m,
(*ii*) $\mathrm{Area}(\Sigma)=2\pi N$, *where* N *is an integer with*

$$N \;\geq\; m(m+1).$$

(*iii*) *There exists a holomorphic map*

$$\Psi : S^2 \;\to\; \mathcal{I}_m$$

into the hermitian symmetric space $\mathcal{I}_m \equiv \mathrm{SO}(2m+1)/\mathrm{U}(m)$ *such that*

$$f \;=\; \pi\circ\Psi,$$

where

$$\pi : \mathcal{I}_m \;\to\; S^{2m} \tag{5.2}$$

is the projection $\mathrm{SO}(2m+1)/\mathrm{U}(m)\to\mathrm{SO}(2m+1)/\mathrm{SO}(2m)$. *Furthermore, this map* Ψ *is everywhere orthogonal to the fibres of* π *and the metric induced on* S^2 *by* f *coincides with the one induced by* Ψ.

This theorem asserts that minimal 2-spheres in the euclidean sphere are in a strong sense algebraic, they correspond explicitly to certain rational curves in the rational variety $\mathcal{I}_m$. Note that $\mathcal{I}_m$ can be realized as the space of complex m-planes in $\mathbf{C}^{2m+1}=\mathbf{R}^{2m+1}\otimes_{\mathbf{R}}\mathbf{C}$ which are totally isotropic with respect to the complex quadratic form on $\mathbf{C}^{2m+1}$ obtained by extending the metric on $\mathbf{R}^{2m+1}$.

Note that part (i) in Calabi's theorem generalizes Almgren's result.

The first non-trivial case of this result, where $n=4$, is already interesting. Here the map (5.2) is the twistor fibration

$$\pi : \mathbf{P}^3_{\mathbf{C}} \;\to S^4$$

whose horizontal complex 2-plane field is a homogeneous contact structure on $\mathbf{P}^3_{\mathbf{C}}$. Every holomorphic curve tangent to this structure pushes down to a minimal surface in S^4. By the theorem above every minimal 2-sphere in S^4 which is non-totally geodesic has area $2\pi N$ for $N\geq 6$. Furthermore, every twisted cubic in $\mathbf{P}^3_{\mathbf{C}}$ is isometric to a minimal 2-sphere in S^4 whose area is exactly 12π.

The construction of this lifting Ψ is gorgeous and quite explicit. Let's realize the map (5.1) in local coordinates as a conformal map

$$\psi : \Delta \ \to \ S^n$$

whose induced metric is

$$\mathrm{d}s^2 \ = \ 2F|\mathrm{d}z|^2.$$

Then ψ is a minimal surface if and only if

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\psi \ = \ -F\psi, \tag{5.3}$$

where we are considering ψ as a map $\psi : \Delta \to \mathbf{R}^{n+1}$ with $|\psi|^2 \equiv 1$. The conformality of ψ is equivalent to

$$\frac{\mathrm{d}\psi}{\mathrm{d}z}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}z} \ = \ 0. \tag{5.4}$$

Note that by (5.3) and the fact that $\psi\cdot\frac{\mathrm{d}\psi}{\mathrm{d}z} = 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}z}\right) \ = \ 0.$$

Therefore

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}z}\cdot\frac{\mathrm{d}\psi}{\mathrm{d}z}\right)\mathrm{d}z^2$$

is a global holomorphic 2-form on the Riemann surface Σ, and therefore if $\Sigma = S^2$, it must be identically zero. Proceeding by induction shows that

$$\frac{\mathrm{d}^k\psi}{\mathrm{d}z^k}\cdot\frac{\mathrm{d}^\ell\psi}{\mathrm{d}z^\ell} \ = \ 0 \qquad \text{for all } k+\ell > 0. \tag{5.5}$$

Therefore, the complex k-plane corresponding to the simple vector

$$\tau_k \ \equiv \ \frac{\mathrm{d}\psi}{\mathrm{d}z}\wedge\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}\wedge\cdots\wedge\frac{\mathrm{d}^k\psi}{\mathrm{d}z^k}$$

is totally isotropic. Furthermore, using (5.3) again one sees that the $2k$-jet of ψ at every point is spanned by ψ and the vectors in τ_k and $\overline{\tau}_k$. It now follows that the image of ψ is linearly full in some $\mathbf{R}^{2m+1}$.

If we now set

$$F_k \ = \ |\tau_k|^2$$

one can show that $\frac{1}{F_m}\tau_m$ is holomorphic, and the mapping Ψ is then defined in Plücker coordinates by

$$\Psi \;=\; \frac{1}{F_m}\tau_m.$$

Incidentally, the functions F_k satisfy the recursion relations

$$F_0 \;=\; 1, \qquad F_1 \;=\; F,$$

$$F_{k+1} \;=\; \frac{F_k^2}{F_{k-1}}\left(\frac{\mathrm{d}}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}\bar{z}}\log F_k + F\right) \tag{5.6}$$

for $1 \le k \le m$ (and $F_{m+1} \equiv 0$), very close to those given in (2.1) above.

This spectacular, highly original work engendered well over a decade of intense research on minimal surfaces and harmonic mappings by people such as Barbosa, Bryant, Burns, Chern, Eells, Guest, Lemaire, Salamon, Wood, the author, and many others (See the Bourbaki article [16] for references).

6. Rigidity of Arithmetic Groups

In 1960, Gene Calabi and Edoardo Vesentini wrote a paper which was foundational in the modern theory of arithmetic groups and rigidity. They considered smooth compact quotients

$$X = D/\Gamma$$

by a discrete group of holomorphic isometries, where D is a Cartan domain, that is, a bounded symmetric, homogeneous domain in $\mathbf{C}^n$. These domains are simply connected, and decompose into products of irreducible ones. Their main technical result is a vanishing theorem for the cohomology groups $H^q(X,\Theta)$, where Θ is the tangent sheaf. They proved that when D is irreducible

$$H^q(X,\Theta) \;=\; 0 \qquad \text{for } 0 \le q < \gamma(D) - 1,$$

where $\gamma(D)$ is an integer, independent of Γ, which satisfies

$$\sqrt{2n} < \gamma(D) \le n+1,$$

where $n = \dim_{\mathbf{C}}(D)$.

By the deformation theory of Frölicher–Nijenhuis [11] and Kodaira–Spencer [12], this proves

The Calabi–Vesentini Rigidity Theorem. *When D is irreducible with $\dim_{\mathbf{C}}(D) > 1$, the complex structure on D/Γ is rigid for any co-compact discrete subgroup of holomorphic isometries Γ acting freely and properly discontinuously on D. This rigidity continues to hold when D is reducible provided that no irreducible factor of D has dimension 1.*

The assertion is famously false when $\dim_{\mathbf{C}}(D) = 1$, i.e., when D is the unit disk in $\mathbf{C}$.

The proof of the main technical result proceeds by the Bochner method. It involves a masterful calculation of the curvature term appearing in the formulas of Bochner–Kodaira–Nakano type for harmonic tangent-bundle-valued forms.

Combined with results of Atle Selberg, this theorem led to the following.

The Algebraicity Theorem. *Let D be a Cartan domain with no irreducible factors of dimension 1. Let G be the group of all holomorphic (isometric) transformations of D, represented as a group of real matrices. Let $\Gamma \subset G$ be a discrete subgroup acting on D with smooth compact quotient. Then there exists an inner automorphism of G such that the components of all matrices representing the transform of Γ generate a simple algebraic extension of the rational number field.*

A little before the Calabi–Vesentini work, Selberg announced a related rigidity result for co-compact discrete subgroups of $\mathrm{SL}_n(\mathbf{R})$. A full length paper [25] appeared in 1960 with references to [10]. A summary of this and some subsequent work appeared in [26].

The work of Calabi–Vesentini greatly influenced Andrei Weil who in his 1962 paper [29] proved a best possible deformation rigidity result for co-compact lattices in real semi-simple Lie groups. The work of Selberg, Calabi–Vesentini and Weil marked the beginning of one of the major developments in modern mathematics, with early contributions by A. Borel, J. Tits, J.-P. Serre, Harish-Chandra, H. Garland, M. S. Raghunathan, among many others. (See [24] for an account in 1972.) This field, concerned with lattices in Lie groups, culminated in famous work of G. Mostow and then G. Margulis.

The notion of Mostow rigidity is stronger than the deformation rigidity discussed above. Deformation rigidity says that under appropriate assumptions on $\Gamma \subset G$, the only continuous deformations $\Gamma_t \subset G$, $|t| < 1$, come from families of inner automorphisms of G. Mostow's *strong rigidity* asserts that given $\Gamma \subset G$ and $\Gamma' \subset G'$ (again with appropriate assumptions), any abstract isomorphism $\Gamma \cong \Gamma'$ extends to an isomorphism $G \cong G'$. Mostow was more explicitly influenced by the work of Selberg than the analytic approach of Calabi–Vesentini. However, Mostow says on page 6 of his monograph [20] that "The phenomenon of strong rigidity for arbitrary lattices first turned up in 1965 in my search for a geometric explanation of deformation-rigidity". The earlier results of Selberg, Calabi–Vesentini, and Andrei Weil on deformation rigidity had a decisive influence on him and in his formulation of strong rigidity.

Calabi's Comments: "The rigidity theorem for discrete, uniform transformation groups of uniform, bounded domains came from a coincidence of contacts. In 1958–59 I was spending a year at the I.A.S., and listened to lectures by Kodaira, in which I finally understood for the first time (after about six years of hearing about them) what cohomology with coefficients in a linear space sheaf is about. At the same time, I remember talking with

A. Selberg about compact Riemannian manifolds with constant negative sectional curvature: to my surprise, he told me then that, *even in 3 dimensions*, he believed that they might be rigid, while I was actually dreaming of finding moduli. I got then the idea of calculating the cohomology with coefficients in the sheaf of germs of Killing vector fields. While writing it up and discussing it with Vesentini, we got the idea of applying the 'vanishing theorem' idea to the cohomology in the sheaf of holomorphic vector fields in the irreducible, locally symmetric Kähler manifolds. The case of the real hyperbolic manifolds then fell by the wayside, because, by about 1961, Dan Mostow had proved their global rigidity."

7. Complex Structures on Products of Spheres

Together with Beno Eckmann, Gene Calabi wrote a paper in 1953 which was transformational in complex geometry. At the time very little was known about complex manifolds outside the Kähler domain. Aside from the examples of H. Hopf, obtained by dividing $\mathbf{C}^n - \{0\}$ by powers of a homothety, most known compact complex manifolds were either projective or deformations of projective. Calabi and Eckmann showed that for all integers $p, q \geq 1$ there exist complex structures on the product of spheres:

$$M_{p,q} \ \equiv \ S^{2p+1} \times S^{2q+1}.$$

These were the first known examples of *simply-connected* compact complex manifolds which were **not algebraic**. (Kodaira's classification of surfaces came over a decade later.) They have remarkable properties. There is a holomorphic fibration

$$M_{p,q} \ = \ S^{2p+1} \times S^{2q+1} \ \xrightarrow{\pi} \ \mathbf{P}^p_{\mathbf{C}} \times \ \mathbf{P}^q_{\mathbf{C}},$$

given by the product $\pi = \pi_1 \times \pi_2$ of the standard Hopf fibrations, whose fibres are elliptic curves. These fibres are homologous to zero, and therefore $M_{p,q}$ cannot carry a Kähler metric. There are many complex subvarieties of $M_{p,q}$, but they are all of the form $\pi^{-1}(V)$, where V is a subvariety of $\mathbf{P}^p_{\mathbf{C}} \times \ \mathbf{P}^q_{\mathbf{C}}$. (Taking V to be smooth gives a wealth of new compact complex manifolds.) Similarly the meromorphic functions on $M_{p,q}$ are all pull-backs of meromorphic functions on $\mathbf{P}^p_{\mathbf{C}} \times \ \mathbf{P}^q_{\mathbf{C}}$.

The subdomain

$$\left(S^{2p+1} - \{x_0\}\right) \times \left(S^{2q+1} - \{y_0\}\right)$$

is particularly interesting. It is a complex manifold diffeomorphic to $\mathbf{R}^{2(p+q+1)}$ which contains an enormous family of compact complex analytic subvarieties. For example, take $\pi^{-1}(W \times W') = \pi_1^{-1}(W) \times \pi_2^{-1}(W')$, where W, W' are subvarieties of $\mathbf{P}^p_{\mathbf{C}}$ and $\mathbf{P}^q_{\mathbf{C}}$ respectively with $x_0 \notin W$ and $y_0 \notin W'$. Any entire holomorphic function must be constant, so $M_{p,q}$ admits no global coordinate chart.

The construction of the complex structure on $M_{p,q}$ is as follows. Note that $S^{2p+1} \subset \mathbf{C}^{p+1}$ carries a field of tangent complex p-planes $\mathcal{H}$ which is invariant under the family of diffeomorphisms given by scalar multiplication by e^{it}. Let H be the "Hopf" vector field which generates this flow. Let $\mathcal{H}'$ and H' be the corresponding objects on S^{2q+1}. At each point $(x,y) \in S^{2p+1} \times S^{2q+1}$ the subspace $\mathcal{H} \times \mathcal{H}'$ is already provided with an almost complex structure J. We extend it to the complement by defining $J(H) = H'$ and $J(H') = -H$. The invariance of J under the flows generated by H and H' is the key to proving its integrability, that is, the vanishing of the Nijenhuis tensor.

This construction of Calabi–Eckmann can be directly generalized to a large and interesting class of spaces, as pointed out by Sid Webster. Let $\Sigma^{2p+1} \subset \mathbf{C}^N$ be a compact submanifold which arises as the local boundary of a complex $(p+1)$-dimensional submanifold. Let $\mathcal{H}$ be the tangential complex p plane field to Σ^{2p+1}. Suppose there is an S^1-action on Σ^{2p+1} generated by a vector field H which is transversal to $\mathcal{H}$ and whose flow preserves $\mathcal{H}$ and its almost complex structure. Let Σ^{2q+1} be another such creature of dimension $2q+1$. Then the construction above gives an integrable complex structure on

$$\Sigma^{2p+1} \times \Sigma^{2q+1}.$$

Such manifolds Σ^{2n-1} can be easily constructed by taking

$$\Sigma^{2n-1} \; = \; S^{2n+1} \cap \{Z \in \mathbf{C}^{n+1} : P(Z) = 0\},$$

where P is a weighted homogeneous polynomial in $(Z_0, ..., Z_n)$, such as $Z_0^{a_0} + \cdots + Z_n^{a_n}$. These include the Brieskorn polynomials whose links Σ are exotic spheres.

Over the years there have been many further constructions of compact complex non-Kähler manifolds. Notable among them is the work of Taubes [27] in complex dimension 3, involving twistor geometry, and the work of Lopez de Medrano, Meersseman, and Verjovsky [17], [18], [19] which represents a highly non-trivial extension of the Calabi–Eckmann construction.

8. Generalizing the Hopf Maximum Principle

In a paper entitled *An extension of E. Hopf's maximum principle with an application to Riemannian geometry*, written in 1958, Calabi introduced the notion of a weak subsolution for a linear elliptic partial differential equation and established the Strong Maximum Principle for such functions. This theorem enabled him to prove a beautiful theorem in global riemannian geometry.

What is particularly remarkable about this paper is that it presages modern viscosity theory (cf. [2]) more than twenty years before its emergence as a major branch of analysis.

Calabi considered differential operators in $\mathbf{R}^n$ of the form

$$L[u] \;=\; \sum_{j,k} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_j b_j(x)\frac{\partial u}{\partial x_j} \quad \text{with } a \;>\; 0,$$

and extended the notion $L[u] \geq 0$ to upper semi-continuous functions u as follows. We say that a C^2-function φ defined near x is a *lower (upper) test function for u at x* if $u - \varphi$ has minimum (resp. maximum) value zero at x. Calabi defined u to be a weak solution to $L[u] \geq c$ at x if for each $\epsilon > 0$ there is a lower test function φ for u at x with $L[\varphi](x) \geq c - \epsilon$.

In modern viscosity theory one requires that for *every upper* test function φ at x, one has $L[\varphi](x) \geq c$. (If upper test functions do not exist, the condition is considered to be satisfied.) This viscosity criterion follows easily from Calabi's and is slightly more general. It allows functions to assume the value $-\infty$ for example. Using this fundamental approach, the school of Crandall, Ishii, Lions, Evans, and others, developed a substantial theory for fully nonlinear equations with important wide-ranging applications. It is an example of the depth and originality of Gene Calabi that he saw this important idea and its usefulness twenty years "before its time".

The first part of Gene's paper proved that any upper semi-continuous function u which satisfies $L[u] \geq 0$ (in his sense) on a domain in $\mathbf{R}^n$, and has an interior maximum point, must be constant. He then showed that on a riemannian n-manifold of non-negative Ricci curvature, the geodesic distance ρ from a point satisfies the inequality $\Delta\rho \leq (n-1)\rho$ in the weak sense. From this he then deduced that on such a manifold, assumed complete, there are no global solutions to the differential inequality $\Delta u \geq f(u)$ where $f(t)$ is a positive increasing function satisfying

$$\int_0^\infty \left\{ \int_0^s f(t)\,\mathrm{d}t \right\}^{\frac{1}{2}} \mathrm{d}s \;<\; \infty.$$

Calabi's Comments: The paper on the maximum principle was really an auxiliary result. The leitmotif of my work at the time was to complete the proof of what was called the Calabi conjectures, looking at the function space of all Kähler metrics in a compact, complex manifold, in a fixed cohomology class, and formulating variational problems that might yield the most 'harmonious' metric in such a space. By the time I announced the problems at the A.M.S. meeting in Baltimore in December 1952 (and again in Amsterdam in 1954), I had learned how little was known about nonlinear elliptic p.d.e.s., and was trying to find *a priori* estimates, using relations motivated by the geometry. The real inspiration came from a paper, presented by Bob Osserman, at an A.M.S. meeting around 1957, on the existence of solutions of the equation $\Delta u = f(u)$ in Euclidean $\mathbf{R}^n$. That was the incident that triggered the result that I am most proud of, "Improper Affine Hyperspheres...", in the Mich. Math. J., 1958. In that paper I needed to interpret sub-harmonic inequalities for non-smooth functions, like Riemannian geodesic distance and Finsler norms. I wrote the paper [5] separately, simply because I thought that it might have broader applications. By the way, the operator and inequality $L[u] = \sum_{jk} a_{jk} \partial^2 u / \partial x_j \partial x_k + \sum_j b_j \partial u / \partial x_j \geq 0$ was first considered by Eberhard Hopf. I first studied it in the book by Bochner and Yano, "Curvature and Betti Numbers". For the rest, I used Osserman's argument literally, except that, while Osserman considered only the Laplace operator in a Euclidean metric, I realized that it applied also in the case of metrics with non-negative Ricci curvature."

Author's Comments: The interaction between Bob Osserman and Gene Calabi at the A.M.S. meeting referred to above, was, in itself, one of the more interesting mathematical exchanges. The two were scheduled to give short presentations in the same session. Gene, however, was delayed and arrived only after Osserman had already left. However, Bob's final result remained suspended on a higher blackboard. Looking at it, Gene recognized exactly what he needed for the problem he was working on.

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