POTENTIAL THEORY ON ALMOST COMPLEX MANIFOLDS

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ABSTRACT

Pseudo-holomorphic curves on almost complex manifolds have been much more intensely studied than their "dual" objects, the plurisubharmonic functions. These functions are standardly defined by requiring that the restriction to each pseudo-holomorphic curve be subharmonic. In this paper subharmonic functions are defined by applying the viscosity approach to a version of the complex hessian which exists intrinsically on any almost complex manifold. Three theorems are proven. The first is a restriction theorem which establishes the equivalence of our definition with the "standard" definition. In the second theorem, using our "viscosity" definitions, the Dirichlet problem is solved for the complex Monge-Ampère equation in both the homogeneous and inhomogeneous forms. These two results are based on theorems found in [HL_3] and [HL_2] respectively. Finally, it is shown that the plurisubharmonic functions considered here agree with the plurisubharmonic distributions. In particular, this proves a conjecture of Nefton Pali.

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0. Introduction.

The purpose of this paper is to develop an intrinsic potential theory on a general almost complex manifold (X, J). Our methods are based on results established in [$\mathrm{HL}_{2,3,4}$]. In particular, we study an extension, to this general situation, of the classical notion of a plurisubharmonic function. For smooth functions φ many equivalent definitions of plurisubharmonicity are available. Several are given in Section 2. For instance, one can require that $i\partial\overline{\partial}\varphi \geq 0$, or $\mathcal{H}(\varphi) \geq 0$ where the complex hessian $\mathcal{H}(\varphi)(V, W) \equiv (\frac{1}{2}V\overline{W} + \frac{1}{2}\overline{W}V - \frac{i}{2}J[V, \overline{W}])(\varphi)$ is a bilinear form on $T_{1,0}X$. Its real form $H \equiv \mathrm{Re}\,\mathcal{H}$ is computed in Section 4 to be $H(\varphi)(v, v) = (vv + (Jv)(Jv) + J[v, Jv])(\varphi)$, yielding a third equivalent condition $H(\varphi) \geq 0$.

At any given point $x \in X$, each of these equivalent definitions only depends on the reduced 2-jet of φ at x. Consequently, one obtains a subset $F(J) \subset J^2(X)$ of the 2-jet bundle of X, which consists of the J-plurisubharmonic jets.

For an upper semi-continuous function u on X, the previous definitions cannot be applied directly to u. However, they can be applied to a smooth "test function" φ for u at x. (See Section 3 for more details.) This yields our general definition of $F(J)$-plurisubharmonic functions.

Our first main result is the Restriction Theorem 5.2 which states that: **if u is $F(J)$-plurisubharmonic and (X', J') is an almost complex submanifold of (X, J), then the restriction of u to X' is $F(J')$-plurisubharmonic.**

An upper semi-continuous function u on (X, J), with the property that its restriction to each pseudo-holomorphic curve in X is subharmonic, will be called *standardly plurisubharmonic* (Definition 6.1). On such curves the complex structure is integrable and all of the many definitions of subharmonic are well known to agree. Consequently, the special case of our restriction theorem, where X' has dimension one, states that each $F(J)$-plurisubharmonic function is plurisubharmonic in the standard sense. The converse is also true due to the classical theorem of Nijenhuis and Woolf, that there exist pseudo-holomorphic curves through any point of X with any prescribed tangent line. This gives our first main result, the equivalence of the two notions of plurisubharmonicity (Theorem 6.2).

The second main result, Theorem 7.5, solves the **Dirichlet problem for F(J)-harmonic functions** in the general almost complex setting. Here the usefulness of the $F(J)$-notion over the standard notion is striking. The result applies simultaneously to both the homogeneous and the inhomogeneous equations, and the proof rests on basic results in [HL_2]. The main statement is essentially the following. Fix a real volume form λ on X which is orientation compatible with J. Let $\Omega \subset X$ be a compact domain with smooth boundary $\partial\Omega$. Fix a continuous functions $f \in C(\overline{\Omega})$ with $f \geq 0$. Given $\varphi \in C(\partial\Omega)$, the Dirichlet Problem asks for existence and uniqueness of J-plurisubharmonic viscosity solutions $u \in C(\overline{\Omega})$ to the equation

$$(i\partial\overline{\partial}u)^n \;=\; f\lambda.$$

with boundary values $u\big|_{\partial\Omega} = \varphi$. Existence and Uniqueness are established whenever Ω has a strictly J-plurisubharmonic defining function. See Theorem 7.4 for full details.

At the end of Section 7 we discuss some other equivalent notions of $F(J)$-harmonic functions.

The hard work in proving the restriction theorem occurs in [HL_3] while the hard work in solving the Dirichlet problem occurs in [HL_2]. In both cases, the key to being able to apply these results to almost complex manifolds is the local coordinate expression given in Proposition 4.5, which states that the subequation $F(J)$ is locally jet equivalent to the standard constant coefficient case on local charts in $\mathbf{R}^{2n} = \mathbf{C}^n$ equipped with a standard complex structure. The methods must be adapted somewhat unless $f \equiv 0$ or $f > 0$ (see Section 7).

On any almost complex manifold there is also the notion of a plurisubharmonic distribution. It has been proved by Nefton Pali [P] that if u is plurisubharmonic in the standard sense, then u is in $L^1_{\text{loc}}(X)$ and defines a plurisubharmonic distribution. Pali conjectures that the converse is also true and proves a partial result in this direction. In Section 8 we prove the full conjecture, thereby showing that on any almost complex manifold all three notions of plurisubharmonicity (standard, viscosity and distributional) are equivalent.

The argument given in Section 8 reduces this nonlinear result to a corresponding result for linear elliptic subequations. This technique applies to any convex subequation which is "second-order complete" in a certain precise sense. In an appendix, linear elliptic operators L with smooth coefficients and no zero-order term are discussed. The notions of a viscosity L-subharmonic function and an L-subharmonic distribution are shown to be equivalent. First we show that an upper semi-continuous function is L-subharmonic in the viscosity sense if and only if it is L-subharmonic in the "classical" sense of being "sub-the-L-harmonics". Such functions are known to be L^1_{loc} and to give an L-subharmonic distribution (Theorem A.6). Conversely, any L-subharmonic distribution has a unique upper semi-continuous, L^1_{loc} representative which is "classically" subharmonic. It is important not only that this representative be unique but also that it can be obtained canonically from the L^1_{loc}-class of the function by taking the essential upper semi-continuous regularization $\widetilde{u}(x) = \text{ess} \limsup_{y \to x} u(y)$. Thus the choice of upper semi-continuous function in the L^1_{loc}-class is independent of the operator L. (It is actually the smallest upper semi-continuous representative.) This fact is essential for the arguments of Section 8.

The proof for this linear case combines techniques from distributional potential theory, classical potential theory and viscosity potential theory. It appears that there is no specific reference for this particular result, so we have included an appendix which outlines the theory. (Also see Section 9 in [HL_4].)

Note 1. We note that when (X, J) is given a hermitian metric, there is a notion of "metric" plurisubharmonic functions, defined via the hermitian symmetric part of the riemannian hessian (cf. [HL_2]). For metrics with the property that every holomorphic curve in (X, J) is minimal (e.g., if the Käher form is closed), these "metric" plurisubharmonic functions are standard (Theorem 9.3). However in Section 9 examples are given which, in the general case, strongly differentiate the metric notion from the intrinsic notion of plurisubharmonicity studied in this paper.

Note 2. In the first appearance of this article we solved the Dirichlet problem for $f \equiv 0$ (and the methods also applied to $f > 0$). Shortly afterward Szymon Pliś posted a paper which also studied the Dirichlet problem on almost complex manifolds [Pl]. His result, which is the analogue in the almost complex case of a main result in [CKNS], is quite different from ours. He considered the case $f > 0$, assumed all data to be smooth, and established complete regularity of the solution. Our result holds for arbitrary continuous boundary data, and interior regularity is known to fail.

Pliś subsequently posted a second version of [Pl] which offered an alternative proof of our result by taking a certain limit of his solutions. He also established Lipschitz continuity of the solution under restrictive hypotheses on the boundary data. In the current version of this paper we allow $f \geq 0$.

It is assumed throughout that J is of class C^∞, however for all results which do not involve distributions, C^2 is sufficient.

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1. Preliminaries

Almost Complex Structures

Let (X, J) be an almost complex manifold. Then there is a natural decomposition

$$TX \otimes_{\mathbf{R}} \mathbf{C} \;=\; T_{1,0} \oplus T_{0,1}$$

where $T_{1,0}$ and $T_{0,1}$ are the i and $-i$-eigenspaces of J respectively. The projections of $TX \otimes_{\mathbf{R}} \mathbf{C}$ onto these complex subspaces are given explicitly by $\pi_{1,0} = \frac{1}{2}(I - iJ)$ and $\pi_{0,1} = \frac{1}{2}(I + iJ)$.

The dual action of J on T^*X shall be denoted by J as well. (For $\alpha \in T_x^*X$, $(J\alpha)(V) \equiv \alpha(JV)$ for $V \in T_xX$.) Then again we have a natural decomposition

$$T^*X \otimes_{\mathbf{R}} \mathbf{C} \;=\; T^{1,0} \oplus T^{0,1}$$

into the i and $-i$-eigenspaces of J respectively with projections again given explicitly by $\pi^{1,0} = \frac{1}{2}(I - iJ)$ and $\pi^{0,1} = \frac{1}{2}(I + iJ)$. Moreover, we have

$$(T_{1,0})^* \;=\; T^{1,0} \qquad \text{and} \qquad (T_{0,1})^* \;=\; T^{0,1}$$

There is also a bundle splitting

$$\Lambda^k T^*X \otimes_{\mathbf{R}} \mathbf{C} \;=\; \bigoplus_{p+q=k} \Lambda^{p,q}X \tag{1.1}$$

where $\Lambda^{p,q}X$ is the $i(p-q)$-eigenspace of J acting as a derivation on $\Lambda^*T^*X \otimes_{\mathbf{R}} \mathbf{C}$. Let $\mathcal{E}^{p,q}X$ denote the smooth sections of $\Lambda^{p,q}X$. Then there are natural operators

$$\partial : \mathcal{E}^{p,q}(X) \;\to\; \mathcal{E}^{p+1,q}(X) \qquad \text{and} \qquad \overline{\partial} : \mathcal{E}^{p,q}(X) \;\to\; \mathcal{E}^{p,q+1}(X)$$

defined by restriction and projection of the exterior derivative d.

Under complex conjugation one has that

$$\overline{\Lambda^{p,q}X} \;=\; \Lambda^{q,p}X$$

and in particular each $\Lambda^{p,p}X$ is conjugation-invariant and decomposes into a real and imaginary part. Let $\Lambda^{p,p}_{\mathbf{R}}X$ denote the real part.

A *pseudo-holomorphic map* $\Phi : (X', J') \to (X, J)$ between almost complex manifolds is a smooth map whose differential Φ_* satisfies $\Phi_* J' = J\Phi_*$ at every point. Thus the pull-back $\Phi^* : \Lambda^*T^*X \to \Lambda^*T^*X'$ is also compatible with the almost complex structures (acting as derivations) and therefore preserves the bigrading in (1.1). It follows that:

The operators ∂ and $\overline{\partial}$ commute with the pullback Φ^* on smooth forms. (1.2)

2-Jets

Denote by $\overline{J}^2(X)$ the vector bundle of 2-jets on an arbitrary smooth manifold X, and let $\overline{J}_x(u) \in \overline{J}_x^2(X)$ denote the 2-jet of a smooth function u at x. The bundle $J^2(X)$ of reduced 2-jets is defined to be the quotient of $\overline{J}^2(X)$ by the trivial line bundle corresponding to the value of the function u. We will be primarily interested in the space of reduced 2-jets in this paper (and so we have chosen a simpler notation for it). The bundle $\mathrm{Sym}^2(T^*X)$ of quadratic forms on TX has a natural embedding as a subbundle of $J^2(X)$ via the hessian at critical points. Namely, if u is a function with a critical point at x, then $V(W(u)) = W(V(u)) + [V,W](u) = W(V(u))$ is a well defined symmetric bilinear form $(\mathrm{Hess}_x u)(V,W)$ on T_xX for arbitrary vector fields V, W defined near x.

Thus there is a short exact sequence of bundles

$$0 \longrightarrow \mathrm{Sym}^2(T^*X) \longrightarrow J^2(X) \longrightarrow T^*X \longrightarrow 0. \tag{1.3}$$

However, for functions u with $(du)_x \neq 0$, there is no natural definition of $\mathrm{Hess}_x u$, i.e., the sequence (1.3) has no natural splitting. There is a natural cone bundle $\mathcal{P}(X) \subset \mathrm{Sym}^2(T^*X) \subset J^2(X)$ defined by

$$\mathcal{P}_x(X) \equiv \{H \in \mathrm{Sym}^2(T_x^*X) : H \geq 0\} \cong \{J_x(u) : u(x) = 0 \text{ and } u \geq 0 \text{ near } x\} \tag{1.4}$$

In local coordinates $x \in \mathbf{R}^N$ for X, the first and second derivatives comprise the reduced 2-jet $J_x(u)$ of a function u. That is,

$$J_x(u) \cong (D_x u, D_x^2 u) \in \mathbf{R}^N \times \mathrm{Sym}^2(\mathbf{R}^N) \stackrel{\mathrm{def}}{=} \mathbf{J}^2(\mathbf{R}^N) \equiv \mathbf{J}^2 \tag{1.5}$$

where $D_x u \equiv (\frac{\partial u}{\partial x_1}(x), ..., \frac{\partial u}{\partial x_N}(x))$ and $D_x^2 u \equiv \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x)\right)$.

The isomorphism (1.5) says that $J_x^2(X) \cong \mathbf{R}^N \times \mathrm{Sym}^2(\mathbf{R}^N) = \mathbf{J}^2(\mathbf{R}^N)$. The standard notation $(p, A) \in \mathbf{R}^N \times \mathrm{Sym}^2(\mathbf{R}^N)$ will be used for coordinates $(p, A) = (Du, D^2u)$ on $\mathbf{J}^2(\mathbf{R}^N)$.

Using these coordinates, we have

$$\mathrm{Sym}_x^2(T^*X) \cong \{(0, A) : A \in \mathrm{Sym}^2(\mathbf{R}^N)\} \subset \mathbf{J}^2 \tag{1.6}$$

and

$$\mathcal{P}_x(X) \cong \{(0, A) : A \geq 0\} \subset \mathbf{J}^2 \tag{1.7}$$

2. The Complex Hessian

Suppose (X, J) is an almost complex manifold. Note that for any real-valued C^2-function u we have $d\overline{\partial}u = \overline{\partial}^2 u + \partial\overline{\partial}u + \epsilon^{2,0}$ where $\epsilon^{2,0}$ is a continuous (2,0)-form. Taking the complex conjugate and adding terms gives $ddu = 0$, which, after taking $(2,0)$- and then $(1,1)$-components, gives us that $\epsilon^{2,0} = -\partial^2 u$ and

$$\partial\overline{\partial}u \;=\; -\overline{\partial}\partial u.$$

Definition 2.1. The $i\partial\overline{\partial}$**-hessian** of a smooth real-valued function u on X is obtained by applying the intrinsically defined real operator

$$i\partial\overline{\partial} : \; C^\infty(X) \;\longrightarrow\; \Gamma(X, \Lambda^{1,1}_{\mathbf{R}} X), \tag{2.1}$$

From (1.2) we conclude the following.

Proposition 2.2. *The $i\partial\overline{\partial}$-hessian operator commutes with pull-back under any pseudo-holomorphic map between almost complex manifolds.*

There is a second useful way of looking at $i\partial\overline{\partial}u$. Recall that a (complex-valued) bilinear form $\mathcal{B}$ on a complex vector space is **hermitian** if $\mathcal{B}(V, W)$ is complex linear in the first variable and complex anti-linear in the second variable. It is further called **symmetric** if it satisfies

$$\mathcal{B}(V, W) \;=\; \overline{\mathcal{B}(W, V)}.$$

When, in addition, $\mathcal{B}(V, V) \geq 0$ for all V, we say $\mathcal{B}$ is **positive** and denote this by $\mathcal{B} \geq 0$. Let $\mathrm{HSym}^2(T_{1,0}X)$ denote the bundle of hermitian symmetric forms on $T_{1,0}X$.

Recall the bundle isomorphism

$$\Lambda^{1,1}_{\mathbf{R}} X \;\cong\; \mathrm{HSym}^2(T_{1,0}X) \tag{2.2}$$

sending

$$\beta \in \Lambda^{1,1}_{\mathbf{R}} X \qquad \text{to} \qquad \mathcal{B}(V, W) \equiv -i\beta(V, \overline{W}).$$

We say $\beta \geq 0$ if $\mathcal{B} \geq 0$.

Definition 2.3. Under the isomorphism (2.2) the $i\partial\overline{\partial}$-hessian of u becomes the **complex hessian** $\mathcal{H}(u)$ **of** u. Namely

$$\mathcal{H}(u)(V, W) \;=\; (\partial\overline{\partial}u)(V, \overline{W}) \tag{2.3}$$

is a section of $\mathrm{HSym}^2(T_{1,0}X)$, and

$$\mathcal{H} : C^\infty(X) \;\longrightarrow\; \Gamma(X, \mathrm{HSym}^2(T_{1,0}X))$$

will be called the **complex hessian operator**.

The complex hessian can be computed in various ways. The main formula we employ is the following.

Proposition 2.4. *For $u \in C^\infty(X)$ and each $V, W \in \Gamma(X, T_{1,0})$,*

$$\mathcal{H}(u)(V,W) \;=\; \left(\tfrac{1}{2}V\overline{W} + \tfrac{1}{2}\overline{W}V - \tfrac{i}{2}J[V,\overline{W}]\right)(u) \tag{2.4}$$

Remark 2.5. We can express (2.4) more succinctly by saying that: for each $V, W \in \Gamma(X, T_{1,0})$, $\mathcal{H}(V,W)$ is the second-order scalar differential operator

$$\mathcal{H}(V,W) \;=\; \tfrac{1}{2}V\overline{W} + \tfrac{1}{2}\overline{W}V - \tfrac{i}{2}J[V,\overline{W}] \tag{2.4$'$}$$

Proof. First recall that for arbitrary sections V and W of $TX \otimes_{\mathbf{R}} \mathbf{C}$ and any complex 1-form α, the exterior derivative of α satisfies:

$$(d\alpha)\left(V,\overline{W}\right) \;=\; V\left(\alpha(\overline{W})\right) - \overline{W}\left(\alpha(V)\right) - \alpha\left([V,\overline{W}]\right).$$

Now assume that V and W are both of type $1,0$ and take $\alpha = \overline{\partial}u$. Then $\alpha(\overline{W}) = \overline{W}(u)$ while $\alpha(V) = 0$. Since $(\partial\overline{\partial}u)(V,\overline{W}) = (d\alpha)(V,\overline{W})$, we have

$$(\partial\overline{\partial}u)(V,\overline{W}) \;=\; V\left(\overline{W}(u)\right) - \overline{\partial}u\left([V,\overline{W}]\right) \;=\; V\left(\overline{W}(u)\right) - [V,\overline{W}]^{0,1}(u). \tag{2.5}$$

Take $\alpha = -\partial u$ and note that $\alpha(\overline{W}) = 0$ while $\alpha(V) = -V(u)$. Since $-(\partial\overline{\partial}u)(V,\overline{W}) = -(d\alpha)(V,\overline{W})$, we have

$$-(\overline{\partial}\partial u)(V,\overline{W}) \;=\; \overline{W}\left(V(u)\right) + \partial u\left([V,\overline{W}]\right) \;=\; \overline{W}\left(V(u)\right) + [V,\overline{W}]^{1,0}(u). \tag{2.6}$$

Finally, using $Jdu = i(\partial u - \overline{\partial}u)$, or equivalently that $J = i$ on $T_{1,0}X$ and $J = -i$ on $T_{0,1}X$, we see that the average of these last two formulas for $\mathcal{H}(u)(V,\overline{W})$ is given by (2.4). ∎

3. $F(J)$-plurisubharmonic Functions.

It is natural, and useful, to have a definition of plurisubhamonic functions on (X, J) expressed purely in terms of the 2-jets of those functions. Specifically, one would like such functions, when smooth, to be defined by constraining their 2-jets to a subset $F(J) \subset J^2(X)$ of the 2-jet bundle, and then pass to general upper semi-continuous functions by viscosity techniques. In this section we give such a definition using the complex hessian $\mathcal{H}$. First, for smooth functions the concept is straightforward.

Definition 3.1. A smooth real-valued function u on X is called $F(J)$**-plurisubharmonic** if $\mathcal{H}(u) \geq 0$, i.e., the hermitian symmetric bilinear form $\mathcal{H}_x(u)$ is positive semi-definite at all points $x \in X$. Moreover, if $\mathcal{H}_x(u) > 0$ is positive definite at all points $x \in X$, we say that u is **strictly** $F(J)$-plurisubharmonic .

Proposition 2.4 implies that at a point $x \in X$, $\mathcal{H}_x(u)$ depends only on $J_x(u)$, the reduced 2-jet of u at x. In particular, the condition $\mathcal{H}_x(u) \geq 0$ (equivalently $i\partial\overline{\partial}u \geq 0$) depends only on the jet $J_x(u)$ of u at x. Hence we can define plurisubharmonicity for a jet $\mathbf{J} \in J^2_x(X)$ as follows.

Definition 3.2. A jet $\mathbf{J} \in J^2_x(X)$ is said to be $F(J)$**-plurisubharmonic** if for any smooth function u with $J_x(u) = \mathbf{J}$, we have

$$\mathcal{H}_x(u) \geq 0. \tag{3.1}$$

The set of $F(J)$-plurisubharmonic jets on X will be denoted by $F(J)$.

We are now in a position to broaden the notion of $F(J)$-plurisubharmonicity to the level of generality encountered in classical complex function theory. Namely we consider functions $u \in \mathrm{USC}(X)$, the space of upper semi-continuous functions on X taking values in $[-\infty, \infty)$. Take $u \in \mathrm{USC}(X)$ and fix $x \in X$. A function φ which is C^2 in a neighborhood of x is called a **test function for** u **at** x if $u - \varphi \leq 0$ near x and $u - \varphi = 0$ at x.

Definition 3.3. A function $u \in \mathrm{USC}(X)$ is $F(J)$**-plurisubharmonic** if for each $x \in X$ and each test function φ for u at x, one has

$$\mathcal{H}_x(\varphi) \geq 0, \quad \text{i.e.} \quad J_x\varphi \ \in \ F_x(J).$$

Note that $u \equiv -\infty$ is $F(J)$-plurisubharmonic since there exist no test functions for u at any point.

Definition 3.3 should be an extension of Definition 3.1 when u is smooth. For this to be true the following **Positivity Condition** for F (where $\mathcal{P}(X)$ is defined by (1.4))

$$F + \mathcal{P}(X) \ \subset \ F \tag{P}$$

must be satisfied for $F = F(J)$. (See Proposition 2.3 in [HL$_2$] for the details.)

Proposition 3.4. *Each fibre $F_x(J)$ of $F(J)$ is a convex cone, with vertex at the origin, containing the convex cone $\mathcal{P}_x(X)$. In particular, $F(J)$ satisfies the Positivity Condition (P).*

Proof. It is easy to see that each fibre $F_x(J)$ is a convex cone with vertex at the origin in $J^2_x(X)$ since $\mathcal{H}_x(u)$ is linear in u. It remains to show that $F_x(J)$ contains $\mathcal{P}_x(X)$ as defined by (1.4). Recall that each vector field V of type $1,0$ is of the form $V = v - iJv$ where v is a real vector field. If x is a critical point of φ, then it is easy to compute from (2.4) that at x

$$\mathcal{H}(\varphi)(V, V) \ = \ v(v(\varphi)) + (Jv)(Jv(\varphi)) \tag{3.2}$$

for all such $1,0$ vector fields $V = v - iJv$. Now suppose that $\varphi(x) = 0$ and that $\varphi \geq 0$ near the point x. Then by elementary calculus

$$(d\varphi)(x) \ = \ 0 \qquad \text{and} \qquad v(v(\varphi))(x) \ \geq \ 0 \quad \forall\, v \in \Gamma(X, TX). \tag{3.3}$$

Combining (3.2) and (3.3) proves that $\mathcal{P}_x(X) \subset F_x(J)$. ∎

Definition 3.5. A subset $F \subset J^2(X)$ which satisfies both the Positivity Condition (P) and the **Topological Condition:**

$$(i) \ \ F \ = \ \overline{\mathrm{Int} F} \qquad (ii) \ \ F_x \ = \ \overline{\mathrm{Int} F_x} \qquad (iii) \ \ \mathrm{Int} F_x \ = \ (\mathrm{Int} F)_x \tag{T}$$

is called a **subequation** (cf. [HL_2, Def. 3.9]). This condition (T) for $F(J)$ is a consequence of a jet equivalence for the complex hessian which is given in the next section.

Definition 3.6. A function $u \in \mathrm{USC}(X)$ is called F-**subharmonic** on X if for each $x \in X$ and each test function φ for u at x, we have $J_x^2(\varphi) \in F_x$. The set of such functions is denoted $F(X)$.

Starting with classical potential theory on $\mathbf{C}^n$ as motivation, many of the important results concerning plurisubhamonic functions on $\mathbf{C}^n$ were extended to general constant coefficient subequations in euclidean space in one of our first papers on the subject [HL_1]. Most of these results were then generalized to any subequation F on a manifold [HL_2]. One can view this paper as coming full circle back to the complex setting by using viscosity methods to prove a new result in the almost complex case.

4. A Local Coordinate Expression for the Real Form of the Complex Hessian.

The point of this section is to establish a formula for the complex hessian in a real coordinate system on X. We begin by reviewing some standard algebra. The space $\mathrm{HSym}^2(T_{1,0})$ of hermitian symmetric bilinear forms on $T_{1,0}$ has an alternate description. Recall the standard isomorphism on complex vector spaces

$$(T, J) \;\cong\; (T_{1,0}, i) \tag{4.1}$$

given by mapping a real tangent vector $v \in T$ to $V = \frac{1}{2}(v - iJv)$ with inverse $v = 2\mathrm{Re}\, V$.

A real symmetric bilinear form $B \in \mathrm{Sym}^2_{\mathbf{R}}(T)$ is said to be **hermitian** (or J-**hermitian**) if $B(Jv, Jv) = B(v, v)$ for all $v \in T$. Let $\mathrm{HSym}^2_{\mathbf{R}}(T)$ denote the subspace of $\mathrm{Sym}^2_{\mathbf{R}}(T)$ consisting of J-hermitian forms. Now (4.1) induces a (renormalized) isomorphism

$$\mathrm{HSym}^2_{\mathbf{R}}(T) \;\cong\; \mathrm{HSym}^2(T_{1,0}), \tag{4.2}$$

given by mapping $\mathcal{B} \in \mathrm{HSym}^2(T_{1,0})$ to its **real form** $B \in \mathrm{HSym}^2_{\mathbf{R}}(T)$ defined by

$$B(v, w) \;\equiv\; \mathrm{Re}\, \mathcal{B}(v - iJv, w - iJw). \tag{4.3}$$

Of course, it is enough to define the quadratic form

$$B(v, v) \;\equiv\; \mathcal{B}(v - iJv, v - iJv) \tag*{(4.3)$'$}$$

which is real-valued and determines (4.3) by polarization. From (4.3)$'$ it is obvious that

$$\mathcal{B} \;\geq\; 0 \qquad \iff \qquad B \;\geq\; 0. \tag{4.4}$$

Now given any $B \in \mathrm{Sym}^2_{\mathbf{R}}(T)$, the **hermitian symmetric part of** B is defined to be the element

$$B^J(v, v) \;\equiv\; B(v, v) + B(Jv, Jv) \tag{4.5}$$

which belongs to $\mathrm{HSym}^2_{\mathbf{R}}(T)$. (Usually one inserts a $\frac{1}{2}$ in (4.5), but here it is cleaner not to do so.)

Combining this algebra with (2.2) and adding J to the notation, we have three isomorphic vector spaces

$$\Lambda^{1,1}(T(J)) \;\cong\; \mathrm{HSym}^2(T_{1,0}(J)) \;\cong\; \mathrm{HSym}^2_{\mathbf{R}}(T(J)). \tag{4.6}$$

It is important that the last vector space $\mathrm{HSym}^2_{\mathbf{R}}(T(J))$ is a real vector subspace of $\mathrm{Sym}^2_{\mathbf{R}}(T(J))$. We have denoted a triple of elements that corresond under these isomorphisms by β, $\mathcal{B}$ and B respectively. Note that $\beta \geq 0 \iff \mathcal{B} \geq 0 \iff B \geq 0$.

We now apply this to the case of the complex hessian of a function φ at a point $x \in X$ where $\beta = i\partial\overline{\partial}\varphi$ and $\mathcal{B} = \mathcal{H}(\varphi)$. The third element $B \equiv H(\varphi) \in \mathrm{HSym}^2_{\mathbf{R}}(T_x(J_x))$ is called the **real form** of the complex hessian. Note that

$$i\partial\overline{\partial}\varphi \;\geq\; 0 \qquad \iff \qquad \mathcal{H}(\varphi) \;\geq\; 0 \qquad \iff \qquad H(\varphi) \;\geq\; 0. \tag{4.7}$$

The formula (2.4) provides a formula for $H(\varphi)$.

Lemma 4.1. *The real form $H(\varphi)$ of the complex hessian $\mathcal{H}(\varphi)$ is given by the polarization of the real quadratic form*

$$H(\varphi)(v,v) \ = \{vv + (Jv)(Jv) + J\left([v,Jv]\right)\}\varphi \tag{4.8}$$

defined for all real vector fields v (where the vector fields act on functions in the standard way).

Proof. As an operator on φ, $H(v,v)$ is given by

$$H(v,v) \ = \ \mathcal{H}(v - iJv, v - iJv),$$

which can be expanded out using $(2.4)'$ to yield (4.8). ∎

In euclidean space $\mathbf{R}^N$ with coordinates $t = (t_1, ..., t_N)$, let $p = D\varphi$ (evaluated at t), and let $D^2\varphi$ denote both the second derivative matrix $A \equiv ((\frac{\partial^2\varphi}{\partial t_i \partial t_j}))$ as well as the quadratic form

$$A(v,v) \ = \ (D^2\varphi)(v,v) \ = \ \sum_{i,j=1}^{N} \frac{\partial^2\varphi}{\partial t_i \partial t_j}(t)\, v_i v_j \qquad \text{where} \quad v \ \equiv \ \sum_{j=1}^{N} v_j \frac{\partial}{\partial t_j}.$$

A calculation gives the following.

Proposition 4.2. *Suppose that J is an almost complex structure on an open subset $X \subset \mathbf{R}^{2n}$. Let v be a constant coefficient vector field on X (i.e., $v = \sum_j v_j \frac{\partial}{\partial t_j}$ where the v_j's are constants). Then*

$$H(\varphi)(v,v) \ = \ (D^2\varphi)(v,v) + (D^2\varphi)(Jv,Jv) + (D\varphi)\{(\nabla_{Jv}J)(v) - (\nabla_v J)Jv\} \tag{4.9}$$

(where $\nabla_v J$ denotes the standard directional derivative of the matrix-valued function J). Equivalently,

$$H(\varphi) \ = \ (A + E(p))^J \tag*{(4.9)$'$}$$

where $p \equiv D\varphi$, $A \equiv D^2\varphi$ and the section $E \in \Gamma(X, \mathrm{Hom}_{\mathbf{R}}(\mathbf{C}^n, \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n))$ is defined by

$$E(p)(v,v) \ \equiv \ \langle (\nabla_{Jv}J)(v), p\rangle \tag{4.10}$$

where $\langle\cdot,\cdot\rangle$ is the standard real inner product on $\mathbf{C}^n$.

Proof. By (4.8)

$$(H\varphi)(v,v) \ = \ (D^2\varphi)(v,v) + (D^2\varphi)(Jv,Jv) + (D\varphi)\{(\nabla_{Jv}J)(v) + J[v,Jv]\}.$$

Now $[v,Jv] = (\nabla_v J)v$, and $J^2 = -I$ implies that $(\nabla_v J)J + J(\nabla_v J) = 0$. Hence we have $J[v,Jv] = J(\nabla_v J)v = -(\nabla_v J)Jv$, which proves (4.9). To prove $(4.9)'$, note that $A^J(v,v) = A(v,v) + A(Jv,Jv)$ and that $E(p)^J = \langle(\nabla_{Jv}J)(v), p\rangle - \langle(\nabla_v J)(Jv), p\rangle$. ∎

Remark. Using the conventions $v = \sum v_j e_j$, $Jv = \sum v_j J_{jk} e_k$ where $e_j = \frac{\partial}{\partial t_j}$, the reader may wish to derive (4.9) and (4.9)$'$ from (4.8) using matrices.

We now continue with the linear algebra. Consider a finite dimensional real vector space T of dimension $2n$ with two almost complex structures J and J_0, inducing the same orientation on X. Think of J_0 as "standard" or "background" data. Assume we are given $g \in \mathrm{GL}^+(T)$ such that

$$J \ = \ gJ_0g^{-1}$$

Extend g as usual to $GL_{\mathbf{C}}(T \otimes \mathbf{C})$. The induced action on $\mathrm{Sym}^2_{\mathbf{R}}(T)$ is given by

$$(g^*B)(v,w) = B(gv, gw) \tag{4.11}$$

The next result is more than we need but it should clarify the algebra.

Lemma 4.3. *Consider elements*

$$\alpha \in \Lambda^{1,1}_{\mathbf{R}}(T, J), \quad \mathcal{H} \in \mathrm{HSym}^2(T_{1,0}(J)), \quad \text{and} \ \ H \in \mathrm{HSym}^2{}_{\mathbf{R}}(T, J) \subset \mathrm{Sym}^2_{\mathbf{R}}(T).$$

Then the pull-backs by g satisfy

$$g^*\alpha \in \Lambda^{1,1}_{\mathbf{R}}(T, J_0), \quad g^*\mathcal{H} \in \mathrm{HSym}^2(T_{1,0}(J_0)), \quad \text{and} \ \ g^*H \in \mathrm{HSym}^2{}_{\mathbf{R}}(T, J_0) \subset \mathrm{Sym}^2_{\mathbf{R}}(T).$$

Moreover, if the three elements $\alpha, \mathcal{H}$ and H correspond to each other under the isomorphisms (4.6), then so do the elements $g^*\alpha, g^*\mathcal{H}$ and g^*H. Any one of the six is ≥ 0 if and only if all six are ≥ 0. Finally, for any $B \in \mathrm{Sym}^2_{\mathbf{R}}(T)$

$$(g^*B)^{J_0} \ = \ g^*\left(B^J\right). \tag{4.12}$$

Proof. Since $Jg = gJ_0$, we have $g^*J^* = J_0^*g^*$. Finally, to prove (4.12) note that $(g^*B)^{J_0} = g^*B + J_0^*g^*B = g^*B + g^*J^*B = g^*(B + J^*B) = g^*(B^J)$ using (4.5). ■

Applying this to the complex hessian $\mathcal{H}(\varphi)$ and its real form $H(\varphi)$, yields

$$\mathcal{H}(\varphi) \ \geq \ 0 \qquad \iff \qquad (g^*H(\varphi))^{J_0} \ \geq \ 0. \tag{4.13}$$

Example 4.4. (The Standard Complex Structure on $\mathbf{C}^n$). Let J_0 denote the standard complex structure "i" on $\mathbf{C}^n$. With $V \equiv \sum_{j=1}^n c_j \frac{\partial}{\partial z_j}$ the complex hessian $\mathcal{H}_0$ is given by $\mathcal{H}_0(\varphi)(V,V) = \sum_{j,k=1}^n \left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right) c_j \bar{c}_k$. The real form H_0 of this complex hessian can be most succinctly expressed as

$$H_0\varphi \ = \ (D^2\varphi)^{J_0} \ = \ D^2\varphi + J_0^*D^2\varphi \tag{4.14}$$

since $\nabla J_0 \equiv 0$. That is, H_0 is simply the J_0-hermitian symmetric part of the second derivative $D^2\varphi$.

Now the subequation $F(J_0)$ on an open subset X of $\mathbf{C}^n$ is easily computed to be

$$F(J_0) \ = \ X \times \mathbf{C}^n \times \mathcal{P}^{\mathbf{C}} \qquad \text{where}$$
$$\mathcal{P}^{\mathbf{C}} \ \equiv \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : A^{J_0} = A + J_0^* A \geq 0\}.$$

By this definition of $\mathcal{P}^{\mathbf{C}}$ the equivalence (4.13) can be rewritten as

$$\mathcal{H}(\varphi) \ \geq \ 0 \qquad \iff \qquad g^* H(\varphi) \ \in \ \mathcal{P}^{\mathbf{C}}. \tag{4.13$'$}$$

Sometimes it is convenient to refer to the subequation $F(J_0)$ simply as $\mathcal{P}^{\mathbf{C}}$ since

$$\varphi \ \text{ is } J_0 \text{ subharmonic on } \ X \qquad \iff \qquad (D^2\varphi)(x) \in \mathcal{P}^{\mathbf{C}} \quad \forall\, x \in X. \tag{4.15}$$

We now assume our euclidean space $\mathbf{R}^{2n}$ to be equipped with a standard complex structure J_0 and write $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$ as above. We further assume that our variable almost complex structure J can be written in the form $J = gJ_0g^{-1}$ for a smooth map $g : X \to \mathrm{GL}^+_{\mathbf{R}}(\mathbf{R}^{2n})$. (This can be arranged in a neighborhood of any point $x \in X$ by choosing $J_0 = J_x$.) The next result describes $F(J)$ in these coordinates as a perturbation of the standard subequation $\mathcal{P}^{\mathbf{C}}$. It is the key to the two main theorems in this paper.

Proposition 4.5. *Suppose $g : X \to \mathrm{GL}^+_{\mathbf{R}}(\mathbf{R}^{2n})$ defines an almost complex structure $J \equiv gJ_0g^{-1}$ on an open subset $X \subset \mathbf{C}^n$. Let $\mathcal{H}$ denote the complex hessian for J. Then*

$$\mathcal{H}(\varphi) \geq 0 \qquad \iff \qquad g^* D^2\varphi + g^* E(D\varphi) \ \in \ \mathcal{P}^{\mathbf{C}} \tag{4.16}$$

where E is defined by (4.10).

Proof. By (4.9)$'$, $H(\varphi)$ is the J-hermitian part of $D^2\varphi + E(D\varphi)$. Hence, by Lemma 4.3, $g^* H(\varphi)$ is the J_0-hermitian part of $g^*(D^2\varphi + E(D\varphi))$. Therefore,

$$g^* H(\varphi) \in \mathcal{P}^{\mathbf{C}} \qquad \iff \qquad g^* \left(D^2\varphi + E(D\varphi)\right) \in \mathcal{P}^{\mathbf{C}}.$$

Combining this with (4.13)$'$ completes the proof of (4.16). ∎

5. Restriction of $F(J)$-plurisubharmonic Functions.

In this section we prove (in Theorem 5.2) that the restriction of a $F(J)$-plurisubharmonic function to an almost complex submanifold is also plurisubharmonic (as a function on the submanifold). The difficulty of this result is somewhat surprising. First we establish some easier facts.

Proposition 5.1. *Suppose $\Phi : (X', J') \to (X, J)$ is a pseudo-holomorphic map.*

(1) If $u \in C^2(X)$ is $F(J)$-plurisubharmonic on (X, J), then $u \circ \Phi$ is $F(J')$-plurisubharmonic on (X', J').

(2) If $j \in F_x(J)$, then $\Phi^*(j) \in F_{x'}(J')$ where $x = \Phi(x')$, that is,

$$\Phi^* \left(F(J)\right) \;\subset\; F(J'),$$

Proof. Since Φ^* commutes with $i\partial\overline{\partial}$, the pull-back of a smooth $F(J)$-plurisubharmonic function is $F(J')$-plurisubharmonic. The proof of (2) is similar. ∎

Now we state the more difficult result.

THEOREM 5.2. (Restriction). *Suppose that (X', J') is an almost complex submanifold of (X, J). If $u \in \mathrm{USC}(X)$ is $F(J)$-plurisubharmonic on X, then $u\big|_{X'}$ is $F(J')$-plurisubharmonic on X'.*

Proof. Since the result is local, we may choose coordinates which reduce us to the following situation. Suppose that J is an almost complex structure on a neighborhood X of the origin in $\mathbf{C}^n$, which agrees with the standard complex structure J_0 at $z = 0$. Suppose further that $X' = (\mathbf{C}^m \times \{0\}) \cap X$ is a J-almost-complex submanifold. By shrinking X if necessary we can find a smooth mapping $g : X \to \mathrm{GL}^+_{\mathbf{R}}(\mathbf{C}^n)$ with

$$g(0) = \mathrm{Id} \qquad \text{and} \qquad J = gJ_0g^{-1} \quad \text{on } X. \tag{5.1}$$

Block the transformation g as

$$g \;=\; \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \text{with respect to} \qquad \mathbf{C}^n = \mathbf{C}^m \times \mathbf{C}^{n-m}.$$

Because of the next result, we can choose g such that

$$g_{21} \;\equiv\; 0 \quad \text{on the submanifold } X'. \tag{5.2}$$

Lemma 5.3. *By further shrinking X if necessary, the mapping g can be chosen to be of the form*

$$g \;=\; \mathrm{I} + f \quad \text{with } f \text{ complex antilinear.}$$

With this choice, f is unique and $f_{21} (\equiv g_{21}) \equiv 0$ on X'.

Proof. Recall that each $g \in \mathrm{End}_{\mathbf{R}}(\mathbf{C}^n)$ has a unique decomposition $g = h + f_1$ with $h \in \mathrm{End}_{\mathbf{C}}(\mathbf{C}^n)$ complex linear, and $f_1 \in \overline{\mathrm{End}}_{\mathbf{C}}(\mathbf{C}^n)$ complex anti-linear. Since $h(0) = \mathrm{I}$, we may assume, by shrinking X, that $h(x)$ is invertible for each $x \in X$. Define $f \equiv f_1 h^{-1}$. Then (since h and J_0 commute), we have

$$J \;=\; gJ_0g^{-1} \;=\; gh^{-1}J_0hg^{-1} \;=\; gh^{-1}J_0(gh^{-1})^{-1} \qquad \text{and} \qquad gh^{-1} \;=\; \mathrm{I} + f.$$

This proves the first assertion.

For the uniqueness statement, suppose that $J = (\mathrm{I}+f_1)J_0(\mathrm{I}+f_1)^{-1} = (\mathrm{I}+f_2)J_0(\mathrm{I}+f_2)^{-1}$ with both f_1 and f_2 complex anti-linear. Then $(\mathrm{I}+f_2)^{-1}(\mathrm{I}+f_1)$ commutes with J_0, i.e., is complex linear. However, the complex anti-linear part of $(\mathrm{I}+f_2)^{-1}(\mathrm{I}+f_1) = (\mathrm{I}+f_2^2)^{-1}(\mathrm{I}-f_2)(\mathrm{I}+f_1)$ is $(\mathrm{I}+f_2^2)^{-1}(f_1-f_2)$, and so $f_1 - f_2 \equiv 0$.

It remains to prove the last assertion. To begin we block J with respect to the splitting $\mathbf{C}^n = \mathbf{C}^m \times \mathbf{C}^{n-m}$ as above. Then since X' is an almost complex submanifold, the component J_{21} must vanish along X'. Therefore, the 21-component of Jg equals $J_{22}g_{21} = J_{22}f_{21}$, while the 21-component of gJ_0 equals $g_{21}i = f_{21}i = -if_{21}$. Since $Jg = gJ_0$, this proves that $(J_{22}+i)f_{21} = 0$. Finally, $J_{22}(0) = i$, so that $f_{21} = 0$ along X' near the origin. ∎

Note. The last two statements can be seen in another way. An almost complex structure J_0 on a real vector space T is equivalent to a decomposition $T \otimes_{\mathbf{R}} \mathbf{C} = T_{1,0} \oplus T_{0,1}$ with $T_{0,1} = \overline{T_{1,0}}$. Another complex structure J on T, inducing the same orientation, has a similar decomposition, and $T_{1,0}(J)$ is the graph in $T_{1,0} \oplus T_{0,1}$ of a unique complex linear map $f : T_{1,0} \to T_{0,1}$ (or equivalently, a J_0-anti-linear map $f : T \to T$). Suppose now that $S \subset T$ is a J-complex subspace which is also J_0-invariant. Then $S_{1,0}(J) = S_{1,0}$ and so $f\big|_{S_{1,0}} = 0$. This is the condition that $f_{21} = 0$.

Completion of the Proof of Theorem 5.2. We apply Proposition 4.5. First note that (4.16) can be restated, using $p = Du$ and $A = D^2u$ as

$$(p, A) \in F_x(J) \qquad \iff \qquad g(x)^t(A + E_x(p))g(x) \in \mathcal{P}^{\mathbf{C}}(\mathbf{C}^n) \tag{4.16$'$}$$

Set $L_x(p) \equiv g^t(x)E_x(p)g(x)$.

Lemma 5.4. *If $p = (p', p'') \in \mathbf{C}^n = \mathbf{C}^m \times \mathbf{C}^{n-m}$, then for $x \in X' \equiv \mathbf{C}^m \times \{0\}$, $L_x((0, p'')) \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ vanishes when restricted to X' as a quadratic form.*

Proof. Suppose that v is a vector field tangent to X' along X'. Then since X' is an almost complex submanifold, the vector field Jv also has this property. It now follows directly from (4.10) that if $p = (0, p'')$, then $E(p)(v, v) \equiv 0$ along X'. In other words the component

$$E_{11}(0, p'') \ \equiv\ 0 \qquad \text{along} \ \ X'.$$

This together with (5.2) implies that

$$L_{11}(0, p'') \ \equiv\ 0 \qquad \text{along} \ \ X'. \qquad \blacksquare$$

The hypotheses (7.12) of the Restriction Theorem 8.1 in [HL$_3$] are now established. To check this, note first that the matrix function h in (7.12) of [HL$_3$] equals our g^t. Hence, by (5.2) we have $h_{12} = g_{21} = 0$ on X' as required. The g in (7.12) of [HL$_3$] is taken to be the identity, so its (12)-component vanishes on X'. The last part of (7.12) in [HL$_3$] is exactly Lemma 5.4 above. Theorem 5.2 now follows from Theorem 8.1 in [HL$_3$]. ∎

For 1-dimensional almost complex manifolds Theorem 5.2 has a converse, which we investigate in the next section.

6. The Equivalence of $F(J)$-plurisubharmonic Functions and Standard Plurisubharmonic Functions.

In complex dimension one, each almost complex structure is integrable, i.e., each almost complex manifold (Σ, J) of real dimension 2 is a Riemann surface. There are many equivalent definitions for subharmonic functions on a Riemann surface, and Definition 3.3 is one of these. We assume these facts without further discussion.

A "standard" definition of a plurisubharmonic function on a complex manifold makes perfect sense on an almost complex manifold.

Definition 6.1. An upper semi-continuous function u on an almost complex manifold (X, J) is said to be **plurisubharmonic in the standard sense** if its restriction to each holomorphic curve Σ in X is subharmonic.

The Restriction Theorem 5.2 implies the forward implication in the next result. The abundance of holomorphic curves on an almost complex manifold will be used to prove the reverse implication.

THEOREM 6.2. Given $u \in \mathrm{USC}(X, J)$ on an almost complex manifold (X, J),

$$u \ \text{ is } F(J) \text{ plurisubharmonic} \qquad \iff \qquad u \ \text{ is plurisubharmonic in the standard sense.}$$

Consequently, we may simply call these functions ***J*-plurisubharmonic**. The set of all such functions on X will be denoted by $\mathrm{PSH}^J(X)$.

In this section we shall replace the notation $F(J)$ by F^X to emphasize the manifold and to suppress confusion with notation for 2-jets. The jet version of Theorem 6.2 can be stated as follows.

Lemma 6.3.

$$\mathbf{J} \in F_z^X \qquad \iff \qquad i^*(\mathbf{J}) \in F_\zeta^\Sigma \quad \text{for all holomorphic curves } i : \Sigma \to X \text{ with } i(\zeta) = z.$$

Proof. $(\Rightarrow)$: This is the special case of Proposition 5.1(2) saying that $i^*(F_z^X) \subset F_\zeta^\Sigma$.

$(\Leftarrow)$: Pick a smooth function φ with $J_z(\varphi) \equiv \mathbf{J}$. Assume that $i_\Sigma^*(\mathbf{J}) \in F_\zeta^\Sigma$ for all $i_\Sigma : \Sigma \to X$ and $i_\Sigma(\zeta) = z$. We must show that $\mathcal{H}_z(V, V)(\varphi) \geq 0$ for all $V \in (T_{1,0}X)_z$. By [NW] there exists $i : \Sigma \to X$ with $i(\zeta) = z$ and $i_*(\frac{\partial}{\partial\zeta}) = V$. Hence, $\mathcal{H}_z(V, V)(\varphi) = \mathcal{H}_z(i_*(\frac{\partial}{\partial\zeta}), i_*(\frac{\partial}{\partial\zeta}))(\varphi) = \mathcal{H}_\zeta(\frac{\partial}{\partial\zeta}, \frac{\partial}{\partial\zeta})(\varphi \circ i)$ which is ≥ 0 since $J_\zeta(\varphi \circ i) = i^*(J_z\varphi) \in F_\zeta^\Sigma$. ∎

Proof of Theorem 6.2 As noted above we only need to prove $\Leftarrow$. Assume that $u \in \mathrm{USC}(X)$ and that $u \circ i$ is subharmonic on Σ for each holomorphic curve $i : \Sigma \to X$. Pick a point $z \in X$ and a test function φ for u at z. Suppose $i : \Sigma \to X$ is a pseudo-holomorphic curve with $i(\zeta) = z$. Obviously, $\varphi \circ i$ is a test function for $u \circ i$ at ζ. Since $u \circ i$ is subharmonic, $J_\zeta(\varphi \circ i) \in F_\zeta^\Sigma$. By Lemma 6.3 this is enough to imply that $J_z(u) \in F_z^X$, since $i^*(J_z(\varphi)) = J_\zeta(\varphi \circ i)$. ∎

There are advantages to using the concept of $F(J)$-plurisubharmonic over the standard one on an almost complex manifold. This is apparent for example, in the Section 7 on the Dirichlet Problem.

We conclude the section by mentioning a more elementary application illustrating the abundance of plurisubharmonic functions locally. We say that a system of local coordinates $z = (z_1, ..., z_n)$ on X is *standard at* $x \in X$ if $z(x) = 0$ and $J_x \cong J_0 (= i)$.

Proposition 6.4. *Suppose z is a local coordinate system for (X, J) which is standard at x. Then the function $u(z) = |z|^2$ is strictly $F(J)$-plurisubharmonic on a neighborhood of x.*

Proof. By (4.9), we see that, since $Du_0 = 0$, the real form of the J-hermitian hessian at the origin is $H(u)_0 = 2I$. Hence, $H(u)$ is positive definite in a neighborhood of 0. ∎

Corollary 6.5. *Each point of an almost complex manifold has a neighborhood system of domains with strictly pseudo-convex smooth boundaries.*

More precisely, these boundaries are strictly $F(J)$-convex in the sense of [HL$_2$].

7. The Dirichlet Problem.

In this section we consider the Dirichlet problem for the subequation $F(J)$ and for the more general "inhomogeneous" subequation $F(J, f)$ defined by adding the condition $(i\partial\overline{\partial}u)^n \geq f\lambda$ where λ is a fixed volume form on X and $f \in C(X)$ satisfies $f \geq 0$. Recall that by Definition 3.1, $F_x(J) = \{J_x^2 u : i\partial\overline{\partial}u \geq 0 \text{ for } u \text{ smooth near } x\}$, which we state more succinctly by saying

$$F(J) \ \text{ is defined by } \quad i\partial\overline{\partial}u \ \geq \ 0. \tag{7.1}$$

We now fix a (real) volume form λ on X compatible with the orientation induced by J. Then each $f \in C(X)$ with $f \geq 0$ determines a subequation

$$F(J, f) \ \text{ defined by } \quad i\partial\overline{\partial}u \ \geq \ 0 \qquad \text{and} \qquad (i\partial\overline{\partial}u)^n \ \geq \ f\lambda. \tag{7.2}$$

For simplicity we abbreviate $F \equiv F(J, f)$.

Definition 7.1. A smooth function u on (X, J) is F**-harmonic** if $J_x^2(u) \in \partial F_x$ for each point $x \in X$.

Example 7.2. (The Standard Model). In the model case $(X, J) = (\mathbf{C}^n, J_0)$ with $\lambda = \lambda_0 = 2^n \times$ the standard volume form on $\mathbf{C}^n$ and $f_0 \geq 0$ a continuous function, we have

$$D^2u \ \in \ \mathbf{F}(J_0, f_0) \qquad \iff \qquad i\partial\overline{\partial}u \geq 0 \ \text{ and } \ (i\partial\overline{\partial}u)^n \ \geq f_0\lambda_0, \tag{7.3}$$

where the second inequality can be rewritten as

$$\det{}_{\mathbf{C}}\left(\frac{\partial^2 u}{\partial z \partial \overline{z}}\right) \ \geq \ f_0.$$

Furthermore,

$$D^2u \ \in \ \partial\mathbf{F}(J_0, f_0) \qquad \iff \qquad i\partial\overline{\partial}u \geq 0 \ \text{ and } \ (i\partial\overline{\partial}u)^n \ = \ f_0\lambda_0, \tag{7.4}$$

where the equality can be rewritten as

$$\det_{\mathbf{C}}\left(\frac{\partial^2 u}{\partial z \partial \overline{z}}\right) \;=\; f_0.$$

This model subequation $\mathbf{F} \equiv \mathbf{F}(J_0, f_0) \subset \mathrm{Sym}^2(\mathbf{R}^{2n})$ is pure second-order, but not constant coefficient unless f_0 is constant. Rewriting (7.3) using jet coordinates we have that

$$A \in \mathbf{F}_x(J_0, f_0) \qquad \iff \qquad A_{\mathbf{C}} \geq 0 \quad \text{and} \quad \det_{\mathbf{C}}(A_{\mathbf{C}}) \;\geq f_0(x) \tag{7.3$'$}$$

The notion of F-harmonicity can be extended to $u \in \mathrm{USC}(X)$ as follows. Define the **Dirichlet dual** of F to be the set

$$\widetilde{F} \;= \sim(-\mathrm{Int}F) \;=\; -(\sim \mathrm{Int}F)\,. \tag{7.5}$$

One can show that $\widetilde{F}$ is also a subequation, i.e., it is a closed subset of the (reduced) 2-jet bundle $J^2(X)$ satisfying conditions (P) and (T) (cf. [HL$_2$] and Definition 3.5 above).

Note that

$$F \cap \left(-\widetilde{F}\right) \;=\; F \cap (\sim \mathrm{Int}F) \;=\; \partial F. \tag{7.6}$$

For any subequation F the notion of an upper semi-continuous F-subharmonic function is defined by using test functions (see Definition 3.6), and Definition 7.1 can be extended to continuous functions as follows.

Definition 7.3. A function $u \in C(X)$ is F-**harmonic** if u is F-subharmonic and $-u$ is $\widetilde{F}$-subharmonic on X.

The dual $\widetilde{F}$ of $F \equiv F(J, f)$ is defined fibre-wise by:

$$\begin{array}{rl} \text{either} & (1)\;\; i\partial\overline{\partial}(-u) \geq 0 \quad \text{and} \quad \left(i\partial\overline{\partial}(-u)\right)^n \geq f(x)\lambda \\ \text{or} & (2)\;\; i\partial\overline{\partial}(-u) \not\geq 0. \end{array} \tag{7.7}$$

Now suppose that Ω is an open set in X with smooth boundary $\partial\Omega$ and that $\overline{\Omega} = \Omega \cup \partial\Omega$ is compact.

Definition 7.4. We say that **uniqueness holds** for the F-Dirichlet problem (DP) on Ω if given $\varphi \in C(\partial\Omega)$ and $v, w \in C(\overline{\Omega})$ with v and w both F-harmonic on Ω, then

$$v \;=\; w \;=\; \varphi \;\text{ on } \partial\Omega \qquad \Rightarrow \qquad v \;=\; w \;\text{ on } \Omega.$$

We say that **existence holds** for (DP) on Ω if given $\varphi \in C(\partial\Omega)$, the Perron function

$$U \;\equiv\; \sup_{\mathcal{F}(\varphi)} u \qquad \text{where} \qquad \mathcal{F}(\varphi) \;\equiv\; \left\{u \in \mathrm{USC}(\overline{\Omega}) \cap F(\Omega) : u\big|_{\partial\Omega} \leq \varphi\right\}$$

satisfies

$$(1)\;\; U \in C(\overline{\Omega}), \qquad (2) \quad U \text{ is } F\text{-harmonic on } \Omega \quad \text{and} \;\; (3)\;\; U = \varphi \text{ on } \partial\Omega.$$

THEOREM 7.5. (The Dirichlet Problem). *For the subequation $F = F(J, f)$ on an almost complex manifold (X, J), we have the following.*

Uniqueness holds for the F-Dirichlet problem on $(\Omega, \partial\Omega)$ if the almost complex manifold (X, J) supports a C^2 strictly $F(J)$-plurisubharmonic function.

Existence holds for the F-Dirichlet problem if $(\Omega, \partial\Omega)$ has a strictly $F(J)$-plurisubharmonic defining function.

Corollary 7.6. *On any almost complex manifold (X, J) each point has a fundamental neighborhood system of domains $(\Omega, \partial\Omega)$ for which both existence and uniqueness hold for the Dirichlet problem.*

Proof. The idea is to apply the results of [HL_2] to the subequations $F = F(J, f)$ despite the lack of a riemannian metric. Proposition 4.5 above states that $F(J)$ is locally jet equivalent to $\mathcal{P}^{\mathbf{C}} = \mathbf{F}(J_0)$, the standard constant coefficient subequation on $\mathbf{C}^n$, while, more generally, Proposition 7.8 below states that $F(J, f)$ is locally jet-equivalent to $\mathbf{F}(J_0, f_0)$, the subequation described in Example 7.2.

Uniqueness. This is a standard consequence of comparison.

THEOREM 7.7. (Comparison). *If (X, J) supports a C^2 strictly J-plurisubharmonic function, then comparison holds for $F \equiv F(J, f)$, that is, for all $u \in F(X)$, $v \in \widetilde{F}(X)$, and compact subsets $K \subset X$:*

$$u + v \ \leq\ 0 \quad \text{on } \partial K \qquad \Rightarrow \qquad u + v \ \leq\ 0 \quad \text{on } K.$$

Proof. Section 10 in [HL_2] contains a proof that any subequation F which is locally affinely jet equivalent to a constant coefficient subequation $\mathbf{F}$ must satisfy *local weak comparison*. Even though $\mathbf{F}(J_0, f_0)$ is not constant coefficient, it is sufficiently close that the Theorem on Sums used in [HL_2] can be used to prove local weak comparison for $F(J, f)$. This is done in Proposition 7.9 below. Theorem 8.3 in [HL_2] states that *local weak comparison* implies *weak comparison* on X. Thus,

$$\text{Weak comparison holds for } F(J, f) \text{ on any almost complex manifold } (X, J). \tag{7.8}$$

It is easy to see that under fibre-wise sum

$$F(J, f) + F(J) \ \subset\ F(J, f), \tag{7.9}$$

that is, the convex cone subequation $F(J)$ is a monotonicity cone for $F(J, f)$.

Theorem 9.5 in [HL_2] states that if there exists a C^2 strictly $F(J)$-plurisubharmonic function on X, then the *strict approximation* property holds for $F(J, f)$ on X. Finally, Theorem 9.2 in [HL_2] states that for any subequation F, if both weak comparison and strict approximation hold, then **comparison holds** on X. Modulo proving Propositions 7.8 and 7.9 below, this completes the proof of comparison. ∎

Existence. This is a consequence of Theorem 12.4 in [HL_2], since it is straightforward to see from the definition that strict boundary convexity for $F(J)$ is the same as strict boundary convexity for $F(J, \lambda)$. ∎

In the case $F = F(J,f)$, if w is C^2 and $(i\partial\overline{\partial}w)^n \leq f\lambda$, i.e., $J^2w \notin \mathrm{Int}F$, then $v \equiv -w \in \widetilde{F}(X)$ and comparison applies to v, u for any $u \in F(X)$. In other words, $u \leq w$ on $\partial K \Rightarrow u \leq w$ on K. Therefore, as a special case of comparison we have the following.

$$
\begin{array}{l}
\text{For all } u \text{ which are } J\,\text{psh and satisfy } (i\partial\overline{\partial}u)^n \geq f\lambda \text{ in the viscosity sense,} \\
\text{if } w \in C^2(X) \text{ satisfies } (i\partial\overline{\partial}w)^n \leq f\lambda \text{ and } u \leq w \text{ on } \partial K, \text{ then } u \leq w \text{ on } K.
\end{array}
\tag{7.10}
$$

Local Jet-Equivalence with the Standard Model.

Recall the standard model in Example 7.2.

Proposition 7.8. *Choose local coordinates in $\mathbf{R}^{2n} \cong \mathbf{C}^n$ as in Section 4 above. In these coordinates the subequation $F(J,f)$ is locally jet-equivalent to the standard model subequation $\mathbf{F}(J_0, f_0)$ defined by*

$$
\det{}_{\mathbf{C}}\left(\frac{\partial^2 u}{\partial z \partial \overline{z}}\right) \geq f_0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial z \partial \overline{z}} \geq 0,
$$

Here $f_0 = \beta f$ where $\beta > 0$ is a smooth function, independent of f.

By the definition of jet-equivalence this Proposition states that there exists a $\mathrm{GL}^+(\mathbf{R}^{2n})$-valued smooth function h and a $(\mathbf{R}^{2n})^* \times \mathrm{Sym}^2(\mathbf{R}^{2n})$-valued smooth function L such that, with jet coordinates $p = Du$ and $A = D^2u$,

$$
(p, A) \in F_x(J, f) \qquad \Longleftrightarrow \qquad h(x)Ah^t(x) + L_x(p) \in \mathbf{F}_x(J_0, f_0). \tag{7.11}
$$

Proof. We recall the linear algebra from §4 (cf. Lemma 4.3) which involves a finite dimensional real vector space T of dimension $2n$ with two almost complex structures J and J_0, and two volume forms λ and λ_0, all of which are orientation compatible. (We think of J_0 and λ_0 as "standard" or "background" data.) Assume we are given $g \in \mathrm{GL}^+(T)$ such that

$$
J = gJ_0g^{-1}
$$

Let $\mathcal{P}^{\mathbf{C}}(T, J) \subset \mathrm{Sym}^2_{\mathbf{R}}(T)$ consist of all $H \in \mathrm{Sym}^2_{\mathbf{R}}(T)$ such that:

(1) H is J-hermitian, i.e., $J^*H = H$ and (2) $H \geq 0$.

Technically speaking, $\mathcal{P}^{\mathbf{C}}(T, J)$ is not a subequation, but it can be used to define $F(J)$ by setting

$$
F_x(J) \equiv \{J^2_x\varphi : H_x(\varphi) \in \mathcal{P}^{\mathbf{C}}(T, J)\}.
$$

Using this notion we have, as a restatement of Proposition 4.5, where $A \equiv D^2_x\varphi$ and $p \equiv D_x\varphi$, that

$$
(p, A) \in F_x(J) \qquad \Longleftrightarrow \qquad g(x)\,(A + J_0^*A)\,g^t(x) + L_x(p) \in \mathcal{P}^{\mathbf{C}}(\mathbf{C}^n, J_0) \tag{7.12}
$$

where $L_x(p) \equiv g(x)(E_x(p) + J_0^*E_x(p))g^t(x)$ with E defined by (4.8). This is the statement that $F(J)$ and $F(J_0)$ are jet-equivalent.

Now with the notation of Lemma 4.3, set $\alpha = i\partial\overline{\partial}\varphi$, $\mathcal{H} \equiv \mathcal{H}(\varphi)$ and $H \equiv H(\varphi)$. Then $H' \equiv g^*H = g(x)\,(A + J_0^*A)\,g^t(x) + L_x(p)$ is the expression in (7.12), and (7.12) is the statement that $H' \geq 0$ and H' is J_0-hermitian.

Let $\beta > 0$ be defined by

$$g^*\lambda \;=\; \beta\lambda_0. \tag{7.13}$$

Then $g^*(f\lambda) = f\beta\lambda_0 = f_0\lambda_0$ where $f_0 \equiv \beta f$. Therefore,

$$\alpha^n \geq f\lambda \qquad \iff \qquad (g^*\alpha)^n \geq f_0\lambda_0 \qquad \iff \qquad \det_{\mathbf{C}} A'_{\mathbf{C}} \geq f_0 \tag{7.14}$$

where $A'_{\mathbf{C}} \in M_n(\mathbf{C})$ is the matrix representative of $\alpha' = g^*\alpha \in \Lambda^{1,1}(T, J_0)$ with respect to any volume-compatible (i.e., λ_0-compatible) basis of $T_{1,0}(J_0)$. This completes the proof of Proposition 7.8. ∎

Local Weak Comparison for the Standard Model

The Theorem on Sums can be used to prove a weaker form of comparison for $\mathbf{F} \equiv \mathbf{F}(J_0, f)$ defined by:

$$A \in \mathbf{F}_x \qquad \iff \qquad A_{\mathbf{C}} \;\geq\; 0 \qquad \text{and} \qquad \det_{\mathbf{C}} A_{\mathbf{C}} \;\geq\; f(x), \tag{7.15}$$

as well as for $F \equiv F(J, f)$ defined by:

$$(p, A) \in F_x \qquad \iff \qquad g(x)Ag^t(x) + L_x(p) \in \mathbf{F}_x. \tag{7.16}$$

A notion of strictness which is uniform is employed. Given $c > 0$, define $\mathbf{F}^c$ by

$$A \in \mathbf{F}^c_x \qquad \iff \qquad B(A; c) \;\subset\; \mathbf{F}_x \tag{7.17}$$

where $B(A; c)$ denotes the ball in $\mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ about A of radius c. Define F^c by

$$(p, A) \in F^c_x \qquad \iff \qquad g(x)Ag^t(x) + L_x(p) \in \mathbf{F}^c_x. \tag{7.18}$$

Thus F is jet-equivalent to $\mathbf{F}$, and F^c is jet-equivalent to $\mathbf{F}^c$. Fix an open set $U \subset \mathbf{C}^n$.

Proposition 7.9. *If $u \in F^c(U)$ and $v \in \widetilde{F}(U)$, then $u+v$ satisfies the maximum principle.*

Proof. If the maximum principle fails for $u+v$, then (by [HL$_2$], Theorem C.1) the following quantities exist and have the stated properties:

(1) $z_\epsilon \;=\; (x_\epsilon, y_\epsilon) \;\to\; (x_0, x_0)$,

(2) $(p_\epsilon, A_\epsilon) \in F^c_{x_\epsilon}$ and $(q_\epsilon, B_\epsilon) \in \widetilde{F}_{y_\epsilon}$,

(3) $p_\epsilon \;=\; \dfrac{x_\epsilon - y_\epsilon}{\epsilon} \;=\; -q_\epsilon$ and $\dfrac{|x_\epsilon - y_\epsilon\,|^2}{\epsilon} \;\to\; 0$,

(4) $-\dfrac{3}{\epsilon} I \;\leq\; \begin{pmatrix} A_\epsilon & 0 \\ 0 & B_\epsilon \end{pmatrix} \;\leq\; \dfrac{3}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}$.

It is straightforward to show that

$$(q, B) \in \widetilde{F}_y \qquad \Longleftrightarrow \qquad g(y)Bg^t(y) + L_y(q) \in \widetilde{\mathbf{F}}_y. \tag{7.19}$$

(More generally, see Lemma 6.14 in [HL$_2$].) Thus (2) can be written as

$$\begin{aligned} (2)' \quad A'_\epsilon \ &\equiv \ g(x_\epsilon)A_\epsilon g^t(x_\epsilon) + L_{x_\epsilon}(p_\epsilon) \ \in \ \mathbf{F}^c_{x_\epsilon}, \quad \text{and} \\ B'_\epsilon \ &\equiv \ g(y_\epsilon)B_\epsilon g^t(y_\epsilon) + L_{y_\epsilon}(q_\epsilon) \ \in \ \widetilde{\mathbf{F}}_{y_\epsilon}. \end{aligned}$$

By definition of the dual, we have $B'_\epsilon \in \widetilde{\mathbf{F}}_{y_\epsilon}$ if $-B'_\epsilon \notin \mathrm{Int}\mathbf{F}_{y_\epsilon}$, that is:

$$\begin{aligned} (2)'' \quad \text{either} \quad (i) \quad & -B'_\epsilon \notin \mathcal{P}^{\mathbf{C}} \\ \text{or} \quad (ii) \quad & -B'_\epsilon \in \mathcal{P}^{\mathbf{C}} \quad \text{but} \quad \det_{\mathbf{C}}(-B'_\epsilon)_{\mathbf{C}} \ \leq \ f(y_\epsilon). \end{aligned}$$

Recall that $\mathcal{P}^{\mathbf{C}} \equiv \{B : B_{\mathbf{C}} \geq 0\}$.

The calculation on page 442 of [HL$_2$] proves that there exist $P_\epsilon \geq 0$ and a number $\Lambda \geq 0$ such that

$$g(x_\epsilon)A_\epsilon g^t(x_\epsilon) + g(y_\epsilon)B_\epsilon g^t(y_\epsilon) + P_\epsilon \ = \ \frac{\Lambda}{\epsilon}|x_\epsilon - y_\epsilon|^2. \tag{7.20}$$

Setting $A_\epsilon{}'' \equiv A'_\epsilon + P_\epsilon$, this proves

$$(5) \quad -B'_\epsilon \ = \ A_\epsilon{}'' - \frac{\Lambda}{\epsilon}|x_\epsilon - y_\epsilon|^2 - L_{x_\epsilon}(p_\epsilon) - L_{y_\epsilon}(q_\epsilon).$$

By positivity, $A'_\epsilon \in \mathbf{F}^c_{x_\epsilon} \ \Rightarrow \ A_\epsilon{}'' \in \mathbf{F}^c_{x_\epsilon}$, so that

$$(6) \quad \mathrm{dist}\left(A_\epsilon{}'', \sim \mathbf{F}_{x_\epsilon}\right) \ \geq \ c.$$

This implies

$$\begin{aligned} (6)' \quad (i) \quad & \mathrm{dist}\left(A_\epsilon{}'', \sim \mathcal{P}^{\mathbf{C}}\right) \ \geq \ c \quad \text{and} \\ (ii) \quad & \det_{\mathbf{C}}\left(A_\epsilon{}''\right)_{\mathbf{C}} \ \geq \ f(x_\epsilon) + \left(c/\sqrt{n}\right)^n \end{aligned}$$

The second inequality requires proof. Abbreviate $x = x_\epsilon$. It suffices to show that each point on the hypersurface Γ defined by $\det_{\mathbf{C}}A_{\mathbf{C}} = (f^{\frac{1}{n}}(x) + (c/\sqrt{n}))^n$ is distance $\leq c$ from the hypersurface Γ' defined by $\det_{\mathbf{C}}B_{\mathbf{C}} = f(x)$. Note that $B_0 \equiv f^{\frac{1}{n}}(x)I \in \Gamma'$ and the unit normal to Γ' at B_0 is $N = (1/\sqrt{n})I$. Set $A_0 = B_0 + cN = (f^{\frac{1}{n}}(x) + (c/\sqrt{n}))I$. Then $\det_{\mathbf{C}}A_0 = (f^{\frac{1}{n}}(x) + (c/\sqrt{n}))^n$ so that $A_0 \in \Gamma$. Note that $\mathrm{dist}(A_0, \Gamma') = c$ and other points $A \in \Gamma$ have smaller distance to Γ'. Therefore, $\mathrm{dist}(A, \Gamma') \geq c$ implies $\det_{\mathbf{C}}A \geq (f^{\frac{1}{n}}(x) + (c/\sqrt{n}))^n \geq f(x) + (c/\sqrt{n})^n$.

Suppose now that case (i) of $(2)''$ holds, i.e., $-B'_\epsilon \notin \mathcal{P}_{\mathbf{C}}$. Then by $(6)'$(i) we have $\|A_\epsilon{}'' + B'_\epsilon\| = \mathrm{dist}\left(A_\epsilon{}'', -B'_\epsilon\right) \geq c$. By (5),

$$\|A_\epsilon{}'' + B'_\epsilon\| \ = \ \Lambda\|I\|\frac{|x_\epsilon - y_\epsilon|^2}{\epsilon} + \left\|(L_{x_\epsilon} - L_{y_\epsilon})\left(\frac{x_\epsilon - y_\epsilon}{\epsilon}\right)\right\|. \tag{7.21}$$

This converges to zero as $\epsilon \to 0$ by (3), so that this case (i) cannot occur for $\epsilon > 0$ small.

Finally, suppose Case (ii) of $(2)''$ holds. By $(6)'$(ii) first and then $(2)''$(ii) we have:

$$c^{\frac{1}{n}} \ \leq\ \det_{\mathbf{C}}\left({A_\epsilon}''\right)_{\mathbf{C}} - f(x_\epsilon) \ \leq\ \det_{\mathbf{C}}\left({A_\epsilon}''\right)_{\mathbf{C}} - \det_{\mathbf{C}}\left(-{B_\epsilon}'\right)_{\mathbf{C}} + f(y_\epsilon) - f(x_\epsilon), \quad (7.22)$$

where $({A_\epsilon}'')_{\mathbf{C}} \geq 0$ and $(-{B_\epsilon}')_{\mathbf{C}} \geq 0$. Again by (5) and (3), as $\epsilon \to 0$, the RHS of (7.22) approaches zero, so this case cannot occur. ∎

Functions Which are Sub-the-Harmonics.

With regard to the subequation $F(J)$, since we have a notion of $F(J)$-harmonic functions (Definition 7.2), one can also consider function that are "sub" these harmonics.

Definition 7.10. A function $u \in \mathrm{USC}(X)$ is said to be **sub-the-$F(J)$-harmonics** if for each compact set $K \subset X$ and each $F(J)$-harmonic function h on a neighborhood of K,

$$u \ \leq\ h \quad \text{on} \ \ \partial K \qquad \Rightarrow \qquad u \ \leq\ h \quad \text{on} \ \ K.$$

Comparison for $F(J)$ on K implies that each $F(J)$-subharmonic is also sub-the-$F(J)$-harmonics. Since $F(J)$-subharmonicity is a local property, the converse is an elementary consequence of local existence which follows as in Remark 9.7 and Theorem 9.2 in [HL$_4$]. This proves the following.

THEOREM 7.11. *Given a function $u \in \mathrm{USC}(X)$,*

$$u \text{ is } F(J)\text{-subharmonic} \qquad \iff \qquad u \text{ is sub-the-}F(J)\text{-harmonics.}$$

In the language of [HL$_4$] this says that $F(J)^{\mathrm{visc}} = F(J)^{\mathrm{classical}}$.

F-Maximal Equals F-Harmonic.

There is a difference in the meaning of "solution" for our subequation $F = F(J, f)$, depending on whether one is grounded in pluripotential theory or viscosity theory.

Definition 7.12. A function $u \in F(X)$ is **F-maximal** if for each compact set $K \subset X$

$$v \leq u \text{ on } \partial K \quad \Rightarrow \quad v \leq u \text{ on } K$$

for all $v \in \mathrm{USC}(K)$ which are F-subharmonic on $\mathrm{Int}K$.

The maximality property for u is equivalent to comparison holding for the function $w \equiv -u$ (that is, $v + w$ satisfies the zero maximum principle for all functions v which are F-subharmonic).

Now comparison holds for $F(J)$ and Perron functions U are $F(J)$-harmonic on small balls. These facts can be used to show that following.

Proposition 7.13. *Given a function $u \in F(X)$,*

$$u \text{ is } F(J)\text{-harmonic} \qquad \iff \qquad u \text{ is } F(J)\text{-maximal.}$$

8. Distributionally Plurisubharmonic Functions.

So far we have discussed three notions of plurisubharmonicity on an almost complex manifold: the viscosity notion, the standard notion using restriction to holomorphic curves, and the notion of being sub-the-$F(J)$-harmonics. In all three cases we start with the same object – an upper semi-continuous function u. Theorem 6.2 states that these first two notions are equal, while Theorem 7.7 states that the first and third notions are the same. There is a yet another definition of plurisubharmonicity which starts with a distribution $u \in \mathcal{D}'(X)$. (Let $v \geq 0$ stipulate that v is a non-negative measure.)

Definition 8.1. A distribution $u \in \mathcal{D}'(X)$ on an almost complex manifold (X, J) is **distributionally J-plurisubharmonic on** X if

$$\mathcal{H}(V, V)(u) \;\geq\; 0 \quad \text{for all } V \in \Gamma_{\text{cpt}}(X, T_{1,0}) \tag{8.1}$$

or equivalently

$$H(v, v)(u) \;\geq\; 0 \quad \text{for all } v \in \Gamma_{\text{cpt}}(X, TX). \tag{8.2}$$

This distributional notion can not be "the same", but it is equivalent in a sense we now make precise. In what follows we implicitly assume that an $F(J)$-plurisubharmonic function $u \in \text{USC}(X)$ is not $\equiv -\infty$ on any component of X.

THEOREM 8.2.

(a) Suppose u is $F(J)$-plurisubharmonic. Then $u \in L^1_{\text{loc}}(X) \subset \mathcal{D}'(X)$, and u is distributionally J-plurisubharmonic.

(b) Suppose $u \in \mathcal{D}'(X)$ is distributionally J-plurisubharmonic. Then $u \in L^1_{\text{loc}}(X)$, and there exists a unique upper semi-continuous representative $\widetilde{u}$ of the L^1_{loc}-class u which is $F(J)$-plurisubharmonic. Moreover,

$$\widetilde{u}(x) \;=\; \operatorname{ess}\limsup_{y \to x} u(y).$$

Remark. In light of Theorem 6.2, statement (a) above is equivalent to a result of Nefton Pali [P, Thm. 3.9], and statement (b) provides a proof of his Conjecture 1 [P, p. 333]. Pali proves his conjecture under certain assumptions on the distrubution u [P, Thm. 4.1].

Both notions of plurisubharmonicity in this theorem can be reformulated using a family of "Laplacians". To begin we consider the standard complex structure on $\mathbf{C}^n$. Recall (Example 4.4) that the standard subequation $\mathcal{P}^{\mathbf{C}} \subset \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$ is the set of real quadratic forms (equivalently symmetric matrices) with non-negative J_0-hermitian symmetric part. The starting point is the following characterization of $\mathcal{P}^{\mathbf{C}}$. Assume $A \in \text{Sym}^2_{\mathbf{R}}(\mathbf{C}^n)$. Then

$$A \;\in\; \mathcal{P}^{\mathbf{C}} \qquad \iff \qquad \langle A, B\rangle \;\geq\; 0 \quad \forall\, B \in H\text{Sym}^2(\mathbf{C}^n) \;\text{ with } B > 0. \tag{8.3}$$

The proof is left to the reader.

Each positive definite $B \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{R}^N)$ defines a linear second-order operator

$$\Delta_B u \;=\; \langle D^2 u, B\rangle \tag{8.4}$$

which we call the B-Laplacian.

For C^2-fundtion u, the equivalence (8.3) can be restated as a characterization of plurisubharmonic functions.

$$u \;\text{ is psh} \qquad \iff \qquad \Delta_B u \;\geq\; 0 \quad \forall\, B \in H\mathrm{Sym}^2(\mathbf{C}^n) \;\text{ with } B > 0. \tag{8.3$'$}$$

Let

$$H_B \;\equiv\; \{A \in \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n) : \langle A, B\rangle \;\geq\; 0\} \tag{8.5}$$

be the subequation associated to the differential operator Δ_B. Then (8.3) can be restated as

$$\mathcal{P}^{\mathbf{C}} \;=\; \bigcap \bigl\{H_B : B \in H\mathrm{Sym}^2(\mathbf{C}^n) \text{ with } B > 0\bigr\}\,. \tag{8.3$''$}$$

Adopting the standard viscosity definition (using C^2-test functions) for H_B-subharmonic upper semi-continuous functions u, it is immediate from (8.3)$''$ that for a C^2-function φ, which is a test function for u, we have

$$D^2_x\varphi \in \mathcal{P}^{\mathbf{C}} \qquad \iff \qquad D^2_x\varphi \in H_B \quad \forall\, B \in H\mathrm{Sym}^2(\mathbf{C}^n) \text{ with } B > 0. \tag{8.6}$$

This proves the following.

Proposition 8.3. *An upper semi-continuous function u defined on an open subset X of $\mathbf{C}^n$ is $\mathcal{P}^{\mathbf{C}}$-plurisubharmonic if and only if it is Δ_B-subharmonic for every B-Laplacian, where $B \in H\mathrm{Sym}^2(\mathbf{C}^n)$ with $B > 0$.*

This proposition can be extended to $F(J)$-plurisubharmonic functions on any almost complex manifold (X, J), because of Proposition 4.5 (jet-equivalence). Suppose that $g : X \to \mathrm{GL}^+_{\mathbf{R}}(\mathbf{R}^{2n})$ defines an almost complex structure $J = gJ_0g^{-1}$ as in Proposition 4.5, and let $E : X \to \mathrm{Hom}_{\mathbf{R}}(\mathbf{C}^n, \mathrm{Sym}^2_{\mathbf{R}}(\mathbf{C}^n))$ be defined as in (4.8).

Definition 8.4. Given $B \in H\mathrm{Sym}^2(\mathbf{C}^n)$ with $B > 0$, define the B-Laplacian by

$$L_B u \;=\; \langle gBg^t, D^2 u\rangle + \langle E^t(gBg^t), Du\rangle. \tag{8.7}$$

THEOREM 8.5. *An upper semi-continuous function u on X is $F(J)$-plurisubharmonic if and only if it is viscosity L_B-subharmonic for all $B \in H\mathrm{Sym}^2(\mathbf{C}^n)$ with $B > 0$.*

Proof. Apply (4.15) and Definition 8.4. ∎

The parallel to Theorem 8.5 for distributional subharmonicity is also true. The proof is left to the reader.

Definition 8.6. A distribution $u \in \mathcal{D}'(X)$ is **distributionally L_B-subharmonic** if $L_B u$ is a non-negative measure on X.

THEOREM 8.7. *A distribution $u \in \mathcal{D}'(X)$ on an almost complex manifold (X, J) is distributionally J-plurisubharmonic if and only if locally, with L_B defined by (8.7), u is distributionally L_B-subharmonic for each $B \in H\mathrm{Sym}^2(\mathbf{C}^n)$ with $B > 0$.*

Combining Theorems 8.5 and 8.7 reduces Theorem 8.2 to the linear analogue for L_B. This theorem is treated in the Appendix – completing the proof of Theorem 8.2. ■

Corollary 8.8. *The concept of distributional J-plurisubharmonicity on an almost complex manifold (X, J) is equivalent to the notion of standard J-plurisubharmonicity.*

Proof. Apply Theorem 8.2 and the restriction Theorem 6.1. ■

9. The Non-Equivalence of Hermitian and Standard Plurisubharmonic Functions.

Whenever an almost complex manifold (X, J) is given a hermitian metric, i.e., a riemannian metric $\langle \cdot, \cdot \rangle$ such that J_x is orthogonal for all x, there is an induced notion of *hermitian plurisubharmonicity* defined via the riemannian hessian (cf. [HL_2]). If the associated Kähler form $\omega(v, w) = \langle Jv, w \rangle$ is closed, then the hermitian plurisubharmonic functions agree with the intrinsic ones studied in this paper. However, in general they are not the same. Proofs of these two assertions form the content of this section.

To begin we recall that on a riemannian manifold $(X, \langle \cdot, \cdot \rangle)$ each smooth function φ has a *riemannian hessian* $\mathrm{Hess}\,\varphi$ which is a section of $\mathrm{Sym}^2(T^*X)$ defined on (real) vector fields v and w by

$$(\mathrm{Hess}\,\varphi)(v, w) \;=\; vw\varphi - (\nabla_v w)\varphi.$$

where ∇ denotes the Levi-Civita connection. If we are given J, orthogonal with respect to $\langle \cdot, \cdot \rangle$, then

$$(\mathrm{Hess}^{\mathbf{C}}\varphi)(v, w) \;\equiv\; (\mathrm{Hess}\,\varphi)(v, w) + (\mathrm{Hess}\,\varphi)(Jv, Jw) \tag{9.1}$$

is a well defined hermitian symmetric form on (TX, J), i.e., a section of $\mathrm{HSym}^2{}_{\mathbf{R}}(TX)$. A function $\varphi \in C^2(X)$ is then defined to be **hermitian plurisubharmonic** if $\mathrm{Hess}^{\mathbf{C}}_x\varphi \geq 0$ for all $x \in X$. This notion carries over to arbitrary upper semi-continuous functions, and these have been used to study complex Monge-Ampere equations in this setting (see [HL_2]).

The natural question is: How are these hermitian psh-functions related to the intrinsic ones? In one important case they coincide.

THEOREM 9.1. *Suppose that $(X, J, \langle \cdot, \cdot \rangle)$ is an almost complex hermitian manifold whose associated Kähler form is closed. Then the space of hermitian plurisubharmonic functions on X coincides with the space of standard plurisubharmonic functions.*

Remark 9.2. Manifolds of this type are important in symplectic topology. Suppose that (X, ω) is a given symplectic manifold and J is an almost complex structure on X such that

$$\omega(v, Jv) \;>\; 0 \qquad \text{for all tangent vectors } v \neq 0.$$

(Such J always exist.) Then we can define an associated hermitian metric by $\langle v, w \rangle = \frac{1}{2}\{\omega(v, Jw) + \omega(w, Jv)\}$. The triple (X, J, ω) is sometimes called a *Gromov manifold.*

By a standard calibration argument, Theorem 9.1 is a direct consequence of the following more general result.

THEOREM 9.3. *Suppose that $(X, J, \langle\cdot,\cdot\rangle)$ is an almost hermitian manifold in which all holomorphic curves are minimal surfaces (mean curvature 0). Then the space of hermitian plurisubharmonic functions on X coincides with the space of standard plurisubharmonic functions.*

Proof. By Definition 4.8 we have

$$H(\varphi)(v,v) \;=\; \{vv + (Jv)(Jv) + J([v,Jv])\}\varphi, \tag{9.2}$$

which combined with Definition 9.1 above gives the identity

$$H(\varphi)(v,v) \;=\; (\mathrm{Hess}^{\mathbf{C}}\varphi)(v,v) + \{\nabla_v v + \nabla_{Jv}(Jv) + J([v,Jv])\}\varphi. \tag{9.3}$$

Consider now the restriction of $H(\varphi)$ to a holomorphic curve Σ, and suppose the vector field v is tangent to Σ along Σ. Then Jv and $J[Jv,v]$ are also tangent to Σ along Σ. As we know the restriction of $H(\varphi)$ to Σ agrees with the intrinsic hessian on Σ and is given by the formula (9.2).

We now consider the restriction of the RHS of (9.3). The tangential part $\nabla^T_v v \equiv (\nabla_v v)^T$ is the intrinsic Levi-Civita connection on Σ. This connection is torsion-free and preserves J, i.e., $\nabla^T J = 0$. (It preserves J since it preserves the metric and J is just rotation by $\pi/2$.) Thus

$$\begin{aligned}
\nabla^T_v v + \nabla^T_{Jv}(Jv) + J([v,Jv]) \;&=\; \nabla^T_v v + \nabla^T_{Jv}(Jv) + J(\nabla^T_v(Jv) - \nabla^T_{Jv}v)\\
&=\; \nabla^T_v v + \nabla^T_{Jv}(Jv) + (\nabla^T_v(J^2v) - \nabla^T_{Jv}Jv)\\
&=\; 0.
\end{aligned}$$

Hence, equation (9.3) becomes

$$H(\varphi)(v,v) \;=\; (\mathrm{Hess}^{\mathbf{C}}\varphi)(v,v) + \{\nabla^N_v v + \nabla^N_{Jv}(Jv)\}\varphi \tag{9.4}$$

where $\nabla^N_v v \equiv \nabla_v v - \nabla^T_v v$. This gives the following.

Lemma 9.4. *If v is a vector field tangent to a holomorphic curve Σ, then along Σ we have*

$$H(\varphi)(v,v) \;=\; (\mathrm{Hess}^{\mathbf{C}}\varphi)(v,v) + |v|^2 H_\Sigma\cdot\varphi \tag{9.5}$$

where H_Σ is the mean curvature vector field of Σ in the given hermitian metric.

Proof. By definition, the second fundamental form of Σ is $B_{v,v} = \nabla^N_v v$. Also by definition $H_\Sigma = \mathrm{tr}B = B_{e,e} + B_{Je,Je}$ for a (any) unit tangent vector e at the point. Formula (9.5) now follows from (9.4). ∎

Suppose now that all holomorphic curves are minimal. Then since every tangent vector v on X is tangent to a holomorphic curve by [NW], Lemma 9.4 implies that the subequations defined by $H(\varphi) \geq 0$ and $\mathrm{Hess}^{\mathbf{C}}(\varphi) \geq 0$ are the same. This completes the proof of Theorem 9.3. ∎

Now in general things are not so nice. Here are some examples which show that the notions of hermitian and standard plurisubharmonicity are essentially unrelated.

Example 9.5. Consider

$$\mathbf{C}^2 \;=\; \mathbf{R}^4$$

with coordinates $X = (z, w) = (x, y, u, v)$ and with the hermitian metric

$$ds^2 \;=\; \frac{|dX|^2}{(1+|X|^2)^2} \tag{9.6}$$

This is just the spherical metric in stereographic projection. Consider the holomorphic curves

$$\Sigma_x \;=\; \{(x,0)\} \times \mathbf{C} \;\subset\; \mathbf{C} \times \mathbf{C} \tag{9.7}$$

Think of these on the 4-sphere. They lie in the geodesic 3-sphere corresponding to the 3-plane $\{x\text{-axis }\} \times \mathbf{C}$. They form a family of round 2-spheres of constant mean curvature which are $\perp$ to the geodesic corresponding to the x-axis, and which fill out the 3-sphere. The mean curvature vector of Σ_x is

$$H_x^{\Sigma} \;=\; \phi(x)\frac{\partial}{\partial x}$$

where

$$x\phi(x) \;>\; 0 \quad \text{for} \quad x \neq 0.$$

Now consider the plurisubharmonic function

$$u(z,w) \;=\; |f(z)|^2$$

where f is holomorphic. Since u is constant on the curve Σ_x we have $\Delta^{\Sigma} u = 0$ and by formula (9.5) we have

$$\mathrm{tr}_{T_p\Sigma}\{\mathrm{Hess}\, u\} \;=\; -H^{\Sigma} \cdot u.$$

Now choose $f(z) = z$ so that $u = |z|^2 = x^2 + y^2$. Then

$$\mathrm{tr}_{T_p\Sigma}\{\mathrm{Hess}\, u\} \;=\; -2x\phi(x) \;<\; 0.$$

We conclude that:

Not every standard plurisubharmonic function on $\mathbf{C}^2$
is hermitian plurisubharmonic for this hermitian metric.

Conversely, we now show that

Not every hermitian plurisubharmonic function on $\mathbf{C}^2$*, with this metric,* *is plurisubharmonic in the standard sense.*

Our example is local. To facilitate computation we make the following change of coordinates. Consider $S^4 \subset \mathbf{R}^5$ (standardly embedded) and let $\Phi : S^4 \to \mathbf{R}^4$ denote stereographic projection, where $\mathbf{R}^4 = \mathbf{C}^2$ is our space above. Fix a point $(a, 0, 0, 0) \in \mathbf{R}^4$ $(a > 0)$. Then there is a unique rotation R of S^4 about the (y, u, v)-plane (i.e., a rotation of the $(x, \{\text{vertical}\})$ 2-plane) which carries $\Phi^{-1}(a, 0, 0, 0)$ to the south pole $(-1, 0, 0, 0, 0,)$. Let $\Psi = \Phi \circ R \circ \Phi^{-1}$ be the change of coordinates. Note that $\Psi(a, 0, 0, 0) = (0, 0, 0, 0)$.

Since R is an isometry, the metric (9.6) is unchanged by this transformation.

Of course the complex structure on $\mathbf{C}^2$ has been conjugated. In particular the holomorphic curve Σ_a in (9.7) has been transformed to the round 2-sphere passing through the origin:

$$\Sigma \;\equiv\; \{(x, 0, u, v) : (x - r)^2 + u^2 + v^2 = r^2\} \tag{9.8}$$

where $r = r(a) > 0$.

Consider now the function

$$u(x, y, u, v) \;=\; \tfrac{1}{2}\left(x^2 + y^2 + u^2 + v^2\right) - Cx$$

for $C > 0$. At the origin $0 \in \mathbf{R}^4$ the hermitian hessian (= the riemannian 4-sphere hessian) agrees with the standard coordinate hessian (since all the Christoffel symbols Γ^k_{ij} vanish at 0). Thus

$$\mathrm{Hess}_0 u = \mathrm{Id},$$

and we conclude that u is hermitian plurisubharmonic in a neighborhood of 0.

On the other hand, the Laplace-Beltrami operator on Σ satisfies

$$\Delta_\Sigma u \;=\; \mathrm{tr}_{T\Sigma}(\mathrm{Hess}\, u) + H_\Sigma \cdot u$$

where H_Σ is the mean curvature vector of the 2-sphere Σ. Since $H_\Sigma = (2/r)\frac{\partial}{\partial x}$, we conclude that

$$(\Delta_\Sigma u)_0 \;=\; 2 - \frac{2}{r}C \;<\; 0 \tag{9.9}$$

if $C > r$. The conformal structure on Σ is the one induced from S^4. Hence (9.9) implies that $u\big|_\Sigma$ is strictly superharmonic on a neighborhood of 0, and the assertion above is proved.

To Summarize: *On the hermitian complex manifold* $(\mathbf{C}^2, ds^2)$ *not every standard plurisubharmonic function is hermitian plurisubharmonic, and conversely, not every hermitian plurisubharmonic function is plurisubharmonic in the standard sense.*

Final Remark. The riemannian hessian is a bundle map

$$\mathrm{Hess} : J^2(X) \;\longrightarrow\; \mathrm{Sym}^2(T^*X) \tag{9.10}$$

and, as such, splits the fundamental exact sequence $0 \to \mathrm{Sym}^2(T^*X) \to J^2(X) \to T^*X \to 0$. The importance of this for nonlinear equations on riemannian manifolds is the following.

Each constant coefficient pure second-order subequation $\mathbf{F}_0 \subset \mathrm{Sym}^2(\mathbf{R}^n)$, which is $\mathrm{O}(n)$-invariant, naturally induces a sub-fibre-bundle $F_0 \subset \mathrm{Sym}^2(T^*X)$, and therefore determines $F \subset J^2(X)$ by

$$F \;=\; \mathrm{Hess}^{-1}(F_0).$$

The situation is analogous on an almost complex manifold (X, J). The real form of the complex hessian is a bundle map

$$H : J^2(X) \;\longrightarrow\; \mathrm{HSym}^2(T^*X) \;\subset\; \mathrm{Sym}^2(T^*X). \tag{9.11}$$

Any $\mathbf{F}_0 \subset \mathrm{HSym}^2(\mathbf{C}^n)$ which is $\mathrm{GL}_{\mathbf{C}}(n)$-invariant and satisfies $\mathbf{F}_0 + \mathcal{P}^{\mathbf{C}} \subset \mathbf{F}_0$ naturally induces a sub- sub-fibre-bundle $F_0 \subset \mathrm{HSym}^2(T^*X)$, and therefore determines $F \subset J^2(X)$ by

$$F \;=\; H^{-1}(F_0).$$

In both cases the resulting subequation is locally jet-equivalent to a constant coefficient equation.

There is "overlap" here corresponding to the cases $\mathrm{Hess}^{\mathbf{C}}(\varphi) \geq 0$ and $H(\varphi) \geq 0$ discussed above. As shown in Example 9.5, for general hermitian metrics these two subequations are not the same.

We note that the map (9.11) can be used to understand the polar cone $F(J)^0 \subset J_2(X)$ (the lower 2-jet bundle). It is the image of $\mathcal{P}^{\mathbf{C}} \subset \mathrm{HSym}^2(TX)$ under the dual map $\mathrm{Hess}^* : \mathrm{HSym}^2(TX) \to J_2(X)$.

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Appendix A. The Equivalence of the Various Notions of Subharmonic for Reduced Linear Elliptic Equations

Consider the reduced linear operator

$$Lu(x) \;=\; a(x) \cdot D^2u(x) + b(x) \cdot Du(x)$$

where a and b are C^∞ on an open set $X \subset \mathbf{R}^n$, and $a > 0$ is positive definite at each point. A C^2-function u on X is said to be *L-subharmonic* if $Lu \geq 0$ and *L-harmonic* if $Lu = 0$. These notions can be generalized using two completely different kinds of "test functions".

Definition A.1. A function $u \in \mathrm{USC}(X)$ is *L-subharmonic in the viscosity sense* if for every $x \in X$ and every test function φ for u at x, $L\varphi(x) \geq 0$ (cf. §3). Let $\mathrm{SH}^{\mathrm{visc}}(X)$ denote the set of these.

Definition A.2. A distribution $u \in \mathcal{D}'(X)$ is *L-subharmonic in the distributional sense* if $(Lu)(\varphi) \equiv u(L^t\varphi) \geq 0$ for every non-negative test function $\varphi \in C^\infty_{\mathrm{cpt}}(X)$, or equivalently, if $Lu \geq 0$, i.e., Lu is a positive measure on X. Let $\mathrm{SH}^{\mathrm{dist}}(X)$ denote the set of these.

We say that u is *viscosity L-harmonic* if both u and $-u$ are viscosity L-subharmonic. We say that u is *distributionally L- harmonic* if $Lu = 0$ as a distribution. In both cases there is a well developed theory of L-harmonics and L-subharmonics.

For example, the L-harmonics, both distributional and viscosity, are smooth. This provides the proof that the two notions of L-harmonic are identical. It is not as straightforward to make statements relating $\mathrm{SH}^{\mathrm{visc}}(X)$ to $\mathrm{SH}^{\mathrm{dist}}(X)$ since they are composed of different objects. The bridge is partially provided by the following third definition of L-subharmonicity.

Definition A.3. A function $u \in \mathrm{USC}(X)$ is *classically L-subharmonic* if for every compact set $K \subset X$ and every L-harmonic function φ defined on a neighborhood of K, we have

$$u \;\leq\; \varphi \quad \text{on } \partial K \qquad \Rightarrow \qquad u \;\leq\; \varphi \quad \text{on } K. \tag{1}$$

Let $\mathrm{SH}^{\mathrm{class}}(X)$ denote the set of these.

We always assume that u is not identically $-\infty$ on any connected component of X.

In both the viscosity case and the distributional case a great number of results have been established including the following.

THEOREM A.4.

$$\mathrm{SH}^{\mathrm{visc}}(X) \;=\; \mathrm{SH}^{\mathrm{class}}(X)$$

THEOREM A.5.

$$\mathrm{SH}^{\mathrm{dist}}(X) \;\cong\; \mathrm{SH}^{\mathrm{class}}(X)$$

Note that Theorem A.4 can be stated as an equality since elements of both $\mathrm{SH}^{\mathrm{visc}}(X)$ and $\mathrm{SH}^{\mathrm{class}}(X)$ are *a priori* in $\mathrm{USC}(X)$. By contrast, Theorem A.5 is not a precise statement until the isomorphism/equivalence is explicitly described.

Theorem A.5 requires careful attention, especially since in applications (such as the one in this paper) the isomorphism sending $u \in \mathrm{SH}^{\mathrm{dist}}(X)$ to $\widetilde{u} \in \mathrm{SH}^{\mathrm{class}}(X)$ is required to produce the same upper semi-continuous function $\widetilde{u} \in \mathrm{USC}(X)$ independent of the operator L.

Separating out the two directions we have:

THEOREM A.6. *If $u \in \mathrm{SH}^{\mathrm{class}}(X)$, then $u \in L^1_{\mathrm{loc}}(X) \subset \mathcal{D}'(X)$ and $Lu \geq 0$, that is, $u \in \mathrm{SH}^{\mathrm{dist}}(X)$.*

THEOREM A.7. *Suppose $u \in \mathrm{SH}^{\mathrm{dist}}(X)$. Then $u \in L^1_{\mathrm{loc}}(X)$. Moreover, there exists an upper semi-continuous representative $\widetilde{u}$ of the L^1_{loc}-class u with $\widetilde{u} \in \mathrm{SH}^{\mathrm{class}}(X)$. Furthermore,*

$$\widetilde{u}(x) \;\equiv\; \overrightarrow{\operatorname{ess}\lim_{y\to x}}\, u(y) \;\equiv\; \lim_{r \searrow 0} \operatorname*{ess\,sup}_{B_r(x)} u$$

is the unique such representative.

The precise statements, Theorem A.6 and Theorem A.7, give meaning to Theorem A.5.

Finally we outline some of the proofs.

Outline for Theorem A.4. Assume $u \in \mathrm{SH}^{\mathrm{visc}}(X)$ and h is harmonic on a neighborhood of a compact set $K \subset X$ with $u \leq h$ on ∂K. Since φ is a test function for u at x_0 if an only if $\varphi - h$ is a test function for $u - h$ at x_0, we have $u - h \in \mathrm{SH}^{\mathrm{visc}}(X)$. Since the maximum principle applies to $u - h$, we have $u \leq h$ on K.

Now suppose $u \notin \mathrm{SH}^{\mathrm{visc}}(X)$. Then there exists $x_0 \in X$ and a test function φ for u at x_0 with $(L\varphi)(x_0) < 0$. We can assume (cf. [HL$_2$, Prop. A.1]) that φ is a quadratic and

$$\begin{aligned} u - \varphi \;&\leq\; -\lambda|x - x_0|^2 \quad \text{for } |x - x_0| \leq \rho \;\text{ and} \\ &=\; 0 \qquad\qquad\qquad \text{at } x_0 \end{aligned}$$

for some $\lambda, \rho > 0$. Set $\psi \equiv -\varphi + \epsilon$ where $\epsilon = \lambda\rho^2$. Then ψ is (strictly) subharmonic on a neighborhood of x_0. Let h denote the solution to the Dirichlet Problem on $B \equiv B_\rho(x_0)$ with boundary values ψ. Since h is the Perron function for $\psi\big|_{\partial B}$ and ψ is L-subharmonic on B, we have $\psi \leq h$ on $\overline{B}$. Hence, $-h(x_0) \leq -\psi(x_0) = \varphi(x_0) - \epsilon < u(x_0)$. However, on ∂B we have $u \leq \varphi - \lambda\rho^2 = -\psi = -h$. Hence, $u \notin \mathrm{SH}^{\mathrm{class}}(X)$. ∎

Outline for Theorem A.6. This theorem is part of classical potential theory, and a proof can be found in [HH], which also treats the hypo-elliptic case. For fully elliptic operators L we outline the part of the proof showing that $u \in L^1_{\mathrm{loc}}(X)$.

Consider $u \in \mathrm{SH}^{\mathrm{class}}(X)$. Fix a ball $B \subset X$, and let $P(x,y)$ be the Poisson kernel for the operator L on B (cf. [G]). Then we claim that for $x \in \mathrm{Int}B$,

$$u(x) \ \leq \ \int_{\partial B} P(x,y)u(y)d\sigma(y) \tag{A.1}$$

where σ is standard spherical measure. To see this we first note that for $\varphi \in C(\partial B)$, the unique solution to the Dirichlet problem for an L-harmonic function on B with boundary values φ is given by $h(x) = \int_{\partial B} P(x,y)\varphi(y)d\sigma(y)$. Since $u \in \mathrm{SH}^{\mathrm{class}}(X)$ we conclude that

$$u(x) \ \leq \ \int_{\partial B} P(x,y)\varphi(y)d\sigma(y)$$

for all $\varphi \in C(\partial B)$ with $u\big|_{\partial B} \leq \varphi$. The inequality (A.1) now follows since $u\big|_{\partial B}$ is u.s.c., and $u\big|_{\partial B} = \inf\{\varphi \in C(\partial B) : u \leq \varphi\}$.

Note that the integral (A.1) is well defined (possibly $= -\infty$) since u is bounded above.

Consider a family of concentric balls $B_r(x_0)$ in X for $r_0 \leq r \leq r_0 + \kappa$ and suppose $x \in B_{r_0}$. Then for any probability measure ν on the interval $[r_0, r_0+\kappa]$ we have

$$u(x) \ \leq \ \int_{[r_0, r_0+\kappa]} \int_{\partial B_r} P_r(x,y)u(y)d\sigma(y)\, d\nu(r) \tag{A.2}$$

where P_r denotes the Poisson kernel for the ball B_r. Let $E \subset X$ be the set of points x such that u is L^1 in a neighborhood of x. Obviously E is open. Using (A.2) we conclude that if $x \notin E$ then $u \equiv -\infty$ in a neighborhood of x (cf. [Ho$_1$, Thm. 1.6.9]). Hence both E and its complement are open. Since we assume that u is not $\equiv -\infty$ on any connected component of X, we conclude that $u \in L^1_{\mathrm{loc}}(X)$.

It remains to show that $Lu \geq 0$. This is exactly Theorem 1 on page 136 of [HH]. ■

Outline for Theorem A.7. In a neighborhood of any point $x_0 \in X$ the distribution $u \in \mathrm{SH}^{\mathrm{dist}}(X)$ is the sum of an L-harmonic function and a Green's potential

$$v(x) \ = \ \int G(x,y)\mu(y) \tag{A.3}$$

where $\mu \geq 0$ is a non-negative measure with compact support. Here $G(x,y)$ is the Green's kernel for a ball B about x_0. It suffices to prove Theorem A.7 for Green's potentials v given by (A.3). The fact that $v \in L^1(B)$ is a standard consequence of the fact that $G \in L^1(B \times B)$ with singular support on the diagonal. Since $G(x,y) \leq 0$, (A.3) defines a point-wise function $v(x)$ near x_0 with values in $[-\infty, 0]$. By replacing $G(x,y)$ with the continuous kernel $G_n(x,y)$, defined to be the maximum of $G(x,y)$ and $-n$, the integrals $v_n(x) = \int G_n(x,y)\mu(y)$ provide a decreasing sequence of continuous functions converging to v. Hence, v is upper semi-continuous. The maximum principle states that $v \in \mathrm{SH}^{\mathrm{class}}(X)$.

Finally we prove that if $u \in L^1_{\mathrm{loc}}(X)$ has a representative $v \in \mathrm{SH}^{\mathrm{class}}(X)$, then $v = \widetilde{u}$. Since

$$\operatorname*{ess\,sup}_{B_r(x)} u \ = \ \operatorname*{ess\,sup}_{B_r(x)} v \ \leq \ \sup_{B_r(x)} v, \tag{A.4}$$

and v is upper semi-continuous, it follows that $\widetilde{u}(x) \le v(x)$.

Applying (A.2) to v and using the fact that $\int P_r(x,y)d\sigma(y) = 1$ (since 1 is L-harmonic) yields

$$\begin{aligned} v(x_0) &\le \frac{1}{\kappa}\int_{[0,\kappa]}\int_{\partial B_r} P_r(x_0,y)v(y)d\sigma(y)\,dr \\ &\le \left(\operatorname*{ess\,sup}_{B_\kappa} v\right)\frac{1}{\kappa}\int_{[0,\kappa]}\int_{\partial B_r} P_r(x_0,y)d\sigma(y)\,dr \\ &= \operatorname*{ess\,sup}_{B_\kappa} v \;=\; \operatorname*{ess\,sup}_{B_\kappa} u \end{aligned}$$

proving that $v(x_0) \le \widetilde{u}(x_0)$. ■

Remark A.8. The construction of $\widetilde{u}$ above is quite general and enjoys several nice properties, which we include here. To any function $u \in L^1_{\text{loc}}(X)$ we can associate its *essential upper semi-continuous regularization*:

$$\widetilde{u}(x) \;=\; \overrightarrow{\operatorname*{ess\,lim}_{y\to x}}\, u(y) \;\equiv\; \lim_{r\searrow 0}\operatorname*{ess\,sup}_{B_r(x)} u$$

This clearly depends only on the L^1_{loc}-class of u.

Lemma A.9. *For any $u \in L^1_{\text{loc}}(X)$, the function $\widetilde{u}$ is upper semi-continuous. Furthermore, for any $v \in \mathrm{USC}(X)$ representing the L^1_{loc}-class u, we have $\widetilde{u} \le v$, and if $x \in X$ is a Lebesgue point for u with value $u(x)$, then $u(x) \le \widetilde{u}(x)$.*

Proof. To show that $\widetilde{u}$ is upper semi-continuous, i.e., $\limsup_{y\to x}\widetilde{u}(y) \le \widetilde{u}(x)$, it suffices to show that

$$\sup_{B_r(x)} \widetilde{u} \le \operatorname*{ess\,sup}_{B_r(x)} u$$

and then let $r \searrow 0$. However, if $B_\rho(y) \subset B_r(x)$, then

$$\widetilde{u}(y) \;=\; \lim_{\rho\to 0}\operatorname*{ess\,sup}_{B_\rho(y)} u \le \operatorname*{ess\,sup}_{B_r(x)} u.$$

Letting $r \searrow 0$ in (A.4) proves that $\widetilde{u}(x) \le v(x)$.

For the last assertion of the lemma suppose that x is a Lebesgue point for u with value $u(x)$, i.e., by definition

$$\lim_{r\to 0}\frac{1}{|B_r(x)|}\int |u(y) - u(x)|\,dy \;=\; 0,$$

and hence the value $u(x)$ must be the limit of the means

$$u(x) \;=\; \lim_{r\to 0}\frac{1}{|B_r(x)|}\int u(y)\,dy \;\le\; \lim_{r\to 0}\operatorname*{ess\,sup}_{B_r(x)} u \;=\; \widetilde{u}(x). \qquad ■$$

Remark A.10. Theorem A.4 holds for any linear operator L with $a \geq 0$ provided that

- L-harmonic functions are smooth,
- The maximum principle holds for $u \in \mathrm{SH}^{\mathrm{visc}}(X)$
- The Perron function gives the unique solution for the Dirichlet problem for L.

Some Historical/Background Remarks. The equivalence of $\mathrm{SH}^{\mathrm{visc}}(X)$ and $\mathrm{SH}^{\mathrm{dist}}(X)$ for linear elliptic operators has been addressed by Ishii [I], who proves the result for continuous functions but leaves open the case where $u \in \mathrm{SH}^{\mathrm{visc}}(X)$ is a general upper semi-continuous function and the case where $u \in \mathrm{SH}^{\mathrm{dist}}(X)$ is a general distribution. The proof that "classical implies distributional" appears in [HH] where the result is proved for even more general linear hypoelliptic operators L. Other arguments that "viscosity implies distributional" are known to Hitoshi Ishii and to Andrzej Swiech. A treatment of mean value characterizations of L-subharmonic functions (again for subelliptic L) can be found in [BL]. A general introduction to viscosity theory appears in [CIL]. A good discussion of the Greens kernel appears in ([G]), and the explicit construction of the Hadamard parametrix is found in [Ho_2, 17.4].

The arguments outlined here are presented in more detail and for general convex subequations on manifolds in [HL].

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