THE AE THEOREM AND ADDITION THEOREMS FOR QUASI-CONVEX FUNCTIONS

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ABSTRACT

The main point of this paper is to prove the following useful result: If the almost everywhere 2-jet of a locally quasi-convex function u satisfies a degenerate elliptic constraint F, then u is F-subharmonic, i.e., u is a viscosity F-subsolution. This AE Theorem makes otherwise difficult results transparent. Some instances of this are presented, including two versions of addition, and a comparison theorem.

1. Introduction.

The main point of this short article is to state and prove a highly useful theorem giving a necessary and sufficient condition for a quasi-convex function to be a subsolution of a degenerate elliptic equation. If the equation is pure second-order with constant coefficients, the result was established in [HL_1, Cor.7.5], and here it is generalized to the fullest extent. As in out previous work (cf. [$\mathrm{HL}_{1,2}$]), we focus on viscosity subsolutions (rather than solutions) and take a "potential theoretic approach" in parallel with, and in fact generalizing, the theory of plurisubharmonic functions in several complex variables. More specifically, fix $X^{\rm open} \subset \mathbf{R}^n$, or more generally a manifold, and consider a closed subset

$$F \ \subset \ J^2(X) \equiv X \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$$

of the bundle of 2-jets of functions on X which satisfies the mild positivity condition (or degenerate ellipticity condition) (P) below. Using test functions, one defines the notion of *F-subharmonicity* (or *viscosity F-subsolution*) for any upper semi-continuous function $u : X \to [-\infty, \infty)$. The set $F(X)$ of these functions enjoys a majority of the important properties of the family of classical subharmonic functions on X.

Recall now that a quasi-convex function on X is twice differentiable at almost every point. The main result is:

The AE Theorem. *Suppose u is locally quasi-convex on a manifold X. Then*

$$J^2_x u \in F \quad \text{a.e.} \qquad \Rightarrow \qquad u \in F(X)$$

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*Partially supported by the N.S.F.

This theorem has many useful applications, and while perhaps not a surprise to experts, it deserves to be highlighted. (We were unable to find it in the literature outside the special case in [HL_1].) It proved, for example, to be a convenient tool in [HL_3] and in characterizing radial subharmonics in [HL_5], a fact which prompted the authors to write this note.

The result applies immediately to prove *addition*. Suppose $F, G \subset J^2(X)$ are closed subsets satisfying (P) as above and let $H \equiv \overline{F+G}$ (fibre-wise sum).

The Basic Addition Theorem. *Suppose u and v are locally quasi-convex on X. Then*

$$u \in F(X) \ \text{ and } \ v \in G(X) \qquad \Rightarrow \qquad u+v \in H(X)$$

Moreover, if F and G are constant coefficients subequations in $\mathbf{R}^n$, the quasi-convex assumptions can be dropped.

A nice application in the constant coefficient case is provided by the comparison result Theorem 6.4 and its generalization Theorem 6.8. A stronger form of the addition theorem is proved in §7.

2. Preliminaries.

There are several (equivalent) ways of defining subsolutions. For the purposes of this paper the most convenient approach is as follows. Let $\text{Sym}^2(\mathbf{R}^n)$ denote the set of symmetric $n \times n$ matrices.

Definition 2.1. Given a real-valued function w defined on an open subset $X \subset \mathbf{R}^n$, a point $x \in X$ is an **upper contact point for** w if there exists $(p, A) \in \mathbf{R}^n \times \text{Sym}^2(\mathbf{R}^n)$ such that

$$w(y) \ \leq \ w(x) + \langle p, y-x\rangle + \tfrac{1}{2}\langle A(y-x), y-x\rangle \qquad \forall\, y \text{ near } x. \tag{2.1}$$

In this case, (p, A) is called an **upper contact jet for** w **at** x.

Given a subset $F \subset X \times \mathbf{R} \times \mathbf{R}^n \times \text{Sym}^2(\mathbf{R}^n) \equiv J^2(X)$, we adopt the following definition of *subsolution*. Let $X \subset \mathbf{R}^n$ be an open subset, and denote by $\text{USC}(X)$ the set of upper semi-continuous functions taking values in $[-\infty, \infty) = \mathbf{R} \cup \{-\infty\}$. Let F_x denote the fibre of F at x.

Definition 2.2. A function $w \in \text{USC}(X)$ is said to be F-**subharmonic on** X if for each point $x \in X$ and every upper contact jet (p, A) for w at x, the jet $(w(x), p, A) \in F_x$. Let $F(X)$ denote the set of functions which are F-subharmonic on X.

Conditions must be placed on F in order for this concept to be useful. Of crucial importance is the condition of **positivity**:

$$(x, r, p, A) \in F \qquad \Rightarrow \qquad (x, r, p, A+P) \in F \quad \forall\ P \geq 0. \tag{P}$$

It is essential in order for every C^2-function w with $(x, w(x), D_x w, D_x^2 w) \in F$ for all $x \in X$ to be subharmonic. The second condition:

$$F \ \text{ is a closed subset} \tag{C}$$

is a necessary requirement for the elementary properties of $F(X)$ involving limits and upper envelopes to hold (see Theorem 2.6 in [HL_2]).

If F satisfies these minimal conditions (P) and (C) we say that F is a **primitive subequation**. The results of this paper hold under (P) and (C) with the exception of comparison results where F is required to be a subequation (see [HL_2]).

There are two extreme cases where the set of upper contact jets of a function w at a point x are essentially completely understood. The first case is where w has no upper contact jets at x. Such is the case if $w(x) = -\infty$ for example. It is also true of the function $w(x) = |x|$ at $x = 0$. The other extreme is when w is twice differentiable at x. By basic differential calculus we have the following.

Lemma 2.3. *Suppose w is twice differentiable at x. If (p, A) is an upper contact jet for w at x, then $p = D_x w$ is unique and $A = D_x^2 w + P$ for some $P \geq 0$. Conversely, for each $P > 0$ $(D_x w, D_x^2 w + P)$ is an upper contact jet for w at x.*

Adding a smooth function ψ to w does not change the set of upper contact points but only the upper contact jets.

Lemma 2.4. *Suppose ψ is smooth. Then*

$$x \ \text{ is an u. c. point for } \ w \qquad \iff \qquad x \ \text{ is an u. c. point for } \ w + \psi$$

$$(p, A) \ \text{ is an u. c. jet for } w \text{ at } x \qquad \iff \qquad (p + D_x\psi, A + D_x^2\psi) \ \text{ is an u. c. jet for } w + \psi \text{ at } x$$

A third elementary fact needed here concerns convex functions.

Lemma 2.5. *If w is convex and twice differentiable at x, then $D_x^2 w \geq 0$.*

Example 2.6. Each function $w \in \mathrm{USC}(X)$ determines a smallest primitive subequation, denoted $\overline{J^+(w)}$, with the property that w is subharmonic. Namely, let $J^+(w)$ denote the set of tuples $(x, w(x), p, A)$ such that (p, A) is an upper contact jet for w at x, and then take the closure $\overline{J^+(w)}$ in $X \times \mathbf{R} \times \mathbf{R}^n \times \mathrm{Sym}^2(\mathbf{R}^n)$. Obviously $J^+(w)$ satisfies condition (P), and hence the closure also satisfies (P). This example $\overline{J^+(w)}$ is in some sense pathological since it does not satisfies the requirements of negativity or the topological regularity required of a subequation in [HL_2].

Recall that a function w is called λ-**quasi-convex** if the function

$$u(y) \ \equiv \ w(y) + \frac{\lambda}{2}|y|^2 \tag{2.2}$$

is convex, and w is called **quasi-convex** if this is true for some $\lambda \geq 0$. Note that the concept of being locally quasi-convex is preserved under diffeomorphisms and therefore makes sense on any manifold.

Any property of convex functions can always be reformulated via (2.2) as a property of quasi-convex functions. Conversely, each result concerning quasi-convex functions can be reformulated as a result about convex functions.

The possible upper contact jets of a (locally) quasi-convex function are fairly well understood. We discuss this in the next section.

3. Upper Contact Jets of Quasi-Convex Functions

Elementary proofs of the results in this section can be found, for example, in the notes [HL_4] posted on the ArXiv.

First Derivatives

For quasi-convex functions, differentiability is automatic at each upper contact point.

Lemma 3.1. (D at UCP). *Suppose w is quasi-convex. If x_0 is an upper contact point for w, then w is differentiable at x_0. Moreover, if (p, A) is any upper contact jet for w at x_0, then $p = D_{x_0}w$ is unique.*

Another even more standard result is called partial continuity of the gradient, or first derivative.

Lemma 3.2. (PC of FD). *Suppose w is quasi-convex and $x_j \to x_0$. If w is differentiable at each x_j and at x_0, then $D_{x_j}w \to D_{x_0}w$.*

Second Derivatives

The results concerning the second-order part of upper contact jets of quasi-convex functions are of a deeper nature. The almost everywhere existence of the first derivative was omitted from the previous discussion because of the following stronger result.

THEOREM 3.3. (Alexandrov). *A locally quasi-convex function is twice differentiable almost everywhere.*

For the next result we need two variations of the notion of an upper contact jet. First, we say that (p, A) is a **strict** upper contact jet for $w \in \mathrm{USC}(X)$ at $x_0 \in X$ if the upper contact inequality (2.1) is strict for $y \neq x_0$. An understanding of the strict upper contact jets will be adequate for our discussion since (p, A) is an upper contact jet if and only if $(p, A + \epsilon I)$ is a strict upper contact jet for all $\epsilon > 0$. Second, we need a notion of upper contact point and jet, which requires the inequality (2.1) to hold globally.

Definition 3.4. Given $w \in \mathrm{USC}(X)$ and $A \in \mathrm{Sym}^2(\mathbf{R}^n)$, a point x is called a **global upper contact point of type A on X** if for some $q \in \mathbf{R}^n$

$$w(y) \ \leq \ w(x) + \langle q, y - x\rangle + \tfrac{1}{2}\langle A(y-x), y-x\rangle \quad \forall\, y \in X. \tag{3.1}$$

Let $C(w, X, A)$ denote the set of all global upper contact points of type A on X for the function w.

Remark 3.5. Note that if w is quasi-convex, then by (D at UCP) each point $x \in C(w, X, A)$ is a point of differentiability and the only q in (3.1) is $q = D_x w$.

THEOREM 3.6. (Jensen-Slodkowski). *Suppose that w is a quasi-convex function possessing a strict upper contact jet (p_0, A_0) at x_0. Let B_ρ denote the ball of radius ρ about x_0. Then there exists $\bar{\rho} > 0$ such that the measure*

$$|C(w, B_\rho, A_0)| \ > \ 0 \qquad \forall\, 0 < \rho \leq \bar{\rho}. \tag{3.2}$$

The four results above yield the following theorem which is easy to apply and adequate for many purposes. It is concerned with the upper contact jets of a quasi-convex function. The order two part of this theorem can be considered a "partial upper semi-continuity of the second derivative". We will use the acronym PUSC of SD.

THEOREM 3.7. (Upper Contact Jets). *Suppose w is quasi-convex with an upper contact jet (p_0, A_0) at a point x_0. Then*

(D at UCP) *w is differentiable at x_0 and $D_{x_0} w = p$.*

Suppose E is a set of full measure in a neighborhood of x_0. Then there exists a sequence $\{x_j\} \subset E$ with $x_j \to x_0$ such that w is twice differentiable at each x_j and

(PC of FD) $D_{x_j} w \ \to \ D_{x_0} w = p_0$,

(PUSC of SD) $D^2_{x_j} w \ \to \ A \leq A_0$.

Proof. By Alexandrov's Theorem, the set of points $x \in E$ where w is twice differentiable, is a set of full measure. In order to apply the Jensen-Slodkowski Lemma we replace (p_0, A_0) by the strict upper contact jet $(p_0, A_0 + \epsilon I)$. Now choose a sequence $\epsilon_j \to 0$, and pick a point $x_j \in B_{\epsilon_j}(x_0)$ such that: (1) $x_j \in E$, (2) w is twice differentiable at x_j, and (3) x_j is a global upper contact point of type $A_0 + \epsilon_j I$ on $B_{\epsilon_j}(x_0)$ for w. By the basic differential calculus fact Lemma 2.3, $D^2_{x_j} w \leq A_0 + \epsilon_j I$. Since w is λ-quasi-convex, we have $D^2_{x_j} w + \lambda I \geq 0$. Thus,

$$-\lambda I \ \leq \ D^2_{x_j} w \ \leq \ A_0 + \epsilon_j I. \tag{3.3}$$

By compactness there is a subsequence such that $D^2_{x_j} w \to A \leq A_0$. ∎

Theorem 3.7 can be stated succinctly in terms of the subset $J^+(w)$ (see Example 2.6) representing the upper contact jets of w, and another subset depending on E Define the subset $J(w, E)$ of the 2-jet bundle $J^2(X)$ to be the set of tuples $(x, w(x), D_x w, D^2_x w + P)$ such that $x \in E$, w is twice differentiable at x, and $P \geq 0$. Then Theorem 3.7 condenses to:

$$\text{If } w \text{ is quasi}-\text{convex and } E \text{ has full measure, then } \ J^+(w) \ \subset \ \overline{J(w, E)}. \tag{3.4}$$

4. The Almost Everywhere Theorem.

The main result of this section is the following. If u is twice differentiable at x, let J^2u denote the 2-jet $(u(x), D_x u, D_x^2 u)$ at x.

THEOREM 4.1. *Suppose that F is a primitive subequation on a manifold X and $u : X \to \mathbf{R}$ is a locally quasi-convex function. Then*

$$J_x^2 u \in F_x \text{ for almost all } x \in X \qquad \Rightarrow \qquad u \text{ is } F \text{ subharmonic on } X. \tag{4.1}$$

Proof. If u is not F-subharmonic on X, then there exists an upper contact jet (p_0, A_0) for u at some point $x_0 \in X$ with $(u(x_0), p, A) \notin F_{x_0}$.

We now apply Theorem 3.4 to u with E taken to be the set of second differentiability points x for u where the 2-jet of u belongs to F_x, i.e., $J_x^2 u \in F_x$. This yields a sequence $x_j \to x_0$ with $J_{x_j}^2 u \to (u(x_0), p_0, A_0 - P)$ with $P \geq 0$. The fact that F is closed, along with the positivity condition, implies that $(u(x_0), p_0, A_0) \in F_{x_0}$, which is a contradiction.■

By the elementary calculus Lemma 2.3 and Alexandrov's Theorem the concluding implication (4.1) can be replaced by the "if and only if" statement:

$$J_x^2 u \in F_x \ \text{ a.e.} \qquad \Longleftrightarrow \qquad u \in F(X) \tag{4.1$'$}$$

5. The Basic Addition Theorem

THEOREM 5.1. *Suppose that F and G are primitive subequations on a manifold X. Then $H \equiv \overline{F + G}$ is also a primitive subequation. Moreover, if u and v are quasi-convex functions on X, then*

$$u \in F(X) \ \text{ and } \ v \in G(X) \qquad \Rightarrow \qquad u + v \in H(X).$$

Moreover, If F and G are constant coefficient (translation invariant) on an open subset $X \subset \mathbf{R}^n$, then the assumption that u and v are quasi-convex can be dropped.

Proof. First note that the sum $F + G$ satisfies condition (P) and therefore so does its closure. Now by Alexandrov's Theorem and Lemma 2.3, $J_x^2 u \in F_x$ and $J_x^2 v \in G_x$ for a.a. $x \in X$. Hence $J_x^2(u + v) = J_x^2 u + J_x^2 v$ belongs to H_x for a.a. x. Now the AE Theorem 4.1 applied to H shows that $u + v \in H(X)$.

The constant coefficient part follows easily from quasi-convex approximation (cf. [HL_1]). ■

6. Comparison in the Constant Coefficient Case

Case I – Pure Second-Order

In this case comparison always holds, and this follows in a straightforward intuitive manner from the Addition Theorem 5.1 along with some algebraic considerations.

We note that a *constant coefficient pure second-order subequation* is any closed subset $F \subset \mathrm{Sym}^2(\mathbf{R}^n)$ which satisfies

$$A \in F \qquad \Rightarrow \qquad A + P \in F \quad \forall\, P \geq 0. \tag{P}$$

The most basic of all examples is

$$\mathcal{P} \ : \ \ \lambda_{\min}(A) \ \geq \ 0$$

where $\lambda_{\min}(A)$ is the minimum eigenvlaue. The dual subequation is

$$\widetilde{\mathcal{P}} \ : \ \ \lambda_{\max}(A) \ \geq \ 0$$

the so-called *subaffine (or co-convex) subequation*. The name subaffine for the subequation $\widetilde{\mathcal{P}}$ is justified by the following discussion. The reader is invited to verify the following facts (or see the proofs of Lemma 3.3a and Corollary 3.5 in [HL_2]):

$$F + \mathcal{P} \ \subset \ F \qquad \iff \qquad \widetilde{F} + \mathcal{P} \ \subset \ \widetilde{F} \qquad \text{and} \tag{6.1}$$

$$F + \widetilde{F} \ \subset \ \widetilde{\mathcal{P}}. \tag{6.2}$$

Let Aff denote the space of all affine functions on $\mathbf{R}^n$. A function $w \in \mathrm{USC}(X)$ will be called a *subaffine function on* X if

$$w \leq a \quad \text{on} \ \ \partial K \qquad \Rightarrow \qquad w \leq a \quad \text{on} \ \ K \qquad \forall\, K^{\mathrm{cpt}} \subset X \ \ \text{and} \ \ \forall\, a \in \mathrm{Aff} \tag{6.3}$$

The final step in the proof of comparison requires the following characterization of $\widetilde{\mathcal{P}}$-subharmonic functions.

Lemma 6.1. ([$\mathbf{HL}_1$]).

$$w \in \widetilde{\mathcal{P}}(X) \qquad \iff \qquad w \ \ \textit{is a subaffine function on} \ \ X.$$

Now putting these results together we have two theorems.

THEOREM 6.2. (The Subaffine Theorem). *If $u \in F(X)$ and $v \in \widetilde{F}(X)$, then*

$$\textit{the sum} \ \ w = u + v \ \ \textit{is a subaffine function on} \ \ X.$$

Proof. By Theorem 5.1 and (6.2), w is $\widetilde{\mathcal{P}}$-subharmonic so that Lemma 6.1 applies. ∎

In order to state comparison for an arbitrary domain $\Omega \subset \mathbf{R}^n$, (i.e., a bounded connected open subset), we note the following strengthening of (6.3). To give the reader some motivation, we point to the examples mentioned in Remark 6.9, which arise in studying the Dirichlet Problem with Prescribed Singularities [HL_6]. We note, however, that this lemma is required even when Ω is the open ball.

Lemma 6.3. *Let $\Omega \subset \mathbf{R}^n$ be any domain, and set $\partial\Omega \equiv \overline{\Omega} - \Omega$. Then*

$$w \leq a \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad w \leq a \quad \text{on} \quad \overline{\Omega} \tag{6.4}$$

for all functions $w \in \mathrm{USC}(\overline{\Omega})$ which are subaffine on Ω.

Proof. We can assume $a \equiv 0$. Exhaust Ω by compact sets $K_1 \subset K_2 \subset \cdots$ and set $U_\delta \equiv \{x \in \overline{\Omega} : w(x) < \sup_{\partial\Omega} w + \delta\}$. Since $w \in \mathrm{USC}(\overline{\Omega})$, each U_δ is an open neighborhood of $\partial\Omega$ in $\overline{\Omega}$. Therefore, for j sufficiently large we have $\partial K_j \subset U_\delta$. Hence by (6.3), for all j sufficiently large we have

$$\sup_{K_j} w \ \leq \ \sup_{U_\delta} w \ \leq \ \sup_{\partial\Omega} w + \delta$$

proving that $\sup_\Omega w \leq \sup_{\partial\Omega} w + \delta$ for all $\delta > 0$. ∎

THEOREM 6.4. (Comparison). *Suppose $\Omega \subset \mathbf{R}^n$ is a domain, and $u, v \in \mathrm{USC}(\overline{\Omega})$. If $u \in F(\Omega)$ and $v \in \widetilde{F}(\Omega)$, then*

$$u + v \ \leq 0 \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad u + v \ \leq \ 0 \quad \text{on} \quad \overline{\Omega}.$$

Proof. Apply Theorem 6.2 and Lemma 6.3, using the affine function $a \equiv 0$ in (6.4). ∎

Both of these two theorems were first proved in [HL_1].

For simplicity and clarity we discussed Case I separately, although it is a subset of the next case.

Case II – Gradient-Free Subequations

Again in this case comparison always holds.

Definition 6.5. A *gradient-free constant coefficient subequation* is a closed subset of $\mathbf{R} \times \mathrm{Sym}^2(\mathbf{R}^n)$ satisfying *positivity* and *negativity*:

$$(r, A) \in F \qquad \Rightarrow \qquad (r - s, A + P) \in F \qquad \forall\, s \geq 0 \ \text{ and } \ \forall\, P \geq 0. \qquad (P) \text{ and } (N)$$

Now the most basic example is the subequation

$$\mathbf{R}_- \times \mathcal{P} \qquad \text{where} \ \ \mathbf{R}_- \ \equiv \ (-\infty, 0].$$

The dual subequation is

$$\widetilde{\mathbf{R}_- \times \mathcal{P}} \ = \ (\mathbf{R}_- \times \mathrm{Sym}^2(\mathbf{R}^n)) \cup (\mathbf{R} \times \widetilde{\mathcal{P}}), \tag{6.5}$$

that is,

$$\widetilde{\mathbf{R}_- \times \mathcal{P}} \ : \quad r \le 0 \quad \text{or} \quad \lambda_{\max}(A) \ \ge\ 0. \tag{6.5)$'$}$$

Here the important algebraic facts are as follows. For any set $F \subset \mathbf{R} \times \mathrm{Sym}^2(\mathbf{R}^n)$:

$$F \ \text{ is a subequation} \qquad \iff \qquad \widetilde{F} \ \text{ is a subequation} \tag{6.6}$$

in which case

$$F + \widetilde{F} \ \subset\ \widetilde{\mathbf{R}_- \times \mathcal{P}} \tag{6.7}$$

Let Aff^+ denote the space of functions of the form

$$a^+ \ =\ \max\{a, 0\} \qquad \text{where} \ \ a \in \mathrm{Aff} \ \ \text{is affine.}$$

These will be referred to as the *affine plus functions* on $\mathbf{R}^n$. The enhancement of Lemma 6.1 characterizes the $(\widetilde{\mathbf{R}_- \times \mathcal{P}})$-subharmonics.

Lemma 6.6. *One has* $w \in (\widetilde{\mathbf{R}_- \times \mathcal{P}})(X) \quad \iff \quad w \in \mathrm{USC}(X)$ *and*

$$w \le a^+ \quad \text{on} \quad \partial K \qquad \Rightarrow \qquad w \le a^+ \quad \text{on} \quad K \qquad \forall\, K^{\mathrm{cpt}} \subset X \quad \text{and} \quad \forall\, a^+ \in \mathrm{Aff}^+$$

Such functions will be referred to as *subaffine plus functions on* X. Here as before, if $\Omega \subset \mathbf{R}^n$ is any domain and w is any upper semi-continuous function on $\overline{\Omega}$ which is subaffine plus on Ω, then

$$w \ \le\ a^+ \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad w \ \le\ a^+ \quad \text{on} \quad \overline{\Omega} \quad \forall\, a^+ \in \mathrm{Aff}^+. \tag{6.8}$$

The enhancements of Theorems 6.2 and 6.4 are now immediate. We assume $F \subset \mathbf{R} \times \mathrm{Sym}^2(\mathbf{R}^n)$ is a gradient-free subequation.

THEOREM 6.7 (The Subaffine Plus Theorem). *If* $u \in F(X)$ *and* $v \in \widetilde{F}(X)$, *then*

$$\textit{the sum} \ \ w \equiv u + v \ \ \textit{is subaffine plus on} \ \ X.$$

Proof. Apply the Addition Theorem 5.1, (6.7) and Lemma 6.6. ∎

THEOREM 6.8. (Gradient-Free Comparison). *Suppose* $\Omega \subset \mathbf{R}^n$ *is a domain, and* $u, v \in \mathrm{USC}(\overline{\Omega})$. *If* $u \in F(\Omega)$ *and* $v \in \widetilde{F}(\Omega)$, *then*

$$u + v \ \le 0 \quad \text{on} \quad \partial\Omega \qquad \Rightarrow \qquad u + v \ \le\ 0 \quad \text{on} \quad \overline{\Omega}.$$

Proof. Apply Theorem 6.7 and then (6.8). Note that $0 \in \mathrm{Aff}^+$ is an affine plus function.∎

Remark 6.9. If Ω is a domain and $x_1, ..., x_k \in \Omega$, then $\Omega - \{x_1, ..., x_k\}$ is also a domain, and the theorems above apply to this domain as well. This is particularly useful in our study of the Dirichlet Problem with Prescribed Singularities [HL_6] on a domain Ω with smooth boundary.

7. A Stronger Form of the Addition Theorem

THEOREM 7.1. *Suppose u and v are quasi-convex and that the sum $w \equiv u + v$ has an upper contact jet (p_0, A_0) at $x_0 \in \mathbf{R}^n$. Then x_0 is an upper contact point for both u and v. Furthermore:*

First Derivatives

(D at UCP). *Both u and v are differentiable at x_0, and the upper contact jets are all of the form $(D_{x_0}u, -)$ and $(D_{x_0}w, -)$ respectively.*

(DPC at FD). *For any sequence $x_j \to x_0$ with both u and v differentiable at each x_j,*

$$D_{x_j}u \to D_{x_0}u \qquad \text{and} \qquad D_{x_j}v \to D_{x_0}v$$

Second Derivatives

(PUSC of SD). *For each set E of full measure near x_0 there exists a sequence $x_j \to x_0$ with $x_j \in E$ and both u and v twice differentiable at x_j, such that*

$$D^2_{x_j}u \ \to\ A \qquad \text{and} \qquad D^2_{x_j}v \ \to\ B \qquad \textit{with} \quad A + B \le A_0.$$

Remark 7.2. In particular, given an upper contact jet (p_0, A_0) for the sum $w \equiv u + v$ at x_0, there exist two jets $(x_0, u(x_0), p, A') \in \overline{J^+(u)}$ and $(x_0, v(x_0), q, B) \in \overline{J^+(v)}$ which sum to $(x_0, w(x_0), p_0, A_0)$. (Take $A' \equiv A + P$ where $P \equiv A_0 - A - B \ge 0$.)

Proof. To show that x_0 is an upper contact point for u and v we can assume that they are both convex because of Lemma 2.4. Every point is a flat lower contact point for v by the Hahn-Banach Theorem. That is, there exists $q \in \mathbf{R}^n$ such that

$$v(x_0) + \langle q, y - x_0\rangle \ \le\ v(y) \qquad \text{for all } y \text{ near } x_0. \tag{7.1}$$

Hence,

$$u(y) \ =\ w(y) - v(y) \ \le\ u(x_0) + \langle p_0 - q, y - x_0\rangle + \tfrac{1}{2}\langle A_0(y - x_0), y - x_0\rangle$$

proving that u has upper contact jet $(p_0 - q, A_0)$ at x_0. The remaining first derivative results for u are just restatements of Lemmas 3.1 and 3.2.

To prove (PUSC of SD) we apply Theorem 3.4 to $w = u + v$ with upper contact jet (p_0, A_0) at x_0 and take E to be the set of points where both u and v are twice differentiable. This yields a sequence $x_j \to x_0$, $x_j \in E$ with $D_{x_j}w \to p_0$ and $D^2_{x_j}w \to A_0 - P$ with $P \ge 0$. Moreover,

$$D_{x_j}u \ \to\ D_{x_0}u = p, \qquad D_{x_j}v \ \to\ D_{x_0}v = q, \qquad \text{and} \qquad p + q \ =\ p_0.$$

In order to extract a subsequence of $\{x_j\}$ such that the second derivatives of u and v converge, i.e.,

$$D^2_{x_j}u \ \to\ A \qquad \text{and} \qquad D^2_{x_j}v \ \to\ B$$

we prove compactness.

We can assume that u and v are both λ-quasi-convex near x_0 with the same $\lambda >> 0$. This implies that at each point $x \in E$ near x_0,

$$-\lambda I \ \leq\ D_x^2 u \qquad \text{and} \qquad -\lambda I \ \leq\ D_x^2 v$$

Since $D_{x_j}^2 w \to A_0 - P$, we can assume that $D_{x_j}^2 w \leq A_0 - P + \epsilon I$ for all j. Hence,

$$D_{x_j}^2 u \ =\ D_{x_j}^2 w - D_{x_j}^2 v \ \leq\ A_0 - P + \epsilon I + \lambda I$$

providing the upper bound for $D_{x_j}^2 u$. Hence we can assume $D_{x_j}^2 u$ converges to A and similarly $D_{x_j}^2 v \to B$. Moreover, $A + B = A_0 + P$. ■

8. Strict Comparison

Given a subequation F, recall the dual subequation $\widetilde{F} \equiv \sim (-F)$ (see [HL$_2$]). For quasi-convex functions, comparison always holds if one of the functions is "strict". Each subequation $G \subset \mathrm{Int}F$ provides an adequate notion of F-strictness.

THEOREM 8.1. (Strict Comparison). *Suppose G and F are subequations on a manifold X with $G \subset \mathrm{Int}F$. Then for any domain $\Omega \subset\subset X$, if $u, v \in \mathrm{USC}(\overline{\Omega})$ are locally quasi-convex on Ω with $u \in G(\Omega)$ and $v \in \widetilde{F}(\Omega)$, comparison holds, i.e.,*

$$u + v \ \leq\ 0 \quad \text{on } \partial\Omega \qquad \Rightarrow \qquad u + v \ \leq\ 0 \quad \text{on } \Omega \tag{ZMP}$$

Moreover, if F and G are constant coefficient on $\mathbf{R}^n$, then the assumption that u and v are quasi-convex can be dropped.

Proof. If (ZMP) fails for $w = u + v$, then w has an interior maximum point x_0 in Ω, with $m \equiv w(x_0) > 0$ but $w \leq 0$ on $\partial\Omega$. At x_0, w has upper contact jet $(p_0, A_0) = (0, 0)$. Applying Theorem 7.1 in local coordinates yields:

(1) $(u(x_0), p, A) \in G_{x_0}$ and $(v(x_0), q, B) \in \widetilde{F}_{x_0}$,

(2) $u(x_0) + v(x_0) \ =\ m >\ 0$,

(3) $p + q \ =\ 0$.

(4) $A + B \ =\ -P$ with $P \geq 0$.

This contradicts $G_{x_0} \subset \mathrm{Int}F_{x_0}$ since

$$(v(x_0), q, B) \in \widetilde{F}_{x_0} \qquad \Longleftrightarrow \qquad (-v(x_0), -q, -B) \ =\ (u(x_0) - m, p, A + P) \notin \mathrm{Int}F_{x_0}$$

and $\mathrm{Int}F_{x_0}$ satisfies (N) and (P). Again the constant coefficient part follows easily from quasi-convex approximation (cf. [HL$_1$]). ■

Note 8.2. The constant coefficient case of Theorem 8.1 was established in [HL$_2$], Corollary C.3, using the Theorem on Sums ([CIL], [C]).

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