SUMMARY OF RESULTS FROM PRIOR NSF SUPPORT

Geometric problems in conformal analysis, dynamics and probability

DMS 1608577, 5/13/16 - 5/31/19, \$258,424

INTELLECTUAL MERIT OF PREVIOUS WORK

My recent work has focused on (1) quasiconformal mappings and their applications to geometric function theory and dynamics, and (2) computational geometry and optimal meshing (using ideas motivated by hyperbolic geometry, Riemann surfaces and conformal mappings). The bullet points below detail some of the specific results obtained.

- **True trees are dense** [21]**:** I show any compact, connected set $K \subset \mathbb{R}^2$ can be approximated in the Hausdorff metric by the critical points of a Shabat polynomial p (polynomials with only ± 1 as critical values). For such a polynomial, $T = p^{-1}([-1,1])$ is a finite tree (called a "true tree") and the proof shows such trees are dense in all planar continua.
- **Dynamical dessins are dense** [34]**:** Using this result, Kevin Pilgrim and I prove that Julia sets of post-critically finite polynomials are dense in all planar continua. See [46], [61].
- **Quasiconformal folding** [22]**:** This extends the "true trees are dense" result to entire functions: given a locally finite, unbounded planar tree T satisfying some natural conditions, I construct $f \in \mathcal{S}$ with $S(f) = \{\pm 1\}$ so that $T' = f^{-1}([-1,1])$ approximates T. Here $S(f)$ is the singular set of f (closure of critical values and asymptotic values), $\mathcal{S}$ denotes the Speiser class of transcendental (non-polynomial) entire functions with $S(f)$ finite, and $\mathcal{S}_{k,n} \subset \mathcal{S}$ are functions with k critical values and n finite asymptotic values. $\mathcal{B}$ denotes the Eremenko-Lyubich class of functions with $S(f)$ compact. QC-folding is a method converting certain trees and graphs into entire functions with precise control of the singular values. Some applications (and the corresponding graphs) are listed below. See also [10], [40], [59], [73]:

  ▶ A wandering domain in $\mathcal{B}$ (Dennis Sullivan [84] proved non-existence for rational functions; similar proofs are given in [39], [44] for $\mathcal{S}$, but the $\mathcal{B}$ case had remained open).

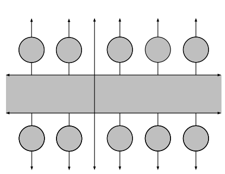
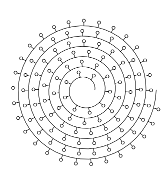
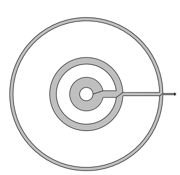
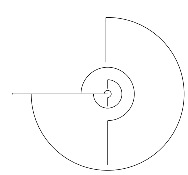
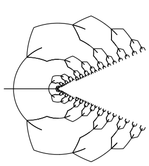
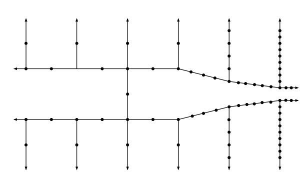
  ▶ A $f \in \mathcal{S}_{3,0}$ so that $\limsup_{r\to\infty} \log m(r,f)/\log M(r,f) = -\infty$ (counterexample in $\mathcal{S}$ to Wiman's 1916 conjecture, see also [45]; m, M are the min, max of $|f|$ on $\{|z| = r\}$).

  ▶ A $f \in \mathcal{S}_{3,0}$ so that area$(\{z : |f(z)| > \epsilon\}) < \infty$ for all ϵ (this is a strong counterexample to the area conjecture of Eremenko and Lyubich).

  ▶ A $f \in \mathcal{S}_{2,0}$ whose escaping set has no non-trivial path components (a counterexample to the strong Eremenko conjecture in $\mathcal{S}$; improves the example in [76] for $\mathcal{B}$).

  ▶ I disprove A. Epstein's order conjecture [24]: if $f, g \in \mathcal{S}$ and $\psi \circ f = g \circ \phi$ for some QC maps ψ, ϕ, then $\rho(f) = \rho(g)$ where $\rho = \limsup_{r\to\infty} \log\log M(r,f)/\log r$.

  ▶ A $f \in \mathcal{S}_{2,0}$ that grows as fast as desired on $\mathbb{R}$. See [65] for examples in $\mathcal{S}_{3,0}$.

- **Models for $\mathcal{B}$ and $\mathcal{S}$:** If Ω is a disjoint union of smooth, unbounded Jordan domains (called tracts) and $F : \Omega \to \{|z| > R\}$ is a holomorphic covering map on each tract, then (Ω, F) is called a model. Eremenko and Lyubich observed [39] that if $f \in \mathcal{B}$ then $(\Omega = \{|f| > R\}, f|_\Omega)$ is a model for large R; these are the EL-models. In [23] I prove that every model can be extended from $\Omega(\rho) = \{|F| > R + \rho\}$ to a $K(\rho)$-quasi-regular map on $\mathbb{C}$; thus every model can be approximated by an EL-model with the same number of tracts, answering a question of Rempe-Gillen [74]. In [27] I derive the analogous (and harder) result for $\mathcal{S}$, and investigate differences between these two classes.

• **The smallest transcendental Julia set** [28]**:** I construct a transcendental entire function f whose Julia set has Hausdorff dimension 1. This had been open since 1975 when Baker [5] proved that $\mathrm{Hdim}(\mathcal{J}(f)) \geq 1$ for all such f. Moreover, my example has finite spherical 1-measure, and packing dimension 1 (the first transcendental example with $\mathrm{Pdim}(\mathcal{J}) < 2$).

• **Small Julia sets in $\mathcal{S}$:** The previous example cannot be in the Eremenko-Lyubich class: Stallard [82], [83] proved that $\{\dim(\mathcal{J}(f)) : f \in \mathcal{B}\} = (1, 2]$. Using a refinement of QC folding, Simon Albrecht and I [29] have shown that $\inf\{\dim(\mathcal{J}(f)) : f \in \mathcal{S}\} = 1$. These are the first Speiser class examples with dimension < 2.

• **Prescribing postsingular dynamics:** Kirill Lazebnik and I prove that given any infinite discrete set S of complex numbers, any $\epsilon > 0$, and any map $h : S \to S$, there is a meromorphic f that approximates h on S in the following way: there is a bijection ψ between S and $P(f)$, the postsingular set of f, so that $f = \psi \circ h \circ \psi^{-1}$ on $P(f)$, $|\psi(z) - z| < \epsilon$ and $|\psi(z) - z| = o(1)$ as $|z| \nearrow \infty$. This is the transcendental analog of DeMarco, Koch and McMullen's result [55] for S finite and f rational. Taking $\epsilon = 0$ is open in both cases.

• **New families of 4-manifolds:** The almost-Kähler metrics on a given 4-manifold always sweep out an open subset in the moduli space of all anti-self-dual metrics. In [33], Claude LeBrun and I construct families of 4-manifolds where this subset is non-empty, but not closed. This surprising result hinges on an unexpected link between harmonic measure on certain hyperbolic 3-manifolds and self-dual harmonic 2-forms on associated 4-manifolds.

• **QC dimension distortion** [31]**:** H. Hakobyan, M. Williams and I show that if $E \subset \mathbb{R}^n$ is Ahlfors regular of dimension d and $f : \mathbb{R}^n \to \mathbb{R}^n$ is QC then $\mathrm{Hdim} f(y+E) = \mathrm{Hdim}(E)$ for a.e. $y \in \mathbb{R}^n$ (this also holds in Carnot groups). For a QC $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $S \subset \mathbb{R}$, we prove $\inf_{y \in S} \mathrm{Hdim}(f(\mathbb{R} \times \{y\})) \leq 2/(d+1)$ and $\inf_{x \in \mathbb{R}} \mathrm{Hdim}(f(\{x\} \times S)) \leq 2d/(d+1)$ and prove sharpness, extending work and answering questions from [6], [7]. We also build $E \subset \mathbb{R}$ with $\mathrm{Hdim}(E) = 1$, and a QC map f so that $f(E \times [0, 1])$ contains no rectifiable sub-arcs; this is the first uncountable example of such a set (impossible if E has positive length).

• **The NOT theorem:** I prove in [25] that any PSLG (planar straight line graph) with n vertices has a $O(n^{2.5})$ conforming **n**on-**o**btuse **t**riangulation (called a NOT for brevity; non-obtuse means all angles $\leq 90°$, conforming means the edges of the triangulation cover the edges of the PSLG). Giving any polynomial bound was a long standing open problem. The NOT theorem improves a famous $O(n^3)$ bound of Eldesbrunner and Tan [38] for conforming Delaunay triangulations, and also improves a variety of other optimal triangulation results.

• **The inverse Voronoi problem** [25]**:** Given a PSLG, I construct a point set V of size $O(n^{2.5})$, so that the Voronoi diagram of V covers Γ (the Voronoi diagram consists of points in $\mathbb{R}^2$ that are closest to two or more points of V). This gives the first polynomial time solution to this machine learning problem stated in [77]. An alternate formulation is to think of placing cell phone towers in several countries bounded by a total of n segments, so that a cell phone used in any country is always closest to a tower in that same country; this is possible with $O(n^{2.5})$ towers (and at least $\simeq n^2$ are needed in some cases).

• **Optimal quad-meshing:** I prove in [26] that every PSLG with n vertices has a conforming quadrilateral mesh with $O(n^2)$ elements, and all angles between $120°$ and $60°$ (except for smaller angles of the PSLG which remain unchanged). The complexity and angle bounds are all sharp. See [14] by Bern and Eppstein, who gave the upper angle bound for polygons.

BROADER IMPACT OF PREVIOUS WORK

• **Developing infrastructure for academic and industrial computing:** My meshing results enhance the suite of available automatic meshing algorithms available for research and industry, and improve practical computational methods in various ways. For example, condition numbers for certain matrices associated with general triangulations grow exponentially with the size of the mesh, but only linearly for NOTs, [90]; the finite element method on a NOT leads to a matrix that is symmetric, positive definite and negative off the diagonal, giving a linear system that is easier to solve [81]. Other practical advantages of NOTs are described in [35] (maximum principles for discrete PDE's), [11] (Hamilton-Jacobi equations), [53], [78] (finding geodesics on a triangulated surface), [1], [87], [88] (meshing space-time), [15], [81] (dual triangulations). My result on the inverse Voronoi problem is cited in [8], a paper dealing with the optimal placement of heat sinks on integrated circuits (removing excess heat is one of the primary bottlenecks in circuit design). The same fundamental problem of efficient packing by Voronoi cells also occurs in biological growth models [85], geographic information systems [92], and facility location problems [62], [91].

My work on optimal meshing depends on my earlier work on numerical conformal mapping [19], the medial axis (coming from computational geometry) and the "iota map" (coming from convex hull is hyperbolic 3-space). Potential applications of rapid conformal and QC mapping include automated face recognition (which enhances privacy and security), medical imaging, obstacle avoidance for robots (or self-driving cars), among others.

• **Building interdisciplinary connections:** The interdisciplinary character of the problems in the proposal can serve as a bridge between researchers with common interests but different backgrounds. For example, my papers on fast conformal mapping [19] and meshing [20], [25], [26], was specifically written to be accessible to both mathematicians and computer scientists, have appeared in a premier computer science journal, and have been presented at computer science conferences and seminars. My work on an easily computed, combinatorial version of the Riemann map (the so-called iota map, e.g., [17], [18], [19], [20]) has been cited in papers by applied mathematicians (e.g., [9], [41], [42]). I co-hosted a graduate workshop on computational geometry which included a mixture of "pure" and "applied" topics, e.g., mini-courses by Scott Sheffield on random geometric structures, by Esther Ezra on geometric set systems, by David Mount on nearest neighbor searches and by Yusu Wang on computational topology and persistent homology. I maintain the website for this workshop, and for other meetings I have organized.

My paper [30] with E. Feinberg and J. Zhang on the behavior of Abel and Cesàro limits has been cited in the economics literature, [50].

• **Educational impact:** The results obtained have been the basis of a series of graduate courses; a set of lecture notes on dynamics and quasiconformal analysis, and another set on conformal mapping and meshing. Some of the results have appeared in my recent book "Fractals in Probability and Analysis" with Yuval Peres. We are currently working on a sequel "Conformal Fractals" that will introduce topics like Julia sets, Kleinian limit sets, harmonic measure, DLA (diffusion limited aggregation) and the Gaussian free field. Another book near completion is a self-contained introduction to planar QC mappings and their applications to complex dynamics (including the folding theorem described in this proposal).

Recently I have mentored a number of undergraduate research projects on topics related to my work: Ahmed Rafiqi used accelerated random walks to calculate conformal maps (2018 Ph.D. from Cornell under John Hubbard); Kevin Sackel worked on QC removability (he won a Churchchill fellowship to Cambridge and is now a PhD student at MIT); Shalin Parekh numerically estimated percolation dimension of random walks on a grid (he spent a year in Geneva in a program run by Stas Smirnov and Wendelin Werner, and is currently a PhD student at Columbia), Christopher Dular implemented my $O(n^2)$ triangulation refinement algorithm (currently at Georgia Tech), and Joe Suk wrote code to implement by "Trues trees are dense" theorem numerically (Ph.D. program at Columbia); this year I am working with undergraduates Emi Brawley, Yugarshi Mondal and Hindy Drillick who all plan to pursue graduate studies in mathematics.

I have supervised 5 Ph.D. dissertations on topics related to my previous and current proposals: Zsuzanne Gönye (geodesics in hyperbolic manifolds), Karyn Lundberg (boundary convergence of conformal maps), Hrant Hakobyan (dimension distortion under QC maps) and Chris Green (numerical conformal mapping), and Kirill Lazebnik (wandering domains). I am currently supervising Jack Burkart (transcendental Julia sets). Most of my students have had some computational aspect to their thesis; this makes them better suited to both academic and non-academic jobs (Green works at Twitter and Lundberg at Lincoln Labs at MIT). Producing mathematicians who can talk to and work with applied mathematicians (or even non-mathematicians) is a form of infrastructure enhancement that makes it easier to transfer decades of mathematical progress into practical solutions of important problems. This point was well illustrated at the 2018 ICM in Rio de Janeiro, where almost a third of all the plenary/prize/public lectures dealt with the interaction of mathematics, computation and applications.

PUBLICATIONS RESULTING FROM RECENT NSF SUPPORT

- Constructing entire functions by quasiconformal folding. *Acta Math.*, 214(1):1–60, 2015.
- A transcendental Julia set of dimension 1. *Invent. Math.*, 212(2) 407–460, 2018.
- Harmonic measure: algorithms and applications, *Proc. Int. Cong. of Math., 2018 Rio de Janeiro*, Vol. 2, 1507–1534.
- Models for the Eremenko–Lyubich class. *J. Lond. Math. Soc. (2)*, 92(1):202–221, 2015.
- Dynamical dessins are dense. with K. Pilgrim, *Rev. Mat. Iberoamer.*, 31(3), 1033–1040, 2015.
- The order conjecture fails in class $\mathcal{S}$. *J. d'Analyse.*, 127(1), 283–302, 2015.
- Models for the Speiser class. *P. Lond. Math. Soc.*, 114(5), 765-797, 2015.
- Nonobtuse triangulations of PSLGs. *Discrete Comput. Geom.* 56(1), 43–92 , 2016.
- Quadrilateral meshes for PSLGs. *Discrete Comput. Geom.* 56(1), 1–42, 2016.
- Frequency of dimension distortion under quasisymmetric mappings, with H. Hakobyan, and M. Williams, *Geometric and Functional Analysis*, 26(2), 379–421, 2016.
- Anti-self-dual 4-manifolds, quasi-Fuchsian groups and almost-Kahler geometry, with Claude LeBrun, *to appear in Communications in Analysis and Geometry.*
- Speiser class Julia sets with dimension near one, with S. Albrecht. submitted to J. d'Analyse.
- Prescribing the postsingular dynamics of meromorphic functions, with K. Lazebnik, preprint

EVIDENCE OF RESEARCH PRODUCTS AND THEIR AVAILABILITY

All preprints are posted on `www.math.sunysb.edu/~bishop/papers`. My webpage also contains lecture notes and slides of lectures related to my research (also links to videos of my lectures when they exist), class notes with links to relevant literature, as well as abstracts of my papers, descriptions of my research and links to workshops I have organized or attended and to related work of other mathematicians. A survey of my recent work was published in the proceedings of the 2018 ICM.

PROJECT DESCRIPTION

The proposal deals with combinatorial constructions in geometric function theory (true trees, quasiconformal folding) and applications of function theory ideas to discrete problems (constructing manifolds, optimal meshing, random growth). Quasiconformal mappings and harmonic measure will play prominent roles. The proposed questions range from long standing (and probably very difficult) conjectures to questions suitable for graduate and even undergraduate students.

1. Trees and triangles

• **True trees:** Given a tree T with n edges drawn in the plane, add edges connecting each vertex to ∞ until we obtain obtain a topological triangulation of the sphere with $2n$ elements. Glue together equilateral triangles using the same pattern of adjacencies to get a conformal sphere. A homeomorphic copy T' of T lives on this sphere (the edges of the triangulation not incident on ∞). One can check that we can map each triangle conformally to either the upper or lower half-plane so that ∞ maps to ∞ and each edge of T' maps to $[-1, 1]$, and so that these maps agree across the boundaries of the triangles. Thus the resulting function p is a n-to-1 holomorphic function from the sphere to itself, branched of order n at ∞, and hence is a polynomial of degree n. Moreover, p only has critical values at ± 1 (these are called Shabat polynomials) and $T' = p^{-1}([-1, 1])$. We call T' the "true form" of T, or a "true tree" for short, see [47], [56], [57], [79].

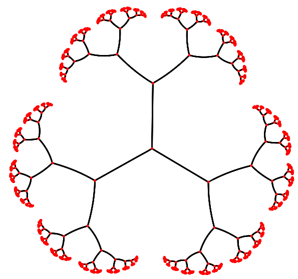
This construction is a special case of Grothendieck's *dessins d'enfants* where a finite graph on a compact surface gives rise to a conformal structure on that surface. It is interesting algebraically because the polynomial p can be taken to have algebraic coefficients; thus the action of Galois groups on the polynomials induces an action on finite planar trees. It is interesting analytically because the true form of a tree gives every edge equal harmonic measure (the first hitting probability of a Brownian motion) from ∞, and the harmonic measure restricted to the two sides of any edge are equal (the tree is "conformally balanced"). It is natural to ask if there is a similar correspondence for infinite trees:

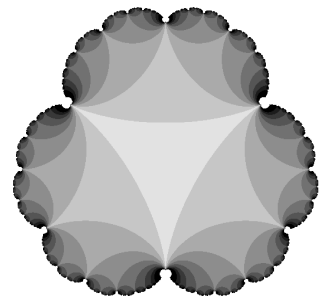
Question 1. *Which locally finite, infinite plane trees T have a true form, i.e., they are ambiently homeomorphic to $f^{-1}([-1, 1])$ for some entire function singular values ± 1?*

The uniformization theorem implies that every such tree occurs as $T = f^{-1}([-1, 1])$ for some holomorphic f defined on either the plane or the unit disk; for us an "infinite true tree" means it corresponds to the plane. There is a solution to this type problem due to Peter Doyle [37] in terms of the recurrence or transience of a random walk on an infinite graph constructed from T, but this criterion is not always easy to apply, so we would like more tractable conditions, such as those given by Cui in [36]. An interesting test case are the infinite random trees coming from DLA; see Question 30.

The infinite 3-regular tree T does not have a true form in the plane; this is deduced using Nevanlinna theory in [36] (but it is not hard to see directly). If we compute the true forms $\{T_n\}$ of nth level truncations of T, the Euclidean size of the edges decay exponentially away from the root, and the truncations $\{T_n\}$ seem to approach a limiting shape; see left figure below, drawn with Don Marshall's `ZIPPER` program. Steffen Rohde and Brent Werness have observed that this limit appears to agree exactly with a fractal curve coming from successive reflections across the boundary of the standard deltoid (see right figure below), but there is, as yet, no explanation of this coincidence (the deltoid fractal arises in anti-holomorphic dynamics; see [60] for the precise definition).

Problem 2. *Prove the leaves of the truncated true trees $\{T_n\}$ limit on the deltoid fractal.*





It is known that the deltoid fractal can also be defined as a limit of curves on conformal spheres constructed by gluing together equilateral triangles in a tree-like pattern. A careful study of these patterns should lead to a proof of the illustrated similarity.

• **Quasiconformal folding:** Recall that a quasiconformal map is a homeomorphism of $\mathbb{C}$ that is absolutely continuous on almost all lines and such that $|\mu| \equiv |f_{\bar{z}}/f_z| \leq k < 1$; this says that infinitesimal circles map to ellipses with uniformly bounded eccentricity (taking $k = 0$ gives a conformal map). The measurable Riemann mapping theorem (MRMT) is the fundamental result that every such μ arises from some QC mapping f. A quasi-regular function is a composition of a holomorphic function and a QC map.

The construction of true trees via the uniformization theorem can be replaced by a quasiconformal construction which gives greater control of the geometry. Using this, I proved my "true trees are dense" theorem [21]: any compact continuum can be approximated by finite true trees in the Hausdorff metric. The proof quickly reduces to approximating trees from a certain nice class by true trees. Which infinite planar trees can be approximated by infinite true trees? The quasiconformal folding theorem provides a partial answer.

The QC-folding theorem say that if T is a locally finite, infinite planar tree that satisfies some mild geometric conditions (easy to check in practice), then we can add finite trees to the vertices of T to get a new tree T' and quasiconformal maps η from each complementary component of T' to the right half-plane so that $g = \cosh \circ \eta$ is quasi-regular map on the whole plane. Moreover, g is holomorphic except on a small neighborhood of the original tree T. Then by the measurable Riemann mapping theorem, we can define a holomorphic $f = g \circ \varphi$ (φ is QC and often close to the identity) whose corresponding tree combinatorially equals T', and whose shape approximates T. The singular values of f are the same as for g, and these can be controlled very precisely in the construction. As noted in the summary of previous work, QC-folding has already settled a number of open questions in function theory and holomorphic dynamics. Below we discuss further potential applications.

• **Belyi and Shabat functions:** A holomorphic function $f : X \to \mathbb{S}^2$ on a Riemann surface X is called a Belyi function if f is branched only over 0, 1 and ∞, and f has no removable singularities at punctures of X. The latter condition implies that f cannot be holomorphically extended to a Riemann surface properly containing X. For example, the polynomials associated to true trees above are Belyi functions for the Riemann sphere.

Question 3. *Do all planar domains have a Belyi function? All open Riemann surfaces?*

If the surface X has a Belyi function f, then the preimages of the upper and lower half-planes are topological triangles in G which are conformally equivalent to equilateral triangles glued (according to arclength) along their edges. Thus the previous question really asks if every open Riemann surface can be constructed from a countable collection of equilateral triangles glued along their boundaries. If we also assume the Belyi functions have no asymptotic values (limits along curves tending to ∞ on the surface) then these triangles are compact, and we get a standard triangulation. More concisely:

Conjecture 4. *Every open Riemann surface has an equilateral triangulation.*

Using QC-folding one can construct quasi-regular analogs of Belyi functions without asymptotic values on any Riemann surface. However, using the measurable Riemann mapping theorem to make this function holomorphic may change the conformal structure of the surface, and it is not obvious whether we can define the desired holomorphic function on the same Riemann surface that we started with. This is not a problem for QC-folding on the plane or sphere, because there is only one conformal plane and one conformal sphere. However, for general compact surfaces, Belyi's theorem [13] implies that X has a Belyi function (= has an equilateral triangulation) iff it is an algebraic curve. There are only countably many such curves, so not all compact Riemann surfaces have a Belyi function. Hence not all conformal structures on a given compact surface can be attained, so the potential problem described above actually occurs.

For non-compact surfaces however, it should be possible to alternate applications of the folding construction on compact sub-surfaces (which we can choose to alter the conformal structure only slightly), with conformal correction maps that "push" the perturbed subsurface back into X. For example, the results of [51] should be helpful here. I am pursuing this approach with Lasse Rempe-Gillen.

If we don't allow a Belyi function to have any poles, then we get a Shabat function: a $f : X \to \mathbb{C}$ so that f is holomorphic on X, branched only over 0 and 1, and has no removable singularities in ∂X.

Question 5. *Do all planar domains have a Shabat function? All open Riemann surfaces?*

In this case, the pre-images of the upper and lower half-planes are non-compact triangles that all have one vertex at ∞ (equivalently, we are decomposing the surface into conformal half-strips). The "fold-and-correct" alternating strategy described above should work here too, but the graphs that arise in the folding steps will be much more intricate.

• **Quantitative Runge's theorem:** Given a finite number of Jordan domains $\{D_j\}_1^N$ with disjoint closures, and complex values $\{c_j\}_1^N$, Runge's theorem provides a sequence of polynomials $\{p_n\}$ that converges to c_j uniformly on D_j, but it gives no control of these polynomials off $D = \overline{\cup D_j}$. By slightly enlarging each domain and connecting the new domain boundaries to each other and to ∞, we can construct a graph G to which quasiconformal folding can be applied. The resulting function f can be chosen to be within ϵ of c_j on D_j and off D, and $|f(z)|$ is controlled in terms of ϵ and the geometry of G. A similar folding construction gives a meromorphic approximation with prescribed poles. Can we formulate a general version that can be applied without constructing the graph by hand each time?

Problem 6. *Prove a quantitative version of Runge's theorem derived from QC-folding. i.e., given a holomorphic function on a neighborhood of a compact set X, give explicit estimates on the approximating entire (or meromorphic) function in terms of the geometry of X, the behavior of f and the desired degree of approximation.*

What about QC-folding versions of Mergelyan's or Arakelyan's approximation theorems?

• **Holomorphic immersions and embeddings:** One of the most famous and well studied problems in complex analysis is the Bell-Narasimhan Conjecture (e.g., page 20, [12]):

Conjecture 7. *Every open Riemann surface has a proper holomorphic embedding in* $\mathbb{C}^2$.

This has been intensely studied and is now known for many examples, e.g., for all circular domains without punctures [43]. It is closely connected to many other problems involving complete or proper embedding or immersions of surfaces in $\mathbb{C}^2$ or $\mathbb{C}^3$ and to problems of constructing minimal surfaces in $\mathbb{R}^3$. It is known that every Riemann surface embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$. The problem with embedding into $\mathbb{C}^2$ is that self-intersections can't always be removed by local changes (they are stable under perturbations).

One tool that has been used to construct examples is Runge's theorem. In [63] it is used to construct functions on a bordered Riemann surface whose derivatives are large on a complicated subset X (a "labyrinth") of interlocking pieces near the boundary. A curve tending to the boundary either must avoid this set (in which case it has long length) or it contains points where the derivative is large; in either case the image can be proven to be very long, and hence the image surface is complete. However, in this construction some regions of X where the Runge approximation might be "too large" must be removed from the surface, changing the conformal type in an unknown way. This is very similar to the problem arising when building Belyi functions on all open surfaces. Perhaps the "conformal correction" methods considered there could be useful here, or the quantitative Runge's theorem described above could be applied. As a first step, it would be interesting to see if known special cases of the Bell-Narasimhan conjecture can be re-proven using QC folding, e.g., results from [2], [3], [48]. I thank Graham Smith for suggesting that QC-folding might be related to minimal surfaces and for pointing me to the fascinating work in the area.

2. Transcendental Dynamics

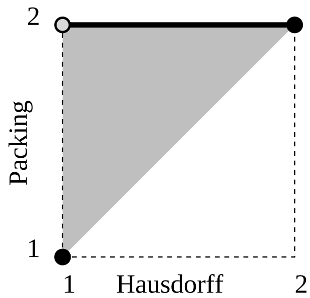
Transcendental dynamics refers to the iteration theory of non-polynomial entire functions. We let $\mathcal{T}$ denote this class of functions. As usual, the Fatou set $\mathcal{F}$ is the maximal open set where the iterates of f form a normal family and the Julia set $\mathcal{J}$ is its complement (and is always non-empty). While similar to polynomial dynamics in many respects, there are several significant differences: wandering domains can exist, Fatou components of any finite or infinite multiplicity may occur, the escaping set $I(f) = \{z : f(z) \to \infty\}$ plays a more prominent role (and has interesting subsets based on rates of escape), the Julia set always contains a non-trivial continuum, and it is generally harder to build "small" Julia sets than "large" ones, in the sense of fractal dimensions.

Recall that Hausdorff, upper Minkowski and packing dimension are defined as

$$\mathrm{Hdim}(K) = \inf\{s : \inf\{\textstyle\sum_j r_j^s : K \subset \cup_j D(x_j, r_j)\} = 0\},$$
$$\overline{\mathrm{Mdim}}(K) = \inf\{s : \limsup_{r\to 0} \inf_N N r^s = 0 : K \subset \cup_{j=1}^N D(x_j, r)\},$$
$$\mathrm{Pdim}(K) = \inf\{s : K \subset \cup_{j=1}^\infty K_j : \overline{\mathrm{Mdim}}(K_j) \le s \text{ for all } j\},$$

and that Hdim $\le$ Pdim. For transcendental Julia sets, these can differ.

• **Dimensions of Julia sets:** I.N. Baker [5] proved in 1975 that a transcendental Julia set must contain a non-trivial continuum and hence has Hausdorff dimension ≥ 1. However, the first example attaining this minimum is in my 2018 paper [28]. It also has packing dimension 1, the first transcendental example with Pdim < 2. The gray triangle below shows the possible pairs $1 \le$ Hdim $\le$ Pdim ≤ 2 for a transcendental Julia set and black denotes all known examples: the vertex $(2, 2)$ is due to Misiurewicz [66] (see also McMullen [64]); the top edge $(t, 2)$, $1 < t < 2$ is due to Stallard [82], [83]; $(1, 1)$ is my example.



Question 8. *For each $1 < s < t < 2$ is there a $f \in \mathcal{T}$ with $\mathrm{Hdim}(\mathcal{J}) = s$ and $\mathrm{Pdim}(\mathcal{J}) = t$? (gray triangle)*

Question 9. *Is there a transcendental Julia set with $\mathrm{Hdim}(\mathcal{J}) = 1, \mathrm{Pdim}(\mathcal{J}) = 2$? (upper left corner)*

Question 10. *For $t \in (1, 2)$, is there a $f \in \mathcal{T}$ with $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Pdim}(\mathcal{J}(f)) = t$? (diagonal edge)*

A current student of mine, Jack Burkart, is working on answering Question 10 in the affirmative for some value of $1 < t < 2$. His argument should give a dense set of such t's and then continuous deformations might allow us to deduce all values $1 < t < 2$ occur.

Although my example has finite 1-dimensional spherical measure, it has infinite 1-dimensional packing measure and it does not lie on any rectifiable curve.

Question 11. *Can a transcendental Julia set lie on a rectifiable curve on the sphere?*

The Julia set of $\tan(z)$ is $\mathbb{R}$, so this can occur for meromorphic functions. Question 11 seems very delicate, and I have ideas for both constructing such an example and for proving it can't exist. The Fatou components in my example are infinitely connected, which leads to the infinite packing measure and the impossibility of connecting the Julia set by a finite length curve. Kisaka and Shishikura [54] have constructed examples in $\mathcal{T}$ with annular Fatou components, and I believe similar examples can also be constructed by QC-folding. Can we combine all these ideas (perhaps with characterizations of rectifiability from [4], [49]) to produce a positive answer to Question 11?

• **Dynamics in the Speiser class $\mathcal{S}$:** As mentioned earlier, the Speiser class consists of transcendental functions that have a finite number of singular values. Each such function has a finite dimensional family of quasiconformal deformations $M_f = \{g = \psi \circ f \circ \varphi\}$ such that g is entire and and ψ, φ are both QC. Hence Speiser functions are similar to polynomials in certain ways (e.g., no wandering domains). Simon Albrecht and I [29] constructed a sequence in $\mathcal{S}$ with $\mathrm{Hdim}(\mathcal{J}) \searrow 1$, but the following is still open:

Question 12. *Is $\{\mathrm{Hdim}(\mathcal{J}(f)) : f \in \mathcal{S}\} = (1, 2]$?*

By results of Rippon and Stallard [75], the Hausdorff dimension of such examples is > 1 and the packing dimension equals 2. The Hausdorff dimension of the Julia set changes continuously over M_f, so Question 12 would follow from:

Conjecture 13. *If $f \in \mathcal{S}$, then $\sup\{\mathrm{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 2$.*

This is an analog of Shishikura's result [80] about dimensions of quadratic Julia sets tending to 2 near generic points in the boundary of the Mandelbrot set (also to the fact that Kleinian limit sets have dimension tending to 2 near most boundary points of Teichmüller space [32]). Possibly Shishikura's proof can be adapted to this case. Conjecture 13 is due to Lasse Rempe-Gillen. In the other direction,

Question 14. *Is there an $f \in \mathcal{S}$ with $\inf\{\mathrm{Hdim}(\mathcal{J}(g)) : g \in M_f\} = 1$?*

Both the above imply $\dim(\mathcal{J})$ can be non-constant on M_f, but even this is unknown:

Question 15. *If $f \in \mathcal{S}$, $g \in M_f$ is $\mathrm{Hdim}(\mathcal{J}(f)) = \mathrm{Hdim}(\mathcal{J}(g))$?*

Before my examples with Albrecht, every known Speiser Julia set had Hausdorff dimension 2, so our examples are the first where this question can even be tested. Recall from above that the escaping set is defined as $I(f) = \{z : |f^n(z)| \to \infty\}$.

Question 16. *Is there an* $f \in \mathcal{S}$ *with* $\dim(I(f)) = 1$*?*

Rempe-Gillen and Stallard [72] gave such an example in the Eremenko-Lyubich class $\mathcal{B}$. The answer would be yes in the Speiser class, if Question 14 could be solved by considering only affine equivalents $af(cz + d) + b$ of a single function $f \in \mathcal{S}$, since Rempe-Gillen has shown $\mathrm{Hdim}(I)$ is invariant under such deformations and it is always bounded above by $\mathrm{Hdim}(\mathcal{J})$. I have some tentative sketches of what the corresponding tree for a folding construction of such a function might look like, but many details remain to be verified.

3. 4-manifolds and critical points of harmonic measure

As mentioned in the summary of previous work, the almost-Kähler metrics on a 4-manifolds always form an open subset of the moduli space of all anti-self-dual metrics, but Claude Lebrun and I have constructed, for the first time, examples where this subset is not closed. (For the precise definitions of almost-Kähler and anti-self-dual, see [33], but these definitions are not necessary for understanding the problems stated below.)

The simplest of many possible examples occurs when Σ is a compact surface and M is a geometrically finite hyperbolic 3-manifold homeomorphic to $\Sigma \times \mathbb{R}$. There is a unique harmonic function u on M that tends to 1 in one infinite end of M and tends to 0 in the other end (this is the harmonic measure of one end of the manifold). The 4-manifold N can be thought of as M times a circle, with the two infinite ends each collapsed to points. LeBrun had shown that N has an almost-Kähler metric iff u has no critical points in M.

Given Σ, the space of associated M's is parameterized by pairs of conformal structures on Σ; when the two structures are the same, then M is represented by a Fuchsian group G acting on the hyperbolic upper half-space $\mathbb{R}^3_+$ and the limit set $\Lambda \subset \mathbb{R}^2$ is a circle. In this case, the harmonic measure function u has no critical points. As we deform one of the structures, the Fuchsian group becomes quasi-Fuchsian and Λ becomes a fractal quasi-circle with complementary components Ω_0, Ω_1. Then u is the quotient of the harmonic measure function $\omega(z, \Omega_1, \mathbb{R}^3_+)$ (the harmonic function on $\mathbb{R}^3_+$ that has boundary values 0 on Ω_0 and 1 on Ω_1). We show that if Σ has high enough genus, then we can always deform G so that Λ approximates a certain explicit "dogbone" curve, and this implies the harmonic measure function has a critical point in $\mathbb{R}^3_+$.

Question 17. *How large a genus is needed? Can we check small examples computationally?*

Question 18. *We actually prove* u *has at least two critical points. Does it have exactly two? What other possibilities are there?*

Question 19. *For surfaces* Σ *where critical points occur, how common are they? Does the set where they occur have infinite volume in Teichmüller space? Is this set dense on the boundary of Teichmüller space? What about the set where critical points do not occur?*

Question 20. *Are there simple necessary or sufficient geometric conditions on a closed Jordan curve which imply the corresponding harmonic measure function has a critical point?*

4. Optimal triangulations and conformal geometry

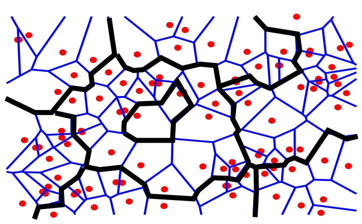
• **Planar meshes:** Earlier we discussed decomposing a Riemann surface into conformal images of equilateral triangles. In computational geometry and meshing for numerical PDE there is also a need to decompose polygonal domains into actual triangles that are as close to equilateral as possible, while keeping the triangulation as simple as possible, e.g., using only a polynomial number of elements (as a function of the number n of boundary segments). Easy examples show that polynomial complexity rules out any uniform lower bound on angles (consider a long narrow rectangle), and hence any upper bound that is less than 90° (since the angles sum to 180°, an upper bound $< 90^\circ$ implies a strictly positive lower bound). Polynomial algorithms giving 90° for simply polygons and larger bounds for PSLGs (planar straight line graphs) were found in the 1990's (see [16], [67], [86]), but the sharp result for PSLGs is more recent: In [25] I proved the existence of polynomial sized, conforming non-obtuse triangulations (NOTs; all angles $\leq 90^\circ$) for any PSLG. If the PSLG has n vertices, my construction gives $O(n^{2.5})$ elements; the best known lower bound is $\simeq n^2$, so a gap exists:

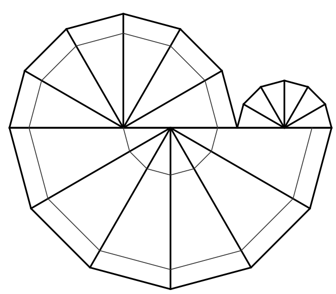
Conjecture 21. *Every PSLG has a conforming NOT with $O(n^2)$ elements.*

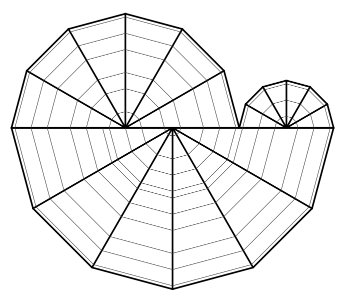
Conjecture 22. *Every PSLG has an $O(n^2)$ conforming Delaunay triangulation.*

Conjecture 23. *Every PSLG has an $O(n^2)$ conforming Voronoi diagram.*

A Delaunay triangulation is defined by the property that any pair of triangles sharing an edge having opposite angles summing to $\leq \pi$. Given a point set V, the corresponding Voronoi digram is the collection of points that are nearest to two or more different points. Conforming means that the edges of the triangulation or diagram covers the edges of the given PSLG, e.g., see the figure at left below. It is obvious that a NOT is also Delaunay, and it is easy to build a conforming Voronoi diagram for a NOT by placing six points in each triangle in a certain way. Thus the first conjecture above implies the second two, but I have never found any approach for the latter cases that simplified the proof, or gave a better estimate, than in the NOT problem. Perhaps all three problems are equivalent to each other. Can we prove they have the same complexity (even if we can't determine exactly what that complexity is)?







Given a PSLG, the NOT algorithm first adds edges to form a dissection by isosceles triangles with good angles (a dissection is like a mesh, except that edges may overlap without being equal; thus some "bad" points are vertices of some triangles but interior to edges of others). Next, the algorithm converts the dissection into a mesh by propagating the "bad" points along paths parallel to the bases of the triangles. An example is illustrated on the right above. If the propagation paths terminate (by leaving the dissection or hitting another vertex), they cut the dissection into a mesh using triangles and quadrilaterals (and the latter can be made into triangles by adding diagonals). However, as shown above, a propagation path may cross the same triangle repeatedly. In order to get the uniform complexity bound, the algorithm bends paths to terminate them faster; to get the desired angle bounds, the amount of bending is limited by constraints that closely resemble keeping a discrete second derivative bounded. This makes the proof of the NOT theorem reminiscent of Pugh's closing lemma: every C^1 vector field has a C^1 perturbation with a closed orbit [69],[70], [71]; this is open for C^2 vector fields and perturbations. Dennis Sullivan asked to make this precise:

Question 24. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a C^2-closing lemma (or suggest a counterexample)?*

The $O(n^2)$ upper bound in Conjecture 21 is a worst case estimate, so most PSLGs with n vertices might have much smaller NOTs. It would be very interesting (and important for applications) to know if the algorithm can do better in better cases:

Problem 25. *Find an algorithm that, given a PSLG Γ, produces a $O(N)$ sized conforming NOT, where N is the size of a minimal conforming NOT for that Γ.*

• **Dimension 3:** Solving the above problems would mostly complete the theory of optimal triangulation in $\mathbb{R}^2$, but the corresponding theory using tetrahedra in $\mathbb{R}^3$ (the really important case for applications) is wide open; there are many examples, heuristics and implementations, but few rigorous results. The main open question in the field is:

Question 26. *Do polyhedra in $\mathbb{R}^3$ have non-obtuse tetrahedralizations of polynomial size?*

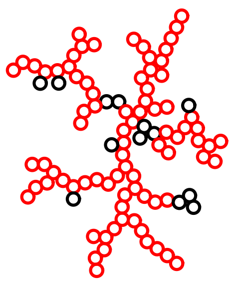
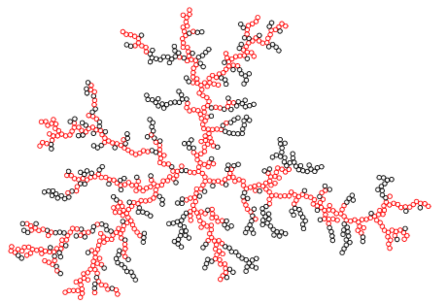
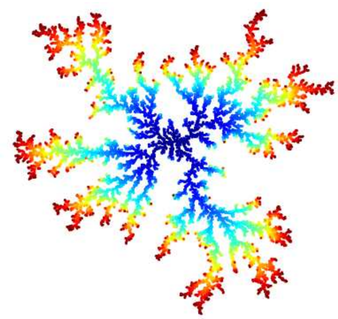
Does this hold for any dihedral angle bound $\theta < 180°$? Even finding an acute tetrahedralization (all angles $< 90°$) of a cube in $\mathbb{R}^3$ was open until recently (the smallest known example uses 1,370 pieces [89]) and there is no acute decomposition for the cube in $\mathbb{R}^4$, [58]. The breakthrough in 2-dimensional meshing was to introduce the idea of a thick/thin decomposition of a polygon that is analogous to the thick/thin decomposition of a Riemann surface; in the thin parts of the polygon, Euclidean geometry is used to create the mesh and in the thick parts hyperbolic geometry is used. Fast, approximate conformal mapping gives the decomposition into thick and thin parts, and the use of the two alternate geometries gives the optimal angles bounds. Can we use analogous ideas in $\mathbb{R}^3$? Can one create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in Perelman's proof of Thurston's geometrization conjecture) to decompose it into pieces and then utilize the "natural" geometries on the different pieces to define meshes? Any progress would have a significant impact on many problems of practical interest.

An intermediate problem between 2 and 3 dimensions is to find NOTs for triangulated surfaces in $\mathbb{R}^3$. The proof of the NOT theorem uses properties of planar geometry that may not hold on a surface with curvature (angles at a vertex not summing to 2π). Is there always a polynomial bound for a non-obtuse refinement of a triangulated surface? What if we assume the surface is the boundary of a convex polyhedron? This case is particularly interesting for many reasons, e.g., the paper of O'Rourke [68] on a special case of Dürer's Problem: does every convex polyhedron in $\mathbb{R}^3$ has a spanning tree of its edges, so that cutting along these edges gives a non-overlapping unfolding in the plane; this famous problem is still open after 500 years. O'Rourke's paper cites the NOT theorem, because his method becomes easier if the faces of the polyhedron are non-obtuse triangles.

5. Random trees and harmonic measure

Earlier we saw true trees, whose edges all have equal harmonic measure from ∞. Next we consider trees with equal length edges that are grown using harmonic measure from ∞.

DLA (diffusion limited aggregation) is defined by fixing a unit disk at the origin and sending in a second unit disk moving by Brownian motion from infinity until it touches the first disk. Successive disks are added in the same way. The main problem is to determine the almost sure growth rate $\alpha = \limsup_n \frac{1}{n} \log \mathrm{diam}(\mathrm{DLA}(n))$. Some DLA clusters with $n = 100, 1000, 10000$ are shown below. The last one is colored according to when the disk was added; the colors on the first two will be explained below.

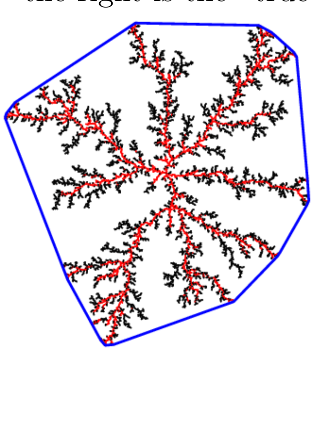
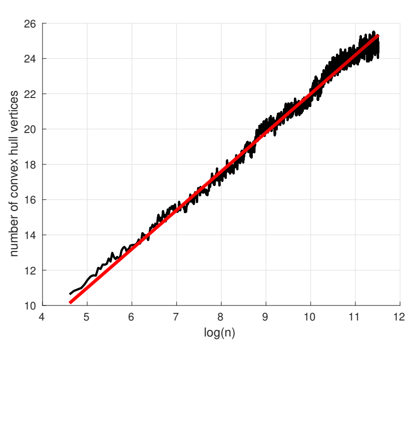




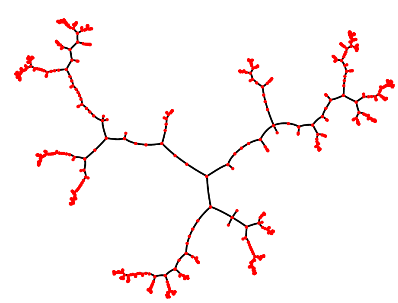
Obviously $\mathrm{diam}(\mathrm{DLA}(n)) \leq 2n$, but Harry Kesten [52] improved this to $O(n^{2/3})$ almost surely; this remains the best known upper bound even 30 years later. The trivial lower bound is $\gtrsim \sqrt{n}$ (consider the areas), and shockingly, this is still the best known:

Conjecture 27. $\lim_n \mathrm{diam}(\mathrm{DLA}(n))/\sqrt{n} = \infty$ *almost surely.*

Consider the convex hull of the disk centers of a DLA cluster. If there is a convex hull vertex where the exterior angle measure is $\theta = \pi/2\lambda > \pi$, then conformal mapping estimates show that the probability of the next disk being added at this vertex is at least $\simeq n^{-\alpha\lambda}$. On the other hand, if the DLA diameter grows like n^{α}, then the convex hull perimeter should increase by 1 about every $n^{1-\alpha}$ steps, leading us to the equation $\alpha\lambda = 1-\alpha$ or $\alpha = 1/(1+\lambda)$. Since $\frac{1}{2} \leq \lambda \leq 1$, we have $\frac{1}{2} \leq \alpha \leq \frac{2}{3}$; the trivial lower bound and Kesten's upper bound.

One way for the convex hull to have (relatively) sharp angles is for it to have few vertices. In the pictures above, a disk is red if its center became a convex hull vertex when it was added; note that these form a large fraction of the small clusters (a fraction that should tend to zero with n, but how quickly?). See below for a plot of a DLA cluster and its convex hull. Also shown is a plot of the number of vertices in the convex hull as a function on $\log n$ (averaged over 100 random trials). The plot clearly looks linear as a function of $\log n$. On the right is the "true tree" version of a DLA with a 1000 vertices (see Question 30).



Question 28. *Is the number of convex hull vertices $O(\log n)$ almost surely? If this is true, can we deduce Conjecture 27? A growth rate $\alpha > 1/2$?*

Experiments indicate the answers to the questions above are yes, at least for small values of n (less than a million). Running larger experiments poses an interesting computational question of its own. The naive method of drawing a DLA cluster takes time $n^2 \log n$. We simulate Brownian motion by a random walk that jumps half the distance to the cluster at each step; this takes an expected $O(\log n)$ steps to approximate the hitting point of the new disk. However, we have to compute the distance to the cluster; this involves finding the minimum of n distances to individual disks, giving quadratic growth in work. Nearest-neighbor search is a well studied problem in computational geometry: given a set $S \subset \mathbb{R}^n$ and point $x \in \mathbb{R}^n$ find the closest point of S to x (or accurately estimate the distance from x to S, which is what we want for the DLA application). This is generally done by pre-processing the set S using dyadic grids, and allows us to find the closest point (and update the data structure) with logarithmic work.

Question 29. *Can we model a DLA cluster of size n in $O(n \log^2 n)$ time?*

Finally, we return to beginning of the proposal; Question 1 asked for practical criteria for deciding if an infinite planar tree has a true form in the plane. We can form an infinite tree from DLA by joining the centers of adjacent disks in a DLA cluster.

Question 30. *Does an infinite DLA tree have a true form in the plane, almost surely?*

Probably this is not true; the figure of the "true DLA tree" above shows the outer edges are much smaller than the inner ones; this tends to support the idea that the limiting tree naturally lives in the hyperbolic disk, rather than the Euclidean plane. How to prove this is correct seems very far from obvious.

BROADER IMPACTS OF THE PROPOSAL

Enhancing computational infrastructure: All of the broader impacts discussed in the summary of previous work (enhanced computational infrastructure, encouraging interdisciplinary research and enhancing STEM education) also apply to the current proposal. Solutions to the 2 and 3 dimensional meshing problems could have a dramatic impact on various aspects of modeling surfaces and 3-dimensional bodies, which in turn have numerous implications for computing in research and manufacturing. The increasing use of finite element methods increases the incentive to improve automatic meshing algorithms. However, many known algorithms can create distorted and even unusable grid elements, so automatic meshing methods with geometric guarantees are essential. As well as improving known theoretical results, I will work to implement my previous algorithms to demonstrate their utility and make them more accessible to potential users.

Educational impact: In the summary of previous work, I described courses, lecture notes, graduate and undergraduate projects related to my work, and the current proposal has similar impacts on the infrastructure of research and education. In particular, I plan to follow-up the 2017 graduate workshop on computational and random geometry with another next year. Graduate students working on problems related to this proposal receive training in aspects of both pure and applied mathematics, participate in seminars in both departments, and become more open to such collaborations; this improves the likelihood they will participate in interdisciplinary and academic/industrial collaborations and improves their ability to motivate and train their own students in the future.

Moreover, the geometric and interdisciplinary nature of the problems in this proposal suggest numerous projects that are accessible and appealing to undergraduates or even high school students; such problems can motivate them to the further study of mathematics, or at least give then a greater appreciation for the potential of mathematics in their own field. Currently I am department coordinator for matching high school students seeking research mentors with faculty members. I have also helped organize Stony Brook's annual Math Day for undergraduates and am involved in starting a geometry lab for undergraduate research here. A few examples of problems related to the proposal that might be suitable for undergraduates to work on include:

• **Drawing a transcendental Julia:** Even though my "dim =1" example is given by an explicit infinite product, it is difficult to get an accurate computer picture because we lack a simple test for being in the Julia or Fatou sets.

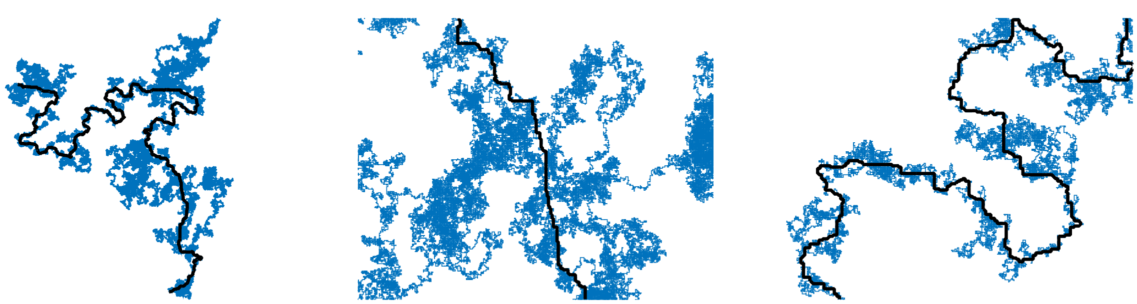
• **Approximate the true form of a planar tree**. Use Don Marshall's `ZIPPER` program to explore how combinatorics influences geometry of true trees. A former undergraduate, Joe Suk, has worked on this, but much more remains to do. For example, compute the true forms of truncations of various infinite trees, as we did for the 3-regular tree. Do the leaves limit on fractal curves that we can recognize from other settings?

• **Implement the NOT algorithm.** Implement the $O(n^{2.5})$ NOT algorithm and its variations for conforming Delaunay triangulations and Voronoi coverings. Study the "triangle flow" for triangulated surfaces, e.g., the Platonic solids.

• **Critical points of harmonic measure:** Compute limit sets of "small" quasi-Fuchsian groups and look for a critical point of the harmonic measure function of one side. This could give a "small" example of the 4-manifold as described in the proposal.

• **Faster DLA:** Speed up DLA experiments suggested in the proposal using nearest neighbor search structures from computational geometry. Test Question 28 at larger scales and formulate measures of how the convex hull deviates from being "round".

• **Brownian geometry:** It is conjectured that the Brownian trace (the set visited by planar Brownian motion during time $[0, 1]$) contains no rectifiable curves, or even curves of Hausdorff dimension 1. A former undergraduate, Shalin Parekh, did experiments using random walks on a square grid that showed the minimal length path connecting widely separated points scaled like dimension ≈ 1.02. See figures below. However, his experiments were fairly small (walks with around 10^5 steps); make his code more efficient and repeat these experiments for longer walks. It would also be interesting to investigate the adjacency graph of the complementary components. Are there any interesting graph theoretic properties? How does the graph diameter grow? It is conjectured by Wendelin Werner that in the limiting case, any two complementary components of the Brownian trace can be connected by a finite path of touching components. What is the correct discrete formulation of this conjecture?



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