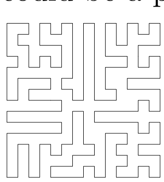
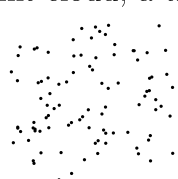
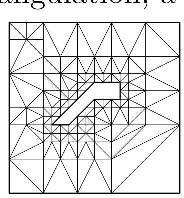
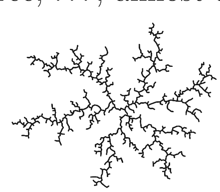
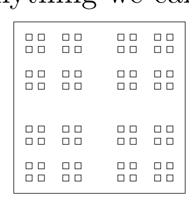
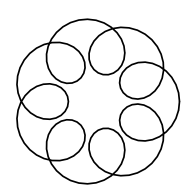
RESULTS FROM PRIOR NSF SUPPORT
Analysis of conformal and quasiconformal maps
DMS 1006309, 6/15/10 - 5/31/13, $200,441

This section gives brief statements of the main results obtained; a more detailed discussion with background will be given in the project description. Preprints giving complete statements and proofs are available at: `www.math.sunysb.edu/~bishop/papers`.

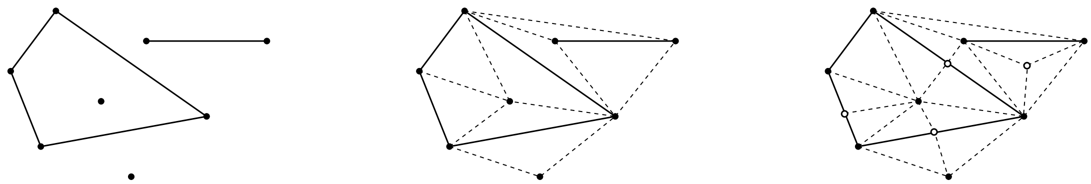
Let $\mathcal{E}$ denote the class of entire functions and $\mathcal{T} \subset \mathcal{E}$ be the transcendental functions (non-polynomials). For $f \in \mathcal{E}$, the singular set $S(f)$ is the closure of its critical values and finite asymptotic values. The Speiser class $\mathcal{S} \subset \mathcal{T}$ denotes functions such that $S(f)$ is finite, and the Eremenko-Lyubich class $\mathcal{B}$ are those where $S(f)$ is bounded. Let $\mathcal{S}^* \subset \mathcal{S}$ be the functions with no finite asymptotic values and the two critical values ± 1. Polynomials with $S(p) = \{\pm 1\}$ are called generalized Chebyshev or Shabat polynomials.

- **Critical points of Shabat polynomials** [18]**:** I show any compact, connected set $K \subset \mathbb{R}^2$ can be approximated in the Hausdorff metric by the critical points of a Shabat polynomial p. $T = p^{-1}([-1,1])$ is a finite tree, the "true form" of one of Grothendieck's *dessins d'enfants*, and this result shows such true forms are dense in all planar continua.
- **The order conjecture** [23]**:** I disprove Adam Epstein's conjecture that two entire functions $f, g \in \mathcal{S}$ that are quasiconformally equivalent (i.e., $\psi \circ f = g \circ \phi$ for some QC maps ψ, ϕ) must have the same order of growth. f has 3 critical values, which is sharp.
- **The area conjecture** [20]**:** I disprove the conjecture of Eremenko and Lyubich [49] that for $f \in \mathcal{S}$, $\int_{f^{-1}(K)} (1+|z|)^{-2} dxdy < \infty$ (finite logarithmic area) whenever K is a compact set disjoint from $S(f)$. In fact, area$(\{z : |f(z)| > \epsilon\}) < \infty$ for all $\epsilon > 0$.
- **A wandering domain in $\mathcal{B}$** [20]**:** I construct $f \in \mathcal{B}$ whose Fatou set has a wandering domain. In 1985 Eremenko-Lyubich [49] and Goldberg-Keen [54] extended Sullivan's "no wandering domains" theorem to $\mathcal{S}$, but situation for $\mathcal{B}$ had remained open until now.
- **The strong Eremenko conjecture fails in $\mathcal{S}$** [20]**:** I build a $f \in \mathcal{S}$ so that the escaping set has no non-trivial path components, by adapting the argument in [78] for $\mathcal{B}$.
- **A Julia set of dimension** 1 [24]**:** I construct $f \in \mathcal{T}$ whose Julia set $\mathcal{J}(f)$ has Hausdorff and packing dimension 1 (and locally finite 1-measure). This has been an open problem since 1975 when Noel Baker [5] proved that $\dim(\mathcal{J}(f)) \geq 1$ for all $f \in \mathcal{T}$.

A planar straight line graph (PSLG) is any finite union of segments and points; a polygon is a special case when the segments meet end-to-end. Besides a polygon, a PSLG could be a point cloud, a triangulation, a tree, . . . ; almost anything we can draw:

A conforming mesh for a PSLG fills the convex hull and has edges and vertices that cover the whole PSLG. Below is a PSLG and a triangulation; we can add extra vertices (Steiner points) to improve the shape of the triangles. Obtaining upper angle bounds is important in many applications, e.g., the finite element method (see broader impact section).



- **Acute triangulation of PSLGs** [21]**:** I show any PSLG with n vertices has a $O(n^{2.5})$ acute triangulation (all angles $< 90°$). Giving any polynomial bound was a long standing open problem in computational geometry (CG). n^2 is the best known lower bound.
- **Delaunay triangulations** [21]**:** I also give a $O(n^{2.5})$ bound for conforming Delaunay triangulations of PSLGs, improving a $O(n^3)$ bound of Eldesbrunner and Tan [43] from 1993 (breaking the n^3 barrier was also a well known open problem in CG).
- **Refining triangulations** [21]**:** Any triangulation of a simple n-gon has a $O(n^2)$ non-obtuse refinement (angles $\leq 90°$), improving an $O(n^4)$ bound of Bern and Eppstein [12].
- **Almost non-obtuse triangulations** [21]**:** I prove every PSLG has a conforming triangulation with $O(n^2/\epsilon^2)$ elements and all angles $\leq 90° + \epsilon$. This improves results of S. Mitchell [69] when $\epsilon = \frac{3}{8}\pi = 67.5°$ and Tan [88] when $\epsilon = \frac{7}{30}\pi = 42°$.
- **Optimal quadrilateral meshes for polygons** [31]**:** I show any polygon has a $O(n)$ quadrilateral mesh with all angles $\leq 120°$ and all new angles $\geq 60°$. Both angle and time bounds are sharp. The upper angle bound is due to Bern and Eppstein [13].
- **Optimal quadrilateral meshes for PSLGs** [22]**:** I prove any PSLG has a conforming $O(n^2)$ quadrilateral mesh with all angles $\leq 120°$ and all new angles $\geq 60°$. This is optimal for both angle bounds and complexity and is the first polynomial time algorithm for quad-meshing PSLGs that gives both upper and lower angle bounds.
- **QC dimension distortion** [33]**:** Hrant Hakobyan and I show that if E has dimension d and $f : \mathbb{R}^2 \to \mathbb{R}^2$ is QC, then at least one component of $f(E \times [0,1])$ has dimension $\leq 2/(d+1)$ and we build examples to show this is sharp. We also show there is a 1-dimensional set $E \subset \mathbb{R}$ and a QC map f so that $f(E \times [0,1])$ contains no rectifiable sub-arcs. No uncountable example was previously known.
- **New proof of Jones' TST** [34]**:** I gave a new proof of Peter Jones' traveling salesman theorem in $\mathbb{R}^2$ characterizing subsets of rectifiable curves by using Crofton's formula (this computes the length of a set E from the number of intersections of E with random lines). Jones' β-numbers are interpreted as square roots of probabilities that a random line hits the set at two well separated points.
- **Conformal mapping in linear time** [30]**:** I prove that if Ω is a simply connected n-gon, then in time $O(n \cdot p \log p)$ we can construct a $(1+\epsilon)$-quasiconformal map $f : \mathbb{D} = \{|z| < 1\} \to \Omega$, where $p = |\log \epsilon|$. This is the asymptotically fastest, globally convergent conformal mapping method known, and is based on ideas from 3-dimensional hyperbolic geometry, quasiconformal mappings and 2-dimensional computational geometry.
- **The Ahlfors iteration** [55]**:** My student, Chris Green, and I proved local quadratic convergence of a numerical conformal mapping method that uses an approximate solution of the Beltrami equation due to Ahlfors. This is the asymptotically fastest method that is both provably convergent and simply enough to implement.

PROJECT DESCRIPTION

We start with constructing holomorphic functions from combinatorial data, apply this to the dynamics of entire functions, then move to interactions between geometric function theory and computational geometry and finally to distortion properties of QC maps and connections to rectifiability, traveling salesman theorems and minimal triangulations.

1. Quasiconformal combinatorics

Quasiconformal (QC) homeomorphisms generalize conformal maps mapping infinitesimal circles to ellipses of bounded eccentricity (usually denoted K); if the map is not 1-to-1, this condition defines a quasiregular (QR) function. K-QC maps satisfy $|\mu_f| = |f_{\overline{z}}/f_z| \leq k < 1$ where $k = (K-1)/(K+1)$ and the measurable Riemann mapping theorem (MRMT) says that every such dilatation μ corresponds to some QC map f. This implies that if g is quasiregular, then there is a QC map ϕ so that $f = g \circ \phi$ is holomorphic; many of our examples are built using this idea.

• **Drawing dessins:** Call a finite tree $T \subset \mathbb{R}^2$ "conformally balanced" if every edge has equal harmonic measure from ∞ and every Borel subset of every edge e has equal measure from either side of e. (Harmonic measure is the first hitting probability of Brownian motion started at ∞.) This might seem impossible to attain, but I prove in [18] that such trees approximate any compact, connected $K \subset \mathbb{R}^2$ in the Hausdorff metric, answering a question of Alex Eremenko (the proof uses a quasiregular construction "fixed" by MRMT as described above). Can we compute such trees explicitly?

If p is a Shabat polynomial (i.e., critical values $= \{\pm 1\}$) then $T = p^{-1}([-1, 1])$ is such a tree and every finite, planar tree occurs (up to homeomorphisms of $\mathbb{R}^2$). This is a special case of Belyi's theorem giving a correspondence between Grothendieck's *dessins d'enfants* (finite graphs drawn on compact surfaces) and Belyi functions (meromorphic functions ramified over $\pm 1, \infty$). In this literature a conformally balanced tree is called the "true form" of the tree. The Shabat polynomials for all trees with ≤ 9 edges are computed in [62], where the author states "The complete study of trees with 10 edges is a difficult work, and probably no one will do it in the foreseeable future".

Problem 1. *Efficiently approximate the true form of a given planar tree.*

This is a conformal mapping and welding problem; Don Marshall has used his `zipper` conformal mapping program to compute true forms of trees with Steffen Rohde in connection with certain random metrics on the 2-sphere known as "quantum gravity" (personal communication). Perhaps my fast conformal mapping algorithms from [30], [55] or the welding methods in [29] could also be used, or the rational approximation ideas in [83].

• **Quasi-trees:** A decomposition $\mathcal{C}$ of a set X is collection of disjoint closed sets whose union is X. If f is a conformal map from $\mathbb{D}^* = \{|z| > 1\}$ to $\Omega = \mathbb{C} \setminus K$ where K is a dendrite (compact, connected and doesn't divide the plane), then $\{f^{-1}(z)\}$, $z \in \partial K$ is a conformal decomposition of $\mathbb{T}$. Characterizing all conformal decompositions is probably extremely difficult, but it may be reasonable to ask

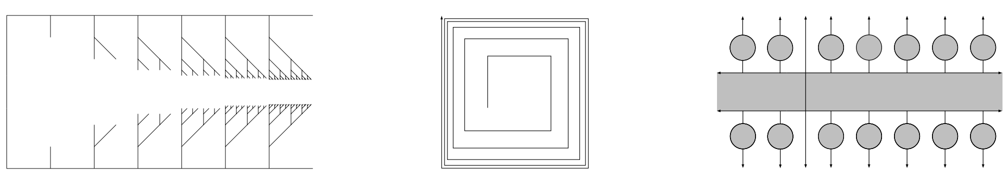
Problem 2. *Characterize decompositions where the $\Omega = f(\mathbb{D}^*)$ is a John domain.*

A domain is John if any two points z, w can be joined an arc $\gamma \subset \Omega$ so $\mathrm{dist}(x, \{z,w\}) = O(\mathrm{dist}(x, \partial\Omega))$ for all $x \in \gamma$. Roughly speaking, John domains generalize quasidisks by allowing non-Jordan boundaries. A known case of the problem is that $K = \partial\Omega$ is a quasiarc iff $\mathcal{C} = \{(x, h(x))\}$ where $h : \mathbb{T} \to \mathbb{T}$ is quasisymmetric (adjacent intervals of equal length map to intervals of comparable length). Another known case is when $\mathbb{T}$ is divided into $2n$ equal intervals and these are identified pairwise by orientation reversing isometries; this corresponds to collapsing the circle to a conformally balanced tree. Problem 1 asks for a way to compute such a collapsing. Problem 2 asks for a reformulation of quasisymmetry from homeomorphisms to decompositions, i.e., a condition that insures the corresponding tree bounds a John domain (with estimates). This is interesting for conformal dynamics, where the decomposition corresponding to a Julia set or Kleinian limit set is often known.

• **Dessins d'adolescents:** For $f \in \mathcal{S}^*$, $T = f^{-1}([-1,1])$ is an unbounded, locally finite tree (we will call these $\mathcal{S}^*$-trees). There is a holomorphic τ from $\Omega = \mathbb{C} \setminus T$ to the right half-plane $\mathbb{H}_r$, that is conformal on each connected component of Ω and satisfies $f = \cosh \circ \tau$ (cosh is the covering map from $\mathbb{H}_r$ to $U = \mathbb{C} \setminus [-1,1]$). Arc-length measure on $\partial\mathbb{H}_r = i\mathbb{R}$ pulls back to a measure (τ-length) on the boundary of each component; every side of T gets τ-length π and every subset gets equal measure from both sides (this is the $\mathcal{S}^*$ version of conformally balanced trees).

Question 3. *Which combinatorial trees occur as $\mathcal{S}^*$-trees? Give one that doesn't.*

Suppose we have a particular embedding of a tree $T \subset \mathbb{R}^2$ and want to build $f \in \mathcal{S}^*$ so $f^{-1}([-1,1])$ approximates T. Set $\Omega = \mathbb{C} \setminus T$ and define $\tau : \Omega \to \mathbb{H}_r$ to be a conformal map from each component, sending ∞ to itself. Then $\cosh \circ \tau$ is holomorphic on Ω, but it is unlikely to be continuous across T. But if T has reasonable geometry (e.g, bounded degree, C^2 arcs as edges, ...) and if all τ-lengths are bounded away from zero, then a method I call "quasiconformal folding" [20] builds a tree T' containing T and a QC map τ from each component of $\Omega' = \mathbb{C} \setminus T'$ to $\mathbb{H}_r$ so that $g = \cosh \circ \tau$ is quasiregular in each component and continuous across T'. By the measurable Riemann mapping theorem there is a QC map ϕ so that $f = g \circ \phi$ is entire. It is easy to check $f \in \mathcal{S}^*$ and $f^{-1}([-1,1]) = \phi^{-1}(T') \approx T$ (often ϕ very close to the identity).



These infinite graphs correspond to functions constructed in [20], [23]; a counterexample to the order conjecture, a function with rapidly spiraling tracts, and a wandering domain in $\mathcal{B}$. They illustrate a key point of QC-folding: to build a function f in $\mathcal{S}$ or $\mathcal{B}$ just draw a tree T and bound (from below) the τ-lengths of it edges (usually easy by standard estimates). It is a simple machine to use (not so simple to build).

• **Folding a pair of pants:** A dynamical application of QC-folding was suggested by Adam Epstein and Lasse Rempe. For a compact surfaces X, suppose $W \subset X$ is connected, open and has no punctures. Epstein [44] defines a finite type map $f : W \to X$

as a holomorphic function such that $S(f)$ is finite, and he showed that many fundamental results of rational dynamics extend to this setting. In [20] I use QC-folding to construct a "pair-of-pants" (a sphere with three disks removed) that has a finite type map into any hyperbolic X (including surfaces that contain W). These are the first "non-trivial" examples of finite type dynamical systems for genus ≥ 2.

Conjecture 4. *Every pair-of-pants has a finite type map into any hyperbolic surface.*

QC-folding gives a finite type quasiregular g on any W, but when we use the measurable Riemann mapping theorem to make it holomorphic, the conformal structure on W changes; does every possible structure occur? An analogous question fails on compact surfaces; on any compact surface there is a quasiregular g ramified over only three points, but correcting the conformal structure only gives countably many distinct Riemann surfaces. This holds since Belyi's theorem [11] says a meromorphic function ramified over three points (a Belyi function) corresponds to a finite graph embedded in the surface and there are only countably many choices for the combinatorics of this graph. However,

Conjecture 5. *Every non-compact Riemann surface has a Belyi function.*

Adam Epstein, Alex Eremenko, Lasse Rempe and I have verified this for all 4-punctured spheres; already a continuum of distinct surfaces. Belyi functions now correspond to uncountably many infinite trees, but some delicate estimates are still needed.

2. Applications to entire functions

• **On trees and growth:** A natural measure of the growth of $f \in \mathcal{E}$ is its order:

$$\rho(f) = \limsup_{z\to\infty} \frac{\log\log|f(z)|}{\log|z|}. \tag{1}$$

The natural parameter spaces of entire functions (at least for dynamics) are the quasiconformally equivalent functions: we say f, F are equivalent (written $f \sim F$) if there are QC maps ϕ, ψ of the plane so that $\psi \circ f = F \circ \phi$. Eremenko and Lyubich [49] proved that if $f \in \mathcal{S}$ has n singular values, then $M(f) = \{F : F \sim f\}$ is a $n+2$ complex dimensional manifold. Adam Epstein asked if the order is constant on each such manifold, but in [23], I build an example where ρ is unbounded on $M(f)$.

Problem 6. *What are the possible ranges of ρ on $M(f)$, $f \in \mathcal{S}$? Is it either a point or unbounded? Can it be all of $(0, \infty)$? Can a minimum or maximum be attained?*

Part of Epstein's motivation in formulating his order conjecture was the even more ambitious question of computing $\rho(f)$ from combinatorial data associated to f. In particular,

Problem 7. *If $f \in \mathcal{S}^*$, compute $\rho(f)$ from the combinatorics of T.*

When $S(f)$ has more than two points, we can connect these points by a finite tree T' and consider $T = f^{-1}(T')$. The counterexample to the order conjecture shows that $\rho(f)$ is not determined by the combinatorics of T, but can we compute the range $\{\rho(g) : g \sim f\}$ from the combinatorics of T? Can we decide if ρ is constant on $M(f)$? What if T is "simple enough", e.g., one infinite ray with bounded size branches attached?

• **Wandering domains that stay close to home:** Just as for polynomials, the Fatou set $\mathcal{F}(f)$ of $f \in \mathcal{T}$ is the maximal open set where the iterates $\{f^n\}$ of f form a normal family; its complement is the Julia set $\mathcal{J}(f)$. One of the most celebrated results in rational dynamics is Dennis Sullivan's "no wandering domains" theorem [87]: every component of the Fatou set is eventually periodic. This was generalized to $\mathcal{S}$ by Eremenko and Lyubich [49], and by Goldberg and Keen [54], but wandering domains can occur for $f \in \mathcal{T}$ (Baker, [6]). All known examples have unbounded orbits, and Eremenko and Lyubich [49] asked

Question 8. *Is there a wandering component for $\mathcal{T}$ that has a bounded orbit? Can a wandering orbit converge to a point?*

One way to answer Question 8 negatively might be find a new proof of Sullivan's theorem where compactness of the orbit replaces the finiteness of singular set; indeed, any alternate proof of Sullivan's theorem would be interesting and helpful. The wandering orbit I construct for $\mathcal{B}$ in [19] uses infinitely many critical points to "compress" the components; hence the orbit must be unbounded. Can we eliminate this? Kisaka and Shishikura [61] constructed a wandering orbit for $f \in \mathcal{T}$ that does not contain any critical points; can their example be replicated in $\mathcal{B}$ using QC-folding? In a different direction, a famous conjecture of Baker asks

Conjecture 9. *If f has order $\rho(f) < 1/2$, then all wandering components are bounded.*

Actually, Baker's original conjecture concerns all Fatou components, but was answered for pre-periodic components by Zheng [94]. Baker's conjecture is open even if $\rho(f) = 0$, although it is known for functions of very slow growth [57], [77] and under various regularity assumptions on the growth (see Hinkkanen's excellent 2008 survey [56]).

• **The most normal entire function:** Baker proved in 1975 that for $f \in \mathcal{T}$, $\mathcal{J}(f)$ contains connected components and hence $\dim(\mathcal{J}) \geq 1$. McMullen [67] gave examples where the Julia set has dimension 2 (even positive area) and Stallard [84], [85] showed that any dimension in $(1, 2]$ could be attained by some $f \in \mathcal{B}$. Is this also true for $\mathcal{S}$? So far, QC-folding has been very effective at reproducing examples from $\mathcal{B}$ (or even $\mathcal{T}$) in $\mathcal{S}$, so I expect this will be possible too. In [24] I show that the Julia set of

$$f(z) = p(z) \prod_k \left(1 - \frac{1}{2}\left(\frac{z}{R_k}\right)^{2^k}\right) \tag{2}$$

has Hausdorff and packing dimension 1; in fact, $\mathcal{J}(f)$ has locally finite 1-measure. Here $R_k \nearrow \infty$ is a carefully chosen sequence and p is a polynomial with $\mathcal{J}(p)$ of small dimension. (I first discovered the "right" geometry using QC-folding, and this led to the formula above.) This is the first known $f \in \mathcal{T}$ with $\dim(\mathcal{J}) = 1$ or $\mathrm{Pdim}(\mathcal{J}) < 2$.

Problem 10. *Show* $\mathrm{Pdim}(\mathcal{J}(f))$ *can attain every value in* $[1, 2]$ *for* $f \in \mathcal{T}$.

One way to get $1 < \mathrm{Pdim}(\mathcal{J}) < 2$ might be to replace the polynomial p in (2) by one such that $1 < \dim(\mathcal{J}(p)) < 2$. If we QC deform p so that $\dim(\mathcal{J}(p))$ sweeps out all dimensions between 1 and 2, does $\mathrm{Pdim}(\mathcal{J}(f))$ also continuously sweep out these values? Do there exist $f \in \mathcal{T}$ giving every possible pair $1 \leq \dim(\mathcal{J}(f)) \leq \mathrm{Pdim}(\mathcal{J}(f)) \leq 2$?

Question 11. *Is there an $f \in \mathcal{T}$ with* $\dim(\mathcal{J}(f)) = 1$ *and* $\rho(f) > 0$*? With* $\rho(f) = \infty$*?*

The function in (2) has order 0 (and grows as slowly as we like, if R_k grows rapidly enough). I expect that positive order examples can be constructed by modifying the terms used in (2). The kth term in (2) is a rescaling of $T_2(z^{2^k})$ where $T_2(z) = 2z^2 - 1$ is the Chebyshev polynomial of degree 2, and certain geometric properties of Shabat polynomials are used in the proof. Replacing T_2 by carefully selected, high degree Shabat polynomials (using my approximation theorem in [18]), should give positive order examples. These examples would have wandering domains and also have very irregular growth, so might be helpful in resolving Baker's conjecture too.

We saw that the order ρ can differ for QC-equivalent functions. What about $\dim(\mathcal{J})$?

Question 12. *If $f, g \in \mathcal{S}$ are QC equivalent is* $\dim(\mathcal{J}(f)) = \dim(\mathcal{J}(g))$*?*

Stallard has shown this fails in $\mathcal{B}$ (even for affinely equivalent functions); can we "QC-fold" her examples into $\mathcal{S}$? The same question is open even in $\mathcal{B}$ for the escaping set

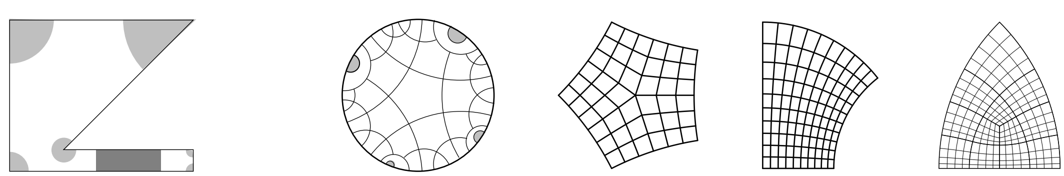
$$I(f) = \{z : f^k(z) \to \infty \text{ as } k \to \infty\}.$$

Question 13 ([76])**.** *If $f, g \in \mathcal{B}$ are QC-equivalent is* $\dim(I(f)) = \dim(I(g))$*?*

3. Computational geometry and conformal analysis

I first describe some problems in computational geometry that were solved using ideas from analysis, as well as some meshing problems that are still open, e.g., sharp bounds for NOTs (= non-obtuse triangulations; all angles $\leq 90°$). I will then turn to some problems in analysis where CG methods may help.

• **NOT theory (not knot theory):** My fast conformal mapping algorithm from [30] puts the Riemann mapping theorem into the available toolkit for proving complexity results in computational geometry. I used it in [31] to give a linear time algorithm for meshing simple polygons by quadrilaterals with optimal angle bounds (all angles $\leq 120°$; all new angles $\geq 60°$; original angles $< 60°$ are left alone). One of the key ideas for both fast mapping and optimal meshing is a thick-thin decomposition of a polygon that is exactly analogous to the thick and thin parts of a Riemann surface. Thin parts of a polygon correspond to pairs of sides which have extremal distance $\leq \epsilon$ inside the polygon. The leftmost figure below shows the thick (white) and thin (shaded) parts of a polygon. The thins parts have simple shapes and are meshed "by hand"; the thick parts are handled by transferred a mesh from the unit disk by a conformal map. The figure shows a partition of the disk and how different shapes in the partition can be quad-meshed. Thus the basic idea is that the thick/thin decomposition divides the polygon into "Euclidean" and "hyperbolic" pieces and we mesh each piece using the relevant geometry.

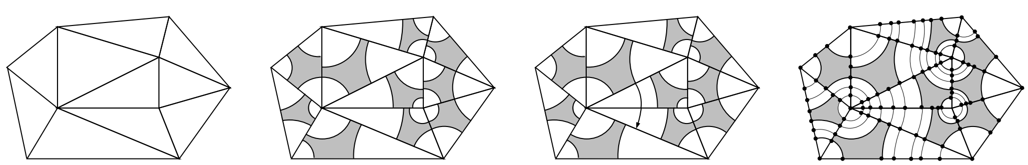


Recently I have extended the optimal meshing results from polygons to planar straight line graphs (a PSLG is any finite union of points and segments; see the results from prior support). Using my fast conformal mapping algorithm, I showed in [22] that any PSLG has a $O(n^2)$ quad-mesh with sharp angle bounds $60°$ and $120°$ (same as for polygons). Cutting each quadrilateral by a diagonal, we get a $O(n^2)$ triangulation with all angles $\leq 120°$, improving known bounds of $187.5°$, Mitchell [69], and $162°$, Tan [88]. The $O(n^2)$ is sharp; some examples require this many triangles. I show in [21] that any bound $90° + \epsilon$ is possible, but my proof gives a constant in $O(n^2)$ that blows up as $\epsilon \to 0$. A famous open problem in computational geometry asks if we can avoid this, i.e.,

Conjecture 14. *Every PSLG has a $O(n^2)$ non-obtuse triangulation (angles $\leq 90°$).*

No polynomial time algorithm constructing NOTs for PSLGs was known until I recently gave an $O(n^{2.5})$ method in [21]. A NOT can always be converted to an acute triangulation (angles $< 90°$) with a comparable number of triangles, [65],[93], but no uniform angle bound $< 90°$ is possible (consider a $1 \times R$ rectangle with R large).

My proof of the $O(n^{2.5})$ NOT-theorem goes as follows (see the picture below). (1) Add extra edges until the PSLG becomes a triangulation. (2) Decompose each triangle into three sectors (the thin parts, white) and a central region (the thick part, shaded). (3) The thin parts are foliated by circular arcs concentric with the vertex and we propagate the vertices of each thick part along the foliation paths until we hit another thick part or hit the convex hull boundary. (4) These paths divide the edges into subsegments that are each the diameter of disk that misses all the original vertices of the PSLG as well as all the newly generated intersections. This set is called a Gabriel set for the PSLG and known circle packing techniques can build a NOT from a Gabriel set [15].



If the triangles form a tree then each propagation path visits each edge at most once; hence $O(n^2)$ points are generated, giving a $O(n^2)$ NOT. But for general triangulations, propagation paths can hit the same edge arbitrarily often. I add $n^{1.5}$ extra points to the PSLG and give an intricate construction to perturb the propagation paths so that they still generate Gabriel points, but so that each path terminates by running into another path after at most $O(n)$ steps; this gives a $O(n^{2.5})$ NOT. The proof adds $n^{1/2}$ extra points for each of $O(n)$ possible obstructions, but a more careful analysis may show that not every obstruction needs the maximal number of extra points.

Dennis Sullivan noted the perturbations in the NOT-proof are reminiscent of Pugh's closing lemma: every C^1 vector field has a small perturbation with a closed orbit [73],[74], [75]. This is open for C^2 vector fields. Dennis asked if this similarity could be made exact:

Question 15. *Can a closing lemma help prove the $O(n^2)$ NOT-theorem? Can the NOT argument help prove a closing lemma?*

• **More NOTty problems:** Non-obtuse triangulation of a PSLG requires $\gtrsim n^2$ triangles in the worst case, but most applications won't involve the worst case, so every time I give the NOT talk for an applied audience, I get asked:

Question 16. *Is there an algorithm that produces an $O(N)$ sized NOT, where N is the minimal number needed for that particular PSLG?*

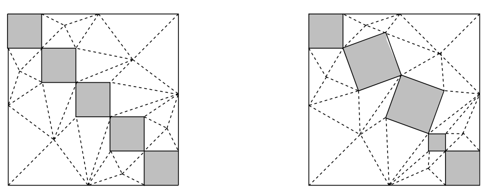
The corresponding theory in $\mathbb{R}^3$ (the really interesting case) is wide open, although there are a multitude of examples, definitions, heuristics and software.

Question 17. *Do polyhedra in $\mathbb{R}^3$ have nonobtuse tetrahedralizations of polynomial size?*

Is this true with any angle bound $< 180°$? Even finding an acute tetrahedralization of a cube in $\mathbb{R}^3$ was an open problem until recently (the smallest known example uses 1,370 pieces [91]) and is impossible in $\mathbb{R}^4$, [63]. The breakthrough in the 2-dimensional case was to introduce hyperbolic geometry into an apparently purely Euclidean problem. Can we use analogous ideas in $\mathbb{R}^3$? One idea is to create a 3-manifold out of a polyhedron, run a Ricci flow on it (as in the proof of Thurston's geometrization conjecture) to decompose it into pieces with geometric structure and then utilize the geometry to find meshes.

• **How sharp is Smirnov?** Next we consider some attacks on analysis problems using computational geometry. In [81] Stas Smirnov proved the conjecture of Astala that $\dim(f(\mathbb{R})) \leq 1+k^2$ for any quasiconformal map f with dilatation $|\mu_f| = |f_{\bar{z}}/f_z| \leq k$. Also see [72]. Sharpness is open, although Astala, Rohde and Schramm used a holomorphic motion of snowflakes to show $d \geq 1 + (.69)k^2$ is possible (personal communication).

Consider two combinatorially equivalent triangulations of the regions below. The piecewise linear map between corresponding triangles is QC and fixes the outer boundary.



Iterating the construction in the shaded squares, we obtain a QC map sending the diagonal to a fractal curve. The maximal dilatation k for the map and the dimension d of the curve are both easy to compute, so every pair of triangulations gives a lower bound. Can the optimal estimate can be approximated by these examples? This should be similar to "fractal approximation" arguments in [17], [38], [59], [66].

Problem 18. *Use available optimization software (e.g.* `CPLEX`*) to minimize $1 + k^2 - d$.*

Either we will get near sharp examples or some insight into what is preventing sharpness; either outcome would be interesting. Smirnov's proof shows that the dilatation of an optimal QC map satisfies the anti-symmetry condition $\mu(\bar{z}) = -\overline{\mu(z)}$. Can this criterion be utilized in our search? If we define "anti-symmetric" triangulations in an appropriate way, does it suffice to search such examples?

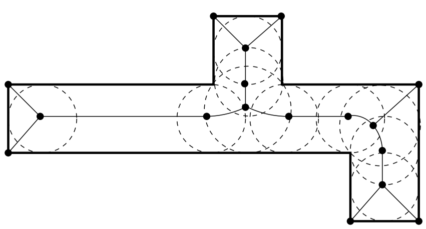
- **How strong is Brennan?** Brennan's famous conjecture [37] asks

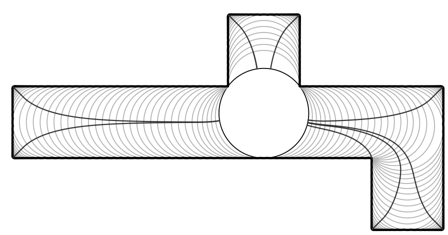
Conjecture 19. *If $f : \Omega \to \mathbb{D}$ is conformal, then $f' \in L^p(\Omega, dxdy)$ for all $p \in [2, 4)$.*

This is related to a number of other important problems; see [9], [16], [39]. Must f' be in weak L^4 (strong Brennan conjecture)? I observed in [26] that this is implied by Astala's celebrated result on area distortion of QC maps [4], if the following is true (stronger Brennan conjecture):

Conjecture 20. *If Ω is simply connected, there is a 2-QC, locally Lipschitz $f : \Omega \to \mathbb{D}$.*

I showed in [25] that there is always a K-QC, locally Lipschitz map $\iota : \Omega \to \mathbb{D}$ with K independent of the domain. This is Thurston's "iota map"; it is the natural isometry between the boundary of the convex hull of Ω^c in the hyperbolic upper half-space and the unit disk (this surface $S \subset \mathbb{R}^3_+$ is called the "dome" of Ω, $\partial S = \partial\Omega$ and iota is the conformal map of the dome to the disk). See [45], [46], [47]. The iota map plays a fundamental role as the initial guess for the fast conformal mapping algorithm in [30] and is used to compute thick-thin decompositions of polygons in linear time. It is fast to compute because it has an alternate definition using ideas from computational geometry. The medial axis of a domain consists of the centers of all internal disks that hit the boundary at least twice (see left below). Fixing one such disk and flowing orthogonally to the boundaries of others (see right below) gives a map from $\partial\Omega$ to a circle. The flow preserves certain cross ratios, and using this, we can compute the image of all n vertices in time $O(n)$. Thus iota gives a "fast but rough" version of the Riemann map.





The iota map has a locally Lipschitz extension to the interior with QC-constant ≤ 7.82 for all domains [27], but > 2.1 for some examples [48]; thus iota does not quite solve Conjecture 20, but other maps may work. Can we efficiently compute the QC-constant of iota for a given domain? If $\mathbf{w} = \{w_1, \dots, w_n\}, \mathbf{z} = \{z_1, \dots, z_n\}$ are n-tuples of distinct points on $\mathbb{T}$ (considered modulo Möbius transformations),

Problem 21. *Compute $d_{QC}(\mathbf{z}, \mathbf{w}) = \inf\{\log K : \exists\ K\text{-}QC\ h : \mathbb{D} \to \mathbb{D}, h(\mathbf{z}) = \mathbf{w}.\}$.*

We know the general Teichmüller form of the extremal map, so it should be possible to approximate K by some sort of Newton's iteration on the space of holomorphic quadratic differentials. We already have efficient methods to map a polygon to the disk both conformally and by the iota map. An efficient computation of d_{QC} would allow us to try to optimize polygons for the worst K; perhaps even lead us to the shape of Brennan counterexample, if one exists. Moreover, there are variants of iota that give better QC-bounds on particular examples; can we search for one that satisfies Conjecture 20?

• **Can a carpenter straighten a chord-arc curve?** A rectifiable curve $\Gamma \subset \mathbb{R}^2$ is called chord-arc if the arclength σ between $x, y \in \Gamma$ satisfies $\sigma(x,y) \le M|x-y|$ for some $M < \infty$. The arclength parameterization satisfies $\gamma_0'(t) = e^{if(t)}$ where $f \in \mathrm{BMO}$, see e.g. [53]. A famous and long standing problem asks if

Conjecture 22. *The space of chord-arc curves is connected in the BMO topology.*

If so, any chord-arc path Γ can be "straightened" to a line segment, i.e., there is a $\gamma : [0,1]^2 \to \mathbb{R}^2$ so that the map $t \to \gamma_t = \gamma(\cdot, t)$ is continuous in the BMO topology, γ_1 is a line segment and $\gamma_0 = \Gamma$. We call γ expansive if all distances increase, i.e., $s < t$ implies $|\gamma(u,s) - \gamma(v,s)| \le |\gamma(u,t) - \gamma(v,t)|$, $\forall u, v \in [0,1]$. Such a motion cannot increase the chord-arc constant M, so is a natural type of motion to consider.

Conjecture 23. *A chord-arc path can be straightened by an expansive motion.*

This is a continuous version of the "carpenter's rule" problem in computation geometry: straighten a polygonal arc without creating self-intersections. An expansive solution was obtained by Connelly, Demaine and Rote [41], and independently by Streinu [86]. The solution in [41] is based on linear programming; given the current position, a set of linear equalities and inequalities determines what motions give a valid motion. Using duality, the authors prove a solution exists unless the arc is already straight (interestingly, the proof reduces to the maximum principle for subharmonic functions). We might prove Conjecture 23 by approximating by polygons and passing to a limit, but we need estimates on how fast points move: a lower bound to make sure the limiting motion actually straightens the curve in finite time, and an upper bound to make sure the motions are equicontinuous in the BMO topology. Does a solution exists with these extra constraints? Can we test it numerically? Is the speed bounded in terms of the chord-arc constant?

4. Quasiconformal distortion

A quasiconformal map is absolutely continuous on almost all lines, so a QC image of $[0,1] \times y$ will be rectifiable for a.e. y, but it is well known that a single segment can have a fractal image (e.g., the von Koch snowflake is a quasicircle). Can every component of $[0,1] \times Y$ have its dimension increased if Y is "big"? How does this depend on Y? We now consider these questions and related problems about rectifiable sets.

• **Simultaneous fractalization:** In [33] Hrant Hakobyan and I prove that if f is QC on $\mathbb{R}^n$, $Y \in \mathbb{R}^{n-1}$ and $d = \dim(Y)$, then (compare to [7])

$$\inf_{y \in Y} \dim(f([0,1] \times \{y\})) \le \frac{n}{d+1}, \tag{3}$$

and we give sharp examples in the plane ($n = 2$). Thus uncountably many parallel segments can be made into "fractals" by a single map. We prove (3) as a special case of

$$\inf_{y \in Y} \dim(f(E \times \{y\}) \le \dim(E) \cdot \dim(f(E \times Y))/\dim(E \times Y). \tag{4}$$

Problem 24. *Is (3) sharp for $n > 2$? Is (4) sharp for general sets E?.*

Equation (4) considers subsets of hyperplanes and translates them in the orthogonal direction. What about general sets and arbitrary translates? The proof of (4) uses the Fuglede d-modulus of a set of measures $\{\nu_\lambda\}$ with respect to a base measure μ. This is defined as $\mathrm{mod}_d(\{\nu_\lambda\},\mu) = \inf_\rho\{\int \rho^d d\mu : \int \rho d\nu_\lambda \geq 1 \quad \forall \lambda\}$, and generalizes conformal modulus. Hakobyan proved that if $E \subset \mathbb{R}^2$ has dimension d, and f is QC then

$$\mathrm{mod}_{2/d}(\{E+y : \dim(f(E+y)) > \dim(E)\}, dxdy) = 0.$$

In other words, the dimension of almost every translate is not increased, in the sense of modulus. What about in the sense of Lebesgue measure?

Conjecture 25. area$(\{y : \dim(f(E+y)) > \dim(E)\} = 0$. *Is this true for* $<$*?*

What about the size of the exceptional set? The dependence on the QC-constant of f? More general linear transformations? An endless supply of questions remains untouched.

• **QC maps destroying rectifiability:** If $n = 2$ and $\dim(Y) = 1$, then (3) says $f(Y \times [0,1])$ has a component of dimension 1; must it be rectifiable? No. In [33] Hakobyan and I prove that $f(Y \times [0,1])$ can be purely un-rectifiable (no rectifiable subarcs). No uncountable example with this property was previously known (for countable sets, see [64]). In fact, positive h-measure examples exist for any h so that $\lim_{t\to 0} h(t)/t = \infty$. My interest in these examples is motivated by:

Conjecture 26. *Every zero area set* $X \subset \mathbb{R}^2$ *has a purely un-rectifiable QC image.*

Roughly, this says there is a QC map whose Jacobian blows up on any null set; thus this question is closely connected to the well studied problem of characterizing Jacobians of QC maps (see [36]). The work with Hakobyan eliminates many sets $X = Y \times [0,1]$ as possible counterexamples, but it would very interesting to eliminate all of them:

Conjecture 27. *If* Y *has zero length,* $Y \times [0,1]$ *has a purely un-rectifiable QC image.*

The construction in [33] covers $Y \times [0,1]$ by thin tubes and uses a QC map to bend the tubes at infinitely many scales. However, the tubes must be "well separated", and hence Y must be porous at infinitely many scales. If I could remove this condition, then it seems likely that Conjecture 26 might follow using a result of Alberti, Csörnyei and Preiss [2], that any null set E has a "Lipschitz product structure". More precisely, for any $\epsilon > 0$, $E = E_x \cup E_y$ where E_x can be covered by "horizontal tubes" $T(f_i, \delta_i) = \{(x,y) : |y - f_i(x)| < \delta_i\}$ where $\sum_i \delta_i < \epsilon$ and $\{f_i\}$ are 1-Lipschitz. E_y is covered by analogous vertical tubes. Constructing a QC map that bends these tubes at infinitely many scales should prove Conjecture 26. The proof in [2] is combinatorial argument about partially ordered sets, and provides an elegant application of combinatorial ideas to analysis.

• **QC maps preserving rectifiability:** Conjecture 26 fails for small QC maps; there is a null set E and a $K > 1$ so that every K-QC image of E contains a rectifiable arc [32]. Such a set is used to build a surface $S \subset \mathbb{R}^3$ that is a quasisymmetric (QS) image of $\mathbb{R}^2$ but not a biLipschitz image [28]. (QS maps generalize QC maps, but we omit the precise definition; BiLipschitz maps preserve distance up to a bounded multiplicative factor).

Problem 28. *Characterize metric spaces that are biLipschitz equivalent to the plane.*

Bonk and Kleiner [35] characterized QS images of $\mathbb{R}^2$; Problem 28 asks what additional condition implies their QS parameterization can be improved to biLipschitz.

The surface S in [28] contains an infinite length curve γ that maps to finite length under any QS map to $\mathbb{R}^2$; clearly there is no such biLipschitz map. The length of the image is estimated using Peter Jones' traveling salesman theorem (TST) [58]; this gives a characterization of rectifiable curves in the plane in terms of "β-numbers" that measure local deviation from a straight line. Roughly speaking, the surface S fails to be a biLipschitz plane because the TST fails on S. Is there a version of TST for biLipschitz images of $\mathbb{R}^2$? Does it characterize biLipschitz images?

Crofton's formula says that the length of $\gamma \subset \mathbb{R}^2$ is $\frac{1}{2}\int \#(\gamma \cap L) dr d\theta$, where L is the line $\{z : \mathrm{Re}(ze^{-i\theta}) = r\}$ and $dr d\theta$ is invariant under isometries of $\mathbb{R}^2$. I recently showed that Jones' TST in $\mathbb{R}^2$ follows easily from Crofton's formula and some simple estimates about dyadic squares (similar to those in Okikiolu's proof of TST [71]). Is there a version of Crofton's formula valid for biLipschitz images of $\mathbb{R}^2$? If so, does this characterize such images? I am studying these problems with Raanan Schul and Mathew Badger (experts on TST and geometric measure theory here at Stony Brook).

• **Can a finite set be infinitely hard to triangulate?** As usual, a triangulation of a finite set V is a maximal set of disjoint open segments with endpoints in V. Some triangulation of V attains the minimal total length (or "weight"), denoted $\mathrm{MWT}(V)$, but we can sometimes decrease $\mathrm{MWT}(V)$ by adding vertices (Steiner points), e.g.,

Let $\mathrm{MWT}(V, n) = \inf\{\mathrm{MWT}(W) : V \subset W, \#(W \setminus V) \leq n\}$. A long standing problem in computer science asks if a finite number of Steiner points suffices, i.e.,

Problem 29. *Is* $\inf_n \mathrm{MWT}(V, n)$ *always attained? Is* $\mathrm{MWT}(V, n)$ *eventually constant?*

Connecting points of V is easier: the smallest length is attained by a Steiner tree with at most $\#(V) - 2$ extra points. Jones' TST approximates this minimum to within a bounded factor, but finding the minimum is NP-hard [52] (but it can be approximated in polynomial time [3], [68]). Finding a minimal triangulation (even without Steiner points) is NP-hard by [70] and Problem 29 asks whether a solution even exists when Steiner points are allowed. I gave a set V where $\mathrm{MWT}(V) > \mathrm{MWT}(V, 1) = \mathrm{MWT}(V, n), n \geq 2$ but the infimum is never attained, so the answer to the first question above is "no". However, my example contains three co-linear points so the question becomes: is the infimum attained for points in general position? If $\mathrm{MWT}(V, n)$ is not eventually constant, and we pass to a minimizing sequence of triangulations, can we use geometric measure theory to describe the limit? To prove that it is actually a finite triangulation? There is a large literature on rectifiable sets that might be applied here, and such a solution would be a remarkable application of "infinite" analysis methods to a "finite" geometry problem.

5. Educational and broader impact of the proposal

Optimal meshing has a number of practical applications. Acute and non-obtuse triangulations make some numerical methods work more efficiently, e.g.,

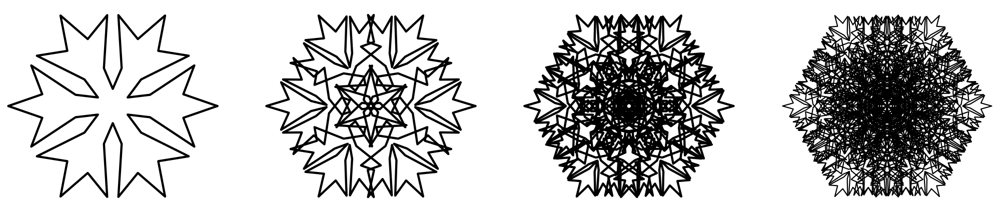
- Maximum principles hold for various discrete PDE's on NOTs, [40].
- Condition numbers for matrices associated with general triangulations grow exponentially with the size of the mesh, but only linearly for NOTs, [92].
- The finite element method on a NOT leads to matrix that is symmetric, positive definite and negative off the diagonal, allowing faster numerical linear algebra [82].
- First order Hamilton-Jacobi equations $u_t + H(\nabla u) = f(x)$ are modeled in [10] using an update that works better with an acute triangulation.
- The Fast Marching Method in [60], [80] for finding geodesics on a triangulated surface is most efficient if the underlying triangulation is acute.
- Meshing space-time using the tent pitcher algorithm of [1], [89], [90] is guaranteed to work for an acute initial triangulation (general cases require extra work).
- Nearest neighbor computer learning uses NOTs, [79].

Other applications are given in [10], [14], [82]. This proposal is closely related to my work on numerical conformal mapping (not explicitly discussed due to lack of space). Conformal mapping is currently the fastest way to solve certain problems in 2-dimensional potential theory, see [42] and its references. Conformal maps, the iota map and the medial axis are used in David Mumford's work on pattern recognition and computer vision.

A more general impact of the proposal is its interdisciplinary character, connecting classical analysis, hyperbolic geometry, computational geometry and numerical analysis. These problems can serve as a bridge between researchers with common interests but different backgrounds. For example, my paper on fast conformal mapping [30] was specifically written to be accessible to both mathematicians and computer scientists and appeared in a premier computer science journal. The iota map and its connections to conformal mapping (introduced in [26], [27], [30], [31]) have already started to appear in the work of some applied mathematicians (e.g., [8], [50], [51]). I spoke on this work at the 2010 Fall Workshop on Computational Geometry, and on NOTs at a keynote address for the 2012 Symposium on Computational Geometry (SoCG). I also organized a workshop at SoCG on the interactions between computational geometry and analysis. This was a sequel to a similar workshop Steve Vavasis and I hosted at Stony Brook in 2007. I maintain websites for both (including videos from the 2007 workshop).

The problems described in the proposal generally have simple statements, a significant geometric and computational component and interesting applications. This makes them attractive to students and I have supervised 4 Ph.D. dissertations on related topics: Zsuzanne Gonye (geodesics in hyperbolic manifolds), Karyn Lundberg (boundary convergence of conformal maps), Hrant Hakobyan (dimension distortion under QC maps) and Chris Green (numerical conformal mapping). These account for four of a total of ten PhD's in analysis at Stony Brook over the last ten years. Many of the problems discussed in the proposal could be used for graduate or undergraduate research projects, e.g.,

• **Search for near sharp examples in Smirnov's $1 + k^2$ theorem**.
• **Numerically estimating the speed in the carpenter's rule motion**.
• **Draw the transcendental Julia set of dimension** 1.
• **Draw a Julia set for a pair-of-pants finite type map**.
• **Approximate the true form of a planar tree**. This is the honors thesis of Kevin Sackel, an undergraduate math/physics major I am advising.
• **Computing optimal QC distance between n-tuples:** This is currently being investigated by Mayank Goswami, a graduate student in applied math.
• **Search for "improvements" to the iota map**.
• **Conformal maps via random walks:** using a trick to allow certain random walks that step past the boundary we can estimate harmonic measure very quickly. This method was used in 2011 by Ahmed Rafiqi (then an undergrad; now in grad school at Cornell) to calculate Schwarz-Christoffel parameters for simply connected polygonal domains and conformal modulus for polygonal annuli.
• **Peano maps on snowflakes:** The figures below were generated by convolving $1/z$ by point masses at the vertices of successive generations of the von Koch snowflake. The limit is a Peano map of the snowflake onto a set with interior. An undergrad, Daniel Levine, is numerically investigating these pictures and the push-forward measure.



• **The Brownian trace is purely unrectifiable:** The trace of 2-dimensional Brownian motion is conjectured to contain no rectifiable arc, even no arcs of dimension $1 + \epsilon$. Draw a N-long random walk on a square lattice and compute the minimal distance D (along the visited sites) between points chosen about distance $\sqrt{N}$ apart. If $D \simeq N$ often, this is evidence that Brownian motion contains rectifiable paths.
• **Implementing the NOT-algorithm** The full polynomial algorithm for finding NOTs of PSLGs is probably too complex to implement, but the special case of refining any polygon triangulation to a non-obtuse one should be easy to code. Use the implementation to study how far the worst case running time is from the "average" behavior.
• **Conformal mapping:** Davis's method and CRDT are simple iterations used in practice to find the parameters in the Schwarz-Christoffel formula, but neither is proven to converge. For pentagons, we can interpret the iterations as acting on $\mathbb{R}^2$. Study their geometry and prove they converge in this special case.
• **Prause's "2-sided" conjecture:** If f_1, f_2 are conformal maps from two sides of a Jordan curve γ to the disk, and if μ is a measure on γ, as a result of joint work with Kari Astala and Stas Smirnov, Istvan Prause conjectured that $(\dim(\mu) - 1)^2 \leq (\dim(f_1(\mu)) - 1)(\dim(f_2(\mu)) - 1)$. Test this numerically for some fractal curves using numerical conformal maps or simulated random walks.

References

[1] R. Abedi, S.H. Chung, J. Erickson, Y. Fan, M. Garland, D. Guoy, R. Haber, J.M. Sullivan, S. Thite, and Y. Zhou. Spacetime meshing with adaptive refinement and coarsening. In *SCG '04: Proceedings of the twentieth annual symposium on computational geometry*, pages 300–309, New York, NY, USA, 2004. ACM.

[2] G. Alberti, M. Csörnyei, and D. Preiss. Structure of null sets in the plane and applications. In *European Congress of Mathematics*, pages 3–22. Eur. Math. Soc., Zürich, 2005.

[3] Sanjeev Arora. Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. *J. ACM*, 45(5):753–782, 1998.

[4] K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994.

[5] I. N. Baker. The domains of normality of an entire function. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 1(2):277–283, 1975.

[6] I. N. Baker. An entire function which has wandering domains. *J. Austral. Math. Soc. Ser. A*, 22(2):173–176, 1976.

[7] Z.M. Balogh, R. Monti, and J. Tyson. Frequency of Sobolev and quasiconformal dimension distortion. to appear in J. Math. Pures Appl.

[8] L. Banjai. Revisiting the crowding phenomenon in Schwarz-Christoffel mapping. *SIAM J. Sci. Comput.*, 30(2):618–636, 2008.

[9] K. Barański, A. Volberg, and A. Zdunik. Brennan's conjecture and the Mandelbrot set. *Internat. Math. Res. Notices*, (12):589–600, 1998.

[10] T.J. Barth and J.A. Sethian. Numerical schemes for the Hamilton-Jacobi and level set equations on triangulated domains. *J. Comput. Phys.*, 145(1):1–40, 1998.

[11] G. V. Belyĭ. Galois extensions of a maximal cyclotomic field. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(2):267–276, 479, 1979.

[12] M. Bern and D. Eppstein. Polynomial-size nonobtuse triangulation of polygons. *Internat. J. Comput. Geom. Appl.*, 2(3):241–255, 1992. Selected papers from the 7th Annual Symposium on Computational Geometry, I (Conway, NH, 1991).

[13] M. Bern and D. Eppstein. Quadrilateral meshing by circle packing. *Internat. J. Comput. Geom. Appl.*, 10(4):347–360, 2000. Selected papers from the Sixth International Meshing Roundtable, Part II (Park City, UT, 1997).

[14] M. Bern and J.R. Gilbert. Drawing the planar dual. *Inform. Process. Lett.*, 43(1):7–13, 1992.

[15] M. Bern, S. Mitchell, and J. Ruppert. Linear-size nonobtuse triangulation of polygons. *Discrete Comput. Geom.*, 14(4):411–428, 1995. ACM Symposium on Computational Geometry (Stony Brook, NY, 1994).

[16] D. Bertilsson. *On Brennan's conjecture in comformal mapping*. PhD thesis, Royal Institute of Technology, Stockholm, Sweden, October 1999.

[17] I. Binder, N. Makarov, and S. Smirnov. Harmonic measure and polynomial Julia sets. *Duke Math. J.*, 117(2):343–365, 2003.

[18] C.J. Bishop. Approximation by critical points of generalized Chebyshev polynomials. preprint 2011.

[19] C.J. Bishop. Approximation in the Eremenko-Lyubich class. preprint 2011.

[20] C.J. Bishop. Constructing entire functions by quasiconformal folding. preprint 2011.

[21] C.J. Bishop. Nonobtuse triangulations of PSLGs. preprint, 2011.

[22] C.J. Bishop. Optimal quadrilateral meshes for PSLGs. Preprint, 2011.

[23] C.J. Bishop. The order conjecture fails in $\mathcal{S}$. preprint 2011.

[24] C.J. Bishop. A transcendental Julia set of dimension 1. preprint 2011.

[25] C.J. Bishop. Divergence groups have the Bowen property. *Ann. of Math. (2)*, 154(1):205–217, 2001.

[26] C.J. Bishop. Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. *Ark. Mat.*, 40(1):1–26, 2002.

[27] C.J. Bishop. An explicit constant for Sullivan's convex hull theorem. In *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pages 41–69. Amer. Math. Soc., Providence, RI, 2004.

[28] C.J. Bishop. An A_1 weight not comparable with any quasiconformal Jacobian. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 7–18. Amer. Math. Soc., Providence, RI, 2007.

[29] C.J. Bishop. Conformal welding and Koebe's theorem. *Ann. of Math. (2)*, 166(3):613–656, 2007.
[30] C.J. Bishop. Conformal mapping in linear time. *Discrete Comput. Geom.*, 44(2):330–428, 2010.
[31] C.J. Bishop. Optimal angle bounds for quadrilateral meshes. *Discrete Comput. Geom.*, 44(2):308–329, 2010.
[32] C.J. Bishop. A set containing rectifiable arcs QC-locally but not QC-globally. *Pure Appl. Math. Q.*, 7(1):121–138, 2011.
[33] C.J. Bishop and H. Hakobyan. Frequency of dimension distortion under quasisymmetric mappings. preprint 2012.
[34] C.J. Bishop and Y. Peres. Fractals in analysis and probability. in preparation.
[35] M. Bonk and B. Kleiner. Quasisymmetric parametrizations of two-dimensional metric spheres. *Invent. Math.*, 150(1):127–183, 2002.
[36] Mario Bonk, Juha Heinonen, and Eero Saksman. The quasiconformal Jacobian problem. In *In the tradition of Ahlfors and Bers, III*, volume 355 of *Contemp. Math.*, pages 77–96. Amer. Math. Soc., Providence, RI, 2004.
[37] J.E. Brennan. The integrability of the derivative in conformal mapping. *J. London Math. Soc. (2)*, 18(2):261–272, 1978.
[38] L. Carleson and P. W. Jones. On coefficient problems for univalent functions and conformal dimension. *Duke Math. J.*, 66(2):169–206, 1992.
[39] L. Carleson and N. G. Makarov. Some results connected with Brennan's conjecture. *Ark. Mat.*, 32(1):33–62, 1994.
[40] P.G. Ciarlet and P.A. Raviart. Maximum principle and uniform convergence for the finite element method. *Comput. Methods Appl. Mech. Engrg.*, 2:17–31, 1973.
[41] R. Connelly, E. D. Demaine, and G. Rote. Straightening polygonal arcs and convexifying polygonal cycles. *Discrete Comput. Geom.*, 30(2):205–239, 2003. U.S.-Hungarian Workshops on Discrete Geometry and Convexity (Budapest, 1999/Auburn, AL, 2000).
[42] T.A. Driscoll and L.N. Trefethen. *Schwarz-Christoffel mapping*, volume 8 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.
[43] H. Edelsbrunner and T. S. Tan. An upper bound for conforming Delaunay triangulations. *Discrete Comput. Geom.*, 10(2):197–213, 1993.
[44] A. Epstein. *Towers of finite type complex analytic maps*. PhD thesis, City University of New York, 1995.
[45] D.B.A. Epstein and A. Marden. Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces. In *Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984)*, volume 111 of *London Math. Soc. Lecture Note Ser.*, pages 113–253. Cambridge Univ. Press, Cambridge, 1987.
[46] D.B.A. Epstein, A. Marden, and V. Markovic. Complex angle scaling. In *Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001)*, volume 299 of *London Math. Soc. Lecture Note Ser.*, pages 343–362. Cambridge Univ. Press, Cambridge, 2003.
[47] D.B.A. Epstein, A. Marden, and V. Markovic. Quasiconformal homeomorphisms and the convex hull boundary. *Ann. of Math. (2)*, 159(1):305–336, 2004.
[48] D.B.A. Epstein and V. Markovic. The logarithmic spiral: a counterexample to the $K = 2$ conjecture. *Ann. of Math. (2)*, 161(2):925–957, 2005.
[49] A. È. Erëmenko and M. Yu. Lyubich. Dynamical properties of some classes of entire functions. *Ann. Inst. Fourier (Grenoble)*, 42(4):989–1020, 1992.
[50] M. Feiszli. *Conformal shape representation*. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–Brown University.
[51] M. Feiszli. Extremal distance, hyperbolic distance, and convex hulls over domains with smooth boundary. *Ann. Acad. Sci. Fenn. Math.*, 36(1):195–214, 2011.
[52] M. R. Garey, R. L. Graham, and D. S. Johnson. The complexity of computing Steiner minimal trees. *SIAM J. Appl. Math.*, 32(4):835–859, 1977.
[53] J.B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1981.

[54] L.R. Goldberg and L. Keen. A finiteness theorem for a dynamical class of entire functions. *Ergodic Theory Dynam. Systems*, 6(2):183–192, 1986.
[55] C. Green. *The Ahlfors iteration*. PhD thesis, Stony Brook University, 2011.
[56] A. Hinkkanen. Entire functions with bounded Fatou components. In *Transcendental dynamics and complex analysis*, volume 348 of *London Math. Soc. Lecture Note Ser.*, pages 187–216. Cambridge Univ. Press, Cambridge, 2008.
[57] A. Hinkkanen and J. Miles. Growth conditions for entire functions with only bounded Fatou components. *J. Anal. Math.*, 108:87–118, 2009.
[58] P. W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.
[59] P. W. Jones and N. G. Makarov. Density properties of harmonic measure. *Ann. of Math. (2)*, 142(3):427–455, 1995.
[60] R. Kimmel and J. A. Sethian. Computing geodesic paths on manifolds. *Proc. Natl. Acad. Sci. USA*, 95(15):8431–8435 (electronic), 1998.
[61] M. Kisaka and M. Shishikura. On multiply connected wandering domains of entire functions. In *Transcendental dynamics and complex analysis*, volume 348 of *London Math. Soc. Lecture Note Ser.*, pages 217–250. Cambridge Univ. Press, Cambridge, 2008.
[62] Yu. Yu. Kochetkov. Planar trees with nine edges: a catalogue. *Fundam. Prikl. Mat.*, 13(6):159–195, 2007.
[63] E. Kopczynski, I. Pak, and P. Przytycki. Acute triangulations of polyhedra and R^n. Preprint.
[64] L.V. Kovalev and J. Onninen. Variation of quasiconformal mappings on lines. *Studia Math.*, 195(3):257–274, 2009.
[65] H. Maehara. Acute triangulations of polygons. *European J. Combin.*, 23(1):45–55, 2002.
[66] N.G. Makarov. Fine structure of harmonic measure. *Algebra i Analiz*, 10(2):1–62, 1998. English translation in St. Petersburg J. Math, 1999.
[67] C. McMullen. Area and Hausdorff dimension of Julia sets of entire functions. *Trans. Amer. Math. Soc.*, 300(1):329–342, 1987.
[68] Joseph S. B. Mitchell. Guillotine subdivisions approximate polygonal subdivisions: a simple polynomial-time approximation scheme for geometric TSP, k-MST, and related problems. *SIAM J. Comput.*, 28(4):1298–1309 (electronic), 1999.
[69] S. A. Mitchell. Refining a triangulation of a planar straight-line graph to eliminate large angles. In *34th Annual Symposium on Foundations of Computer Science (Palo Alto, CA, 1993)*, pages 583–591. IEEE Comput. Soc. Press, Los Alamitos, CA, 1993.
[70] W. Mulzer and G. Rote. Minimum-weight triangulation is NP-hard. *J. ACM*, 55(2):Art. 11, 29, 2008.
[71] K. Okikiolu. Characterization of subsets of rectifiable curves in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 46(2):336–348, 1992.
[72] I. Prause, X. Tolsa, and I. Uriarte-Tuero. Hausdorff measure of quasicircles. *Adv. Math.*, 229(2):1313–1328, 2012.
[73] C.C. Pugh. The closing lemma and structural stability. *Bull. Amer. Math. Soc.*, 70:584–587, 1964.
[74] C.C. Pugh. The closing lemma. *Amer. J. Math.*, 89:956–1009, 1967.
[75] C.C. Pugh. Against the C^2 closing lemma. *J. Differential Equations*, 17:435–443, 1975.
[76] L. Rempe and G.M. Stallard. Hausdorff dimensions of escaping sets of transcendental entire functions. *Proc. Amer. Math. Soc.*, 138(5):1657–1665, 2010.
[77] P. J. Rippon and G. M. Stallard. Functions of small growth with no unbounded Fatou components. *J. Anal. Math.*, 108:61–86, 2009.
[78] G. Rottenfusser, J. Rückert, L. Rempe, and D. Schleicher. Dynamic rays of bounded-type entire functions. *Ann. of Math. (2)*, 173(1):77–125, 2011.
[79] S. Salzberg, A.L. Delcher, D Heath, and S. Kasif. Best-case results for nearest-neighbor learning. *IEEE Trans. Pattern Anal. MAch. Intell.*, 17(6):599–608, 1995.
[80] J. A. Sethian. Fast marching methods. *SIAM Rev.*, 41(2):199–235 (electronic), 1999.
[81] S. Smirnov. Dimension of quasicircles. *Acta Math.*, 205(1):189–197, 2010.

[82] D.A. Spielman and S.-H. Teng. Nearly-linear time algorithms for graph partitioning, graph sparsification, and solving linear systems. In *Proceedings of the 36th Annual ACM Symposium on Theory of Computing*, pages 81–90 (electronic), New York, 2004. ACM.

[83] Herbert R. Stahl. Sets of minimal capacity and extremal domains. preprint, 2012.

[84] G.M. Stallard. The Hausdorff dimension of Julia sets of entire functions. II. *Math. Proc. Cambridge Philos. Soc.*, 119(3):513–536, 1996.

[85] G.M. Stallard. The Hausdorff dimension of Julia sets of entire functions. III. *Math. Proc. Cambridge Philos. Soc.*, 122(2):223–244, 1997.

[86] I. Streinu. Pseudo-triangulations, rigidity and motion planning. *Discrete Comput. Geom.*, 34(4):587–635, 2005.

[87] D. Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985.

[88] T.-S. Tan. An optimal bound for high-quality conforming triangulations. *Discrete Comput. Geom.*, 15(2):169–193, 1996.

[89] S. Thite. Adaptive spacetime meshing for discontinuous Galerkin methods. *Comput. Geom.*, 42(1):20–44, 2009.

[90] A. Üngör and A. Sheffer. Pitching tents in space-time: mesh generation for discontinuous Galerkin method. *Internat. J. Found. Comput. Sci.*, 13(2):201–221, 2002. Volume and surface triangulations.

[91] E. Vanderzee, A.N. Hirani, V. Zharnitsky, and D. Guoy. A dihedral acute triangulation of the cube. *Computational Geometry: Theory and Applications*, (Accepted, and available online), 2009. Also available as a preprint at arXiv as arXiv:0905.3715v4 [cs.CG].

[92] S. A. Vavasis. Stable finite elements for problems with wild coefficients. *SIAM J. Numer. Anal.*, 33(3):890–916, 1996.

[93] L. Yuan. Acute triangulations of polygons. *Discrete Comput. Geom.*, 34(4):697–706, 2005.

[94] J.-H. Zheng. Unbounded domains of normality of entire functions of small growth. *Math. Proc. Cambridge Philos. Soc.*, 128(2):355–361, 2000.