RESULTS FROM PRIOR NSF SUPPORT
Grant number DMS-0103626
Geometry of conformal and quasiconformal mappings
6/15/01 - 5/31/04 $176,388

I have worked on problems involving conformal maps, Kleinian groups and related topics. The main results are summarized below. The website `www.math.sunysb.edu/~bishop/papers` contains all the papers described below. Let $\mathbb{D} = \{|z| < 1\}$ be the unit disk, $\mathbb{D}^* = \{|z| > 1\}$, $\mathbb{T} = \{|z| = 1\}$. If Γ is a closed curve then Ω, Ω^* denote its bounded and unbounded complementary components. For $E \subset \mathbb{T}$, $|E|$ denotes its normalized Lebesgue measure and $\text{cap}(E)$ its logarithmic capacity.

- **Conformal welding:** A circle homeomorphism h is called a generalized conformal welding on $E \subset \mathbb{T}$ (denoted $h \in \text{GCW}(E)$) if $h = g^{-1} \circ f$, where f and g are univalent maps from $\mathbb{D}$, $\mathbb{D}^*$ to disjoint domains Ω, Ω^*, and the composition exists for radial limits of f on E and g on $h(E)$ (this was invented by David Hamilton in [59]; see also [60], [61]). Moreover, h is a (standard) conformal welding (denoted $h \in \text{CW}$) if $E = \mathbb{T}$ and Ω, Ω^* are the two sides of a closed Jordan curve Γ. Not every h is a conformal welding, but in [20] I prove that every h agrees with some $H \in \text{CW}$ off a set E of arbitrarily small Lebesgue measure. Other results in [20] include: every h is in $\text{GCW}(\mathbb{T} \setminus E_1 \cup E_2)$ for some sets with $\text{cap}(E_1) = \text{cap}(h(E_2)) = 0$; if h is log-singular (i.e., $\mathbb{T} = E_1 \cup E_2$ with $\text{cap}(E_1) = \text{cap}(h(E_2)) = 0$) then $h \in \text{CW}$ (this is quite different from the usual sufficient condition of h being quasisymmetric). I also give a new, short proof that quasisymmetric homeomorphisms are conformal weldings using Koebe's circle domain theorem (avoiding the use of the measurable Riemann mapping theorem).
- **Interpolation sets for conformal maps:** In [23] I show that if $E \subset \mathbb{T}$ is compact and has zero logarithmic capacity and g is any homeomorphism of the open disk which extends continuously to E then there is a conformal map of the disk which extends continuously to E and equals g there. This is false if E has positive capacity. The proof is an explicit geometric construction.
- **Factorization and Sullivan's theorem:** In [13], I show any conformal map $f : \mathbb{D} \to \Omega$ can be written as $f = g \circ h$ where h is a K-quasiconformal self-map of the disk and $|g'|$ is bounded away from zero uniformly. We can always take $K < 8$, independent of f [11], [22]. This factorization is closely related to 3-dimensional hyperbolic geometry and actually originates in a theorem of Dennis Sullivan's about boundaries of hyperbolic 3-manifolds (we shall explain this later on).
- **Fast approximation of conformal maps:** Suppose Ω is a plane domain bounded by a simple n-gon P. To apply the Schwarz-Christoffel formula one needs to know the conformal preimages of the vertices, but there is no simple formula for these. In practice they are often found by an iterative method from some initial guess (e.g., n evenly distributed points on the circle). In [29] I show that in time $O(n)$ we can compute n points on the unit circle which

are close to the true preverticies in the sense that there is a K-quasiconformal map of the disk sending our points to the true prevertices and that K is independent of the geometry of P. So far as I know, there was no previous way of making a good initial guess for the conformal prevertices (this is stated in [7], [67]). I also show the CRDT algorithm of Driscoll and Vavasis [49] has the same uniform approximation property (although it is not an $O(n)$ algorithm).

• **Bowen's dichotomy:** Bowen [34] proved that the limit set Λ of a quasiconformal deformation of a cocompact Fuchsian group is either a circle or has Hausdorff dimension > 1. This was extended to cofinite groups by Sullivan [86]; to divergence type groups by myself [10]; and shown to fail for convergence type groups by Astala and Zinsmeister [3], [4], [5]. In the latter case they show a rectifiable, but non-circular, limit set is possible. In [12] I show that if G is a convergence group with bounded injectivity radius then there is a deformation whose limit set has dimension one, but is nowhere rectifiable. The proof depends on estimates from [14] for deforming a Riemann surface with exponential decay of the corresponding Beltrami data.

• **Ruelle's property:** Ruelle [81] proved that if $\{G_t\}$ is an analytic family of deformations of a cocompact Fuchsian group G, then $\dim(\Lambda(G_t))$ is real analytic in t. Astala and Zinsmeister [3], [4], [5] gave examples of convergence groups where this fails. In [21] I give a condition for Ruelle's property to fail and in [28] I show this criterion holds for many infinitely generated groups, e.g., any divergence type group with injectivity radius bounded below.

• **δ-stable groups:** If G is a finitely generated Fuchsian group, then $\dim(\Lambda(G')) = \delta(G')$ for every quasiFuchsian deformation G' of G (δ is the Poincaré exponent). In [15], I give examples of infinitely generated Fuchsian groups G so this holds and other examples for which it fails. The proof depends on estimates of Schwarzian derivatives from [14] and [17].

• **Escaping geodesics:** In [24] I show that $\dim(\Lambda) = \max(\dim(\Lambda_b), \dim(\Lambda_\ell))$ for any Kleinian group; here Λ_b corresponds to geodesics which remain bounded for all time and Λ_ℓ are those which escape to infinity at linear speed. Thus these two opposite behaviors are in some sense generic.

• **Rudin's orthogonality conjecture:** $f \in H^\infty(\mathbb{D})$ is orthogonal if $\{f^n\}_1^\infty$ is an orthogonal sequence in the Hardy space H^2. Walter Rudin conjectured that only inner functions with $f(0) = 0$ are orthogonal, but in [25] I disprove this (C. Sundberg independently did this as well). I also characterize the images of Lebesgue measure under such functions; one surprising consequence is that the Bergman space embeds isometrically in the Hardy space via a composition operator.

• **Minimal sets for quasiconformal maps:** A set E is QS-minimal if $\dim(E) = \inf_f \dim(f(E))$ where the infimum is over all quasisymmetric images of E. In [19] Jeremy Tyson and I obtain new examples by constructing "locally minimal" sets, i.e., sets whose dimension can be lowered, but only by maps with large dilatation. In [18] we answer a question of Juha Heinonen by constructing a set where the dimension can be lowered, but the (positive) infimum is never attained.

• **Miscellaneous:** In [9] I prove that a closed curve in the plane which is homogeneous with respect to biLipschitz self-maps must be a quasicircle, generalizing results from [56], [57], [64]. In [16] (joint work with V. Ya. Gutlyanskii, O. Martio and M. Vuorinen) we give an integral condition on the dilatation of a map in $\mathbb{R}^n$, $n > 2$ which implies differentiability and is slightly weaker than the well known result of Lehto in $n = 2$. This result is partially motivated by a question of Curt McMullen on the rigidity of Kleinian groups; his conjugating maps do not satisfy Lehto's condition in the plane, but their extensions to 3-space satisfy our condition at certain points of the limit set and hence are differentiable at these points. This is another example of 3-dimensional techniques solving a 2-dimensional problem.

PROJECT DESCRIPTION

The proposed work has three parts: (1) constructing conformal collapsing maps which identify certain given sets to points and are conformal away from these sets (special cases include conformal welding problems, Koebe's circle domain conjecture and the construction of certain dynamical objects); (2) the geometry of Kleinian limits sets (e.g., the Ahlfors conjecture, dimension of limit sets, the behavior of escaping geodesics); and (3) connections between 3-dimensional hyperbolic geometry, computational geometry and numerical conformal mappings. We will describe problems covering a broad range of topics and difficulty. Some problems may be too hard to attack currently, but I hope they put the more accessible ones into context and suggest relations between areas that have not previously been connected.

1. Conformal collapsing

• **Moore's theorem:** A decomposition $\mathcal{C}$ of a closed set K is a collection of pairwise disjoint closed sets whose union is all of K. It is called upper semi-continuous if a sequence of elements of $\mathcal{C}$ which converges in the Hausdorff metric must converge to a subset of another element of $\mathcal{C}$. If, in addition, $K = \mathbb{C}$ and all elements of $\mathcal{C}$ are continua which don't separate the plane we call $\mathcal{C}$ a Moore decomposition after R.L. Moore. He proved in [74] that quotienting the plane by such a decomposition (i.e. identify each set to a point) gives the plane again. Given a decomposition $\mathcal{C}$, let $\Omega(\mathcal{C})$ be the interior of the set of singletons and call $\mathcal{C}$ conformal if the quotient map in Moore's theorem can be chosen to be conformal on Ω (in this case we call the quotient map a conformal collapsing). Not every Moore decomposition is conformal: if $\mathcal{C}$ contains only the closed disk $\{|z| \leq 1\}$ and singletons then we would get a conformal map from $\mathbb{D}^*$ to a punctured plane, which is impossible by Liouville's theorem. Which Moore decompositions are conformal? When is the quotient map unique up to Möbius transformations? These questions probably do not have simple answers, but we can seek interesting special cases which do.

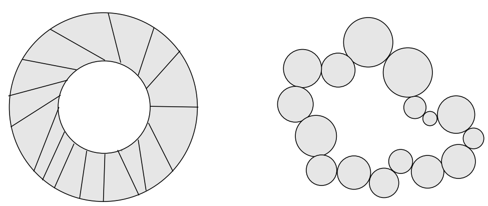
• **Conformal welding and Koebe's theorem:** Let $h : \mathbb{T} \to \mathbb{T}$ be a homeomorphism (all circle homeomorphisms are assumed to be orientation preserving). Foliate the annulus $A = \{1 \leq |z| \leq 2\}$ with smooth curves which connect x to $2h(x)$ for every $x \in \mathbb{T}$. Taking singletons outside A, this gives a Moore decomposition $\mathcal{M}$ of the plane. Note that $\mathcal{M}$ is conformal iff h is a conformal welding (if q is the conformal collapsing take $f(z) = q(z)$ on $\mathbb{D}$ and $g(z) = q(2z)$ on $\mathbb{D}^*$;

then $h = g^{-1} \circ f$ on $\mathbb{T}$). Thus a special case of characterizing conformal Moore decompositions is:

Problem 1. *Characterize the circle homeomorphisms which are conformal weldings. Characterize those for which the curve* Γ *is unique up to Möbius transformations.*

Not every homeomorphism is a conformal welding: the closure of the graph of $\sin(1/x)$ divides the plane into two domains and the conformal maps of $\mathbb{D}$, $\mathbb{D}^*$ onto these domains induce a circle homeomorphism h. This map is not in CW; otherwise one could construct a conformal map from the complement of a line segment to the complement of a point, contradicting Liouville's theorem. On the other hand, $h \in \mathrm{GCW}(\mathbb{T})$, [20]. Other examples not in CW are given in [78], [88]. A famous sufficient condition for h to be a conformal welding is quasisymmetry [1], i.e., there is an $M < \infty$ so that $M^{-1} \leq |h(I)|/|h(J)| \leq M$ whenever $I, J \subset \mathbb{T}$ are adjacent arcs of equal length. This condition can be weakened slightly (see [46], [69], [70]).

Recently, I have made progress using Koebe's circle domain theorem to construct conformal collapsing maps. Koebe's theorem says that any finitely connected plane domain can be conformally mapped to a circle domain (i.e., all boundary components are circles or points). To see the connection, choose n equally spaced points $\{x_k\}_1^n \subset \mathbb{T}$ and let $\Omega_{n,\epsilon}$ be the union of $\mathbb{D}$, $2\mathbb{D}^*$ and an ϵ-neighborhood of the curve connecting x_k to $2h(x_k)$, $k = 1, \dots, n$, from the foliation of A described above. By Koebe's theorem, this domain is conformally equivalent to a circle domain. Taking $\epsilon \to 0$ we obtain a closed chain of tangent circles, whose complement consists of two domains, Ω_n and Ω_n^*. Thus we have conformally collapsed finitely many of the leaves our decomposition to points. If the chains remain bounded as $n \to \infty$, then at most a bounded



number of disks are larger than any given $\delta > 0$. Using this, I prove in [20] that for any h there are non-degenerating sequences of conformal maps $\{f_n\}, \{g_n\}$ so that $|f_n(x) - g_n(h(x))| \to 0$ except for countably many points $x \in \mathbb{T}$.

Conjecture 2. *Every circle homeomorphism* $h \in \mathrm{GCW}(\mathbb{T} \setminus E)$ *for some countable set* E.

The problem is to extract subsequences from the sequences above with boundary values converging except possibly on a countable set. Currently, I can prove this with an exceptional set of zero logarithmic capacity. Perhaps we can make use of other "countability" results such as Moore's triod theorem [75], Collingwood's symmetry theorem for prime ends [44], or the Bagemihl ambiguous point theorem [6]. Some other conformal welding problems that may be tractable include:

Problem 3. *Characterize conformal welding sets (i.e., sets E so that $h \in GCW(E)$ for every h). Is there one of positive logarithmic capacity? Every compact set of zero capacity is a CW-set.*

Problem 4. *Characterize the Fuchsian groups G for which every G-invariant homeomorphism is a conformal welding. This is true for every finitely generated group of the first kind. Is true for any infinitely generated group?*

Conjecture 5. *For every h and $\epsilon > 0$ there is a $H \in \mathrm{CW}$ so that* $\mathrm{cap}(\{h \neq H\}) < \epsilon$.

Peter Jones and I are also interested in generalizing quasisymmetric conformal weldings to more general kinds of conformal collapsings. We say a decomposition $\mathcal{C}$ of the circle is induced by a map $f : \mathbb{T} \to \mathbb{C}$ if $\mathcal{C} = \{f^{-1}(x) : x \in \mathbb{C}\}$. We call $\mathcal{C}$ conformal if f is conformal on $\mathbb{D}^*$.

Problem 6. *Characterize the decompositions of $\mathbb{T}$ which are induced by conformal maps $f : \mathbb{D}^* \to \Omega$, where Ω is a John domain.*

See [77] for the definition and basic properties of John domains. They are an important class of domains which occur in analysis and dynamics [36], [73]. It follows from the standard quasiconformal theory that Ω is the complement of a quasiarc iff it corresponds to a decomposition of $\mathbb{T}$ of the form $\mathcal{C} = \{(x, h(x)) : x \in \mathbb{T}\}$ for some quasisymmetric involution h of the circle. This is a special case of a condition Peter Jones and I call a quasisymmetric decomposition, involving the behavior of logarithmic capacity in the decomposition. We want to prove our condition characterizes John domains and then extend it to Hölder domains if possible (which should have nice applications in complex dynamics).

- **The generalized Moore-Koebe conjecture:** We say a Moore decomposition is a Koebe decomposition if every element is either a closed disk or a point. As noted earlier, not every Moore decomposition can be conformal, but perhaps the following is true.

Conjecture 7. *Every Moore decomposition is conformally equivalent to a Koebe decomposition.*

In particular, only countable many elements would not be collapsed to points, so (in some sense) there should only be countably many obstructions to any Moore decomposition being conformal. We say $\mathcal{M}$ is conformally equivalent to $\mathcal{K}$ if there is a bijection $f : \mathcal{M} \to \mathcal{K}$ such that (1) f is conformal on $\Omega(\mathcal{M})$ and (2) if $\{E_n\} \subset \mathcal{M}$ converges (in the Hausdorff metric) to $E \in \mathcal{M}$ then $f(E_n) \to f(E) \in \mathcal{K}$. One can show Conjecture 7 implies Conjecture 2. It also implies the following famous problem:

Conjecture 8 (Koebe's conjecture)**.** *Every domain is conformally equivalent to a circle domain.*

Koebe's conjecture follows from Conjecture 7 because the decomposition of $\partial\Omega$ into its connected components is a Moore decomposition [74]. The best results on the Koebe conjecture so far are due to He and Schramm [62], [63]. In [20] a sketch is given to show that Conjecture 2 implies any planar domain is conformally equivalent to one whose complement has only countable many components which are not points. Thus conformal welding problems are closely related to Koebe's conjecture.

2. Limit sets of Kleinian groups

We start with some basic definitions. A Kleinian group G is a discrete group of isometries acting on the hyperbolic 3-ball, $\mathbb{B}$. The quotient $M = \mathbb{B}/G$ is a hyperbolic 3-orbifold. A discrete group acting on the hyperbolic disk is called a Fuchsian group. In either case, the accumulation set of any orbit is called the limit set, $\Lambda \subset S^2 = \partial\mathbb{B}$, and it splits into two disjoint subsets: the conical limit set Λ_c (corresponding to the radial segments which return to some compact set modulo G infinitely often) and the escaping limit set Λ_e. The group is called elementary if the limit set is finite. The complement $\Omega = S^2 \setminus \Lambda$ of the limit set is called the ordinary set of G. The critical exponent of the Poincaré series of G is $\delta = \inf\{s : \sum_{g\in G} \exp(-s\rho(0, g(0))) < \infty\}$, and it is a theorem of Peter Jones and myself [30] that $\delta = \dim(\Lambda_c)$ for all non-elementary groups. It is well known (e.g., [87]) that if $\delta \geq 1$ then $\lambda_0 = \delta(2 - \delta)$, where λ_0 is the base eigenvalue of the Laplacian on M. We let $C(\Lambda) \subset \mathbb{B}$ denote the hyperbolic convex hull of the limit set. Then $C(M) = C(\Lambda)/G \subset M$ is called the convex core of M. G is called geometrically finite if the unit neighborhood of $C(M)$ in M has finite volume (this is equivalent to there being a finite sided fundamental domain for G in $\mathbb{B}$). These form a "nice" class of finitely generated groups which are well understood. Unlike the case of Fuchsian groups, there are finitely generated Kleinian groups which are not geometrically finite. These are the geometrically infinite groups. Finally, G is called topologically tame if M is homeomorphic to the interior of a compact manifold with boundary.

• **Building limit sets by conformal collapsing:** One of my motivations for considering conformal collapsing is to build limit sets of Kleinian groups directly, without taking limits of "simpler" groups. One case where I believe this will work are the so called Koebe groups. A geometrically finite group G is a Koebe group if the ordinary set of G has a simply connected G-invariant component (i.e., is a "B-group") and every other component of Ω is a round disk. Koebe groups were introduced by Maskit in [71], [72] where he proved that every geometrically finite B-group is conformally similar to a Koebe group, i.e., is conjugate by a quasiconformal homeomorphism of the sphere which is conformal on the invariant component. (Note the similarity to Conjecture 7.)

Here is a plan to build a Koebe group from scratch. Take a Riemann surface $R = \mathbb{D}/G$ and a collection of disjoint simple geodesics $\{\gamma_k\} \subset R$. Let $\Gamma \subset \mathbb{D}$ be all lifts of $\cup_k \gamma_k$ to the disk and assume that any two components of Γ are at least hyperbolic distance $\delta > 0$ apart.

Conjecture 9. *Suppose $\Gamma \subset \mathbb{D}$ is a δ-separated collection of infinite hyperbolic geodesics (not necessarily invariant under a group). Then there is a continuous map $F : S^2 \to S^2$ which is conformal on $\mathbb{D}^*$ such that (1) F maps each component of Γ to a point, and (2) F maps each* *component of* $\mathbb{D} \setminus \Gamma$ *to a disk. If, in addition,* Γ *is invariant under a Fuchsian group* G*, then for any* $g \in G$, $F \circ g \circ F^{-1}$ *is Möbius, i.e.,* $F(\mathbb{T})$ *is the limit set of a Kleinian group.*

The proof of Conjecture 9 should be similar to my approach to conformal welding: take ϵ-neighborhoods of n components of Γ, apply Koebe's theorem and show the limit exists as $\epsilon \to 0$ and $n \to \infty$. This will require certain estimates using capacity and quasiconformal mappings. One interesting aspect is that unlike Maskit's construction using the Klein-Maskit combination theorem, our group G need not be finitely generated and the geodesics we collapse need not be closed on the quotient surface. In this case the tangent points between round components need not be parabolic fixed points. I am not aware that this possibility has been considered before. If this works out, the next step will be to use conformal collapsing to build limit sets of finitely generated, but geometrically infinite groups (proving such limit sets are locally connected).

• **Where is dimension minimized on the boundary of Teichmuller space?** Another reason for considering Koebe groups is the special role they play on the boundary of Teichmüller space. Fix a finitely generated Fuchsian group G and consider the Hausdorff dimension of the limit set as a function on the corresponding Teichmüller space T_G. In [30] Peter Jones and I proved that this is a lower semi-continuous function on the closure of T_G which only attains its minimum value 1 at G and is equal to its maximum, 2, exactly at the geometrically infinite groups on ∂T_G. Thus the dimension attains a minimum value somewhere on the boundary of Teichmüller space: the value is strictly between 1 and 2 and is attained at some geometrically finite group.

Conjecture 10. *The minimum of* $\dim(\Lambda)$ *on* ∂T_G *is attained at a Koebe group.*

Why Koebe groups? In some sense these are the "simplest" groups on the boundary of T_G, so perhaps their limit sets are "smallest" in terms of dimension. By Maskit's theorem it suffices to prove the first statement of the following conjecture.

Conjecture 11. *If* G' *is conformally similar to a Koebe group* G *then* $\dim(\Lambda(G')) \geq \dim(\Lambda(G))$. *Equality occurs iff the groups are conjugate by a Möbius transformation.*

For Fuchsian groups (which are a special case of Koebe groups), the first part is trivial for topological reasons and the second is Bowen's dichotomy discussed in the summary of previous work. Can we prove Conjecture 11 without using group invariance? We say Ω is a Bowen domain if $\dim(f(\partial\Omega)) \geq \dim(\partial\Omega)$ for every quasiconformal map of S^2 which is conformal on Ω.

Conjecture 12. *The invariant component of a Koebe group is a Bowen domain.*

One can construct examples of Bowen domains by putting a QS-minimal set (see summary of previous work) into the boundary, but other examples are not obvious. Proving a set is minimal for quasiconformal maps usual involves finding a "large" path family in the set. Perhaps these ideas can be modified to study Bowen domains by also allowing certain paths crossing though

Ω. Can we find an "interesting" geometric condition which implies Ω is Bowen (e.g., $\partial\Omega$ has "branching" at all points and all scales)? Assuming the dimension function on $\overline{T_G}$ is minimized at a Koebe group, it is natural to try to identify which group.

Conjecture 13. *The minimizing Koebe group corresponds to collapsing a single geodesic loop.*

Which loop to choose? The shortest one? This seems unlikely in general; if R had several equal length shortest geodesics, there is no reason why the corresponding Koebe groups should all have the same dimension, so by making a small deformation of the surface we could get a new surface with a unique shortest geodesic γ so that collapsing γ did not minimize dimension. On the other hand, perhaps this is true if there is one very short geodesic.

Conjecture 14. *There is $c = c(n) > 0$ so the following holds. Suppose that $R = \mathbb{D}/G$ is a compact surface of genus n and γ is closed simple geodesic on R so that $\ell(\gamma) \leq c\ell(\gamma')$ for any other geodesic γ'. Then collapsing γ yields a Koebe group which minimizes dimension on ∂T_G.*

• **Heat kernels and the Ahlfors conjecture:** After considering the "smallest" groups on the boundary of Teichmüller space, we turn to the "largest" ones: the geometrically infinite groups. Some well known problems state that these groups have the same nice properties that geometrically finite groups do. Among these problems are

Conjecture 15 (Marden's conjecture). *If G is finitely generated, then it is topologically tame.*

Conjecture 16 (The Ahlfors conjecture). *If G is finitely generated, then $\Lambda = S^2$ or $|\Lambda|_2 = 0$.*

Conjecture 17. *If G is finitely generated and non-elementary, then $\delta = \dim(\Lambda)$.*

Conjecture 18. *If G is finitely generated, but geometrically infinite then $\delta = 2$.*

In Conjecture 16, $|\cdot|_2$ denotes 2-dimensional Lebesgue measure. It is known that 15 $\Rightarrow$ 16 $\Rightarrow$ 17 $\Leftrightarrow$ 18 (see [26], [27], [30], [35]). Using heat kernels, Peter Jones and I proved [30] that for finitely generated, geometrically infinite groups, the limit set always has dimension 2 (which is a weaker version of Conjecture 18). If $\delta = 2$ there is nothing to do since we proved $\delta \leq \dim(\Lambda)$ for all non-elementary groups. If $\delta < 2$ then the base eigenvalue satisfies $\lambda_0 > 0$ which implies the heat kernel $k(x, y, t)$ decays exponentially fast in time. If we start a Brownian motion deep inside the convex core (which we can do since the convex core is non-compact in this case), we can use this estimate to prove the Brownian motion never leaves the convex core with high probability. Lifting to the ball, this means that with positive probability, Brownian motion leaves the ball without ever leaving the hyperbolic convex hull of the limit set. Thus Brownian motion hits Λ with positive probability which implies Λ has positive area.

This argument shows that Λ has zero area iff $\lim_{t\to\infty} \int_{C(M)} k(x, y, t)dy = 0$, i.e., iff a Brownian motion eventually exits the convex core of M almost surely. Thus the Ahlfors conjecture can be restated in terms of heat kernel estimates. If G is topologically tame and has a positive lower bound on its injectivity radius then Peter Jones and I proved a more precise rate of decay, namely $\int_{C(M)} k(x,y,t)dy \le C/\sqrt{t}$ [31]. Does this last estimate characterize topological tameness? What if we allow thin parts? The heat kernel approach gives a new way to attack these conjectures and provides a natural way of comparing them.

• **The escaping limit set has full dimension:** Fernández and Melián [54] proved that in an infinite area Riemann surface with no Green's function, the geodesics rays escaping to ∞ (with a given base point) have dimension 1. What about higher dimensions?

Conjecture 19. *If M is an infinite volume hyperbolic n-manifold with no Green's function then the escaping geodesics have dimension $n-1$.*

I can prove this with a positive lower bound on the injectivity radius. The proof involves constructing an appropriate harmonic function with bounds on the gradient. In general we will require a better understanding of harmonic functions in thin parts of hyperbolic manifolds.

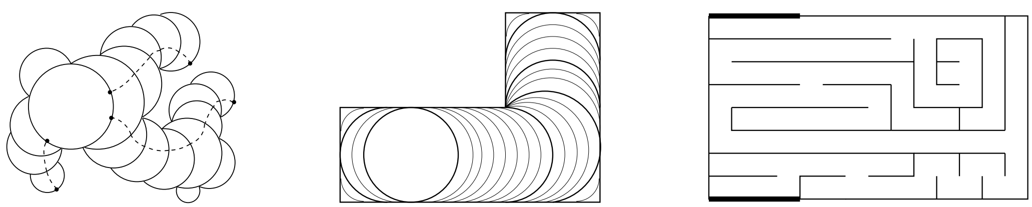
• **The law of the iterated logarithm for Kleinian groups:** Another problem involving harmonic functions in thin parts is to determine the exact Hausdorff guage function for limit sets of geometrically infinite, but topologically tame Kleinian groups (the dimension is 2 by [30]). Peter Jones and I proved in [31] that if G also has injectivity radius bounded away from zero, then the limit set Λ has positive, finite Hausdorff measure for the function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$, extending work of Sullivan [84]. Is this still true if M has thin parts? The proof in [31] depends on a careful estimate of the Green's function for M. If M has a thin part, there is a n^{-3} chance of Brownian motion exiting the thin part at distance n from where it entered. Thus the behavior of Green's function on the manifold should be modeled by the random walk on the integers with these transition probabilities. It is also still open to find the dimension (in terms of guage functions) of the conical limit set, even with a lower bound on the injectivity radius.

3. Computational, hyperbolic and conformal geometry

• **Domes and ι:** Suppose Ω is a simply connected plane domain. The dome of Ω is the surface S_Ω in $\mathbb{R}^3_+$ which is the upper envelope of all semi-spheres whose base is a disk contained in Ω. Equivalently, S_Ω is the boundary of the hyperbolic convex hull in $\mathbb{R}^3_+$ of $\Omega^c = S^2 \setminus \Omega$. Thurston observed that there is an isometry ι from S_Ω with its hyperbolic path metric to the hyperbolic disk. Restricting this map to the boundary $\partial S_\Omega = \partial\Omega$ gives a map $\iota : \partial\Omega \to \mathbb{T}$ (at least if $\partial\Omega$ is a Jordan curve; in general we have to be more careful about the boundary values, but this is standard material). Sullivan (in a special case) [85] and Epstein and Marden (in general) [51] proved there is a K-quasiconformal extension of $\iota : \Omega \to \mathbb{D}$ with K independent of Ω.

Problem 20. *What is the best K for the Sullivan-Epstein-Marden theorem?*

It is currently known that $2.1 < K < 7.82$ (by Epstein-Markovic [52] and myself [22]). The map ι also has a simple geometric description in the plane. Suppose that $\Omega \subset \mathbb{C}$ is a finite union of disks. Then Ω can be written as the union of one "root" disk D_0 and finite number of crescents (domains bounded by two circular arcs). We can map one arc of a crescent to the other by a unique elliptic Möbius transformation. Equivalently, follow the circles which are orthogonal to both boundary arcs. Composing these maps we get an explicit map $\iota : \partial\Omega \to \partial D_0$ (see left figure below). This is the same map as we discussed above. A similar construction can be done for a general simply connected domain, but the details are more involved (using the medial axis, a geometric object we will describe later, one can foliate $\Omega \setminus D_0$ by circular arcs and ι is obtained by following paths orthogonal to this foliation; see center figure below). Thus ι gives a purely geometric way to estimate conformal modulus up to a uniform factor, e.g., we can use ι to estimate the modulus of the path family connecting the two bold arcs in the maze on the right below, and the answer will be accurate up to a factor of at least 7.82. Geometric estimates of conformal quantities are basic tools of geometric function theory; does this type of estimate have any interesting applications?



• **The factorization theorem and Brennan's conjecture:** Knowing the best K in Problem 20 is interesting for classical complex analysis. This is because I proved in [13] that the extension of $\iota : \Omega \to \mathbb{D}$ can be chosen to be locally Lipschitz with a uniform bound. This implies there is a $K < \infty$, so that every conformal map $f : \mathbb{D} \to \Omega$ can be factored as $f = g \circ h$ where h is a K-quasiconformal self-map of the disk and $|g'|$ is bounded away from zero uniformly.

Problem 21. *What is the optimal K for factorization?*

If this holds for $K = 2$ then Astala's sharp L^p estimate [2] for quasiconformal maps implies

Conjecture 22 (Brennan's conjecture)**.** *If $g : \Omega \to \mathbb{D}$ is conformal then $g' \in L^p$ for all $p < 4$.*

Epstein and Markovic have an example where K for Sullivan's theorem is > 2.1, but what about the factorization theorem? Can we prove the factorization theorem with L^p estimates on g' and h' directly without estimating the QC constant? For example, the map ι has a Beltrami dilatation μ of a special form; can this be exploited to give a better estimate than Astala's general result? Is Astala's theorem sharp for maps which preserve the circle, e.g., if $h : \mathbb{T} \to \mathbb{T}$ is M-quasisymmetric, what is the best L^p space the derivative of a quasiconformal extension of h to the disk can be in? If such estimates fail then we should be able to find a domain with bad estimates and try to iterate the geometry to obtain a counterexample to Brennan's conjecture.

• **The parameter problem:** Is it useful to have an easily computed rough approximation for the Riemann map? If Ω is bounded by a simple polygon P with vertices $\mathbf{v} = \{v_1, \ldots, v_n\}$, then the conformal map of the disk onto Ω is given by the Schwarz-Christoffel formula $f(z) = A + C\int^z \prod_{k=1}^n (1 - \frac{w}{z_k})^{\alpha_k - 1} dw$, where $\pi\alpha_k$ is the interior angle at vertex v_k and $\mathbf{z} = \{z_1, \dots, z_n\}$ are the preimages of the vertices. To apply the formula we must first know the points $\mathbf{z}$, which are usually found by some iterative procedure. My results imply that $d_{QC}(\mathbf{z}, \iota(\mathbf{v})) \leq \log 7.82$, where $d_{QC}(\mathbf{z}, \mathbf{w})$ is the minimal $\log K$ so that there is a K-QC map of the disk mapping $\mathbf{z}$ to $\mathbf{w}$. Moreover, I show in [29] that if P has n vertices then $\mathbf{w} = \iota(\mathbf{v})$ can be computed with work $\leq Cn$. The novel feature is that the constants $K = 7.82$ and C are independent of n and the geometry of the polygon. So far as I know, there are no results of this type already known. The standard practice when solving for $\mathbf{z}$ is to start with n equally spaced points on the circle. Is $\iota(\mathbf{v})$ a better starting point in practice, i.e., does it lead to a solution more often or more quickly using known iteration methods?

One such method is due to Davis [47] Suppose the n angles are fixed. Given a current guess $\mathbf{w}$ for the prevertices one computes the Schwarz-Christoffel map with these parameters and compares the resulting polygon $P_{\mathbf{w}}$ to the given polygon P. The spacing between the prevertices is increased or decreased according to whether the side lengths of $P_{\mathbf{w}}$ are too short or too long compared to the sides of P. As Howell points out in [67], lengthening the side of a polygon with given angles can sometimes decrease its harmonic measure, and so Davis's method can fail to converge even if we start arbitrarily close to the correct answer. However, it does seem to work in many cases, and is nice because it uses the geometry of the problem (rather than calling a general non-linear equation solver). Can we use more sophisticated geometric properties of harmonic measure to improve it?

One possible improvement is to use the ι map. Given n distinct points on the circle, we can triangulate the disk with $n-2$ triangles and these points as vertices. Now form $n-3$ quadrilaterals with domain $\mathbb{D}$ by using vertices from adjacent triangles. The logarithms of the conformal moduli of these quadrilaterals determines the n points up to a Möbius transformation and hence identifies n-tuples (modulo Möbius transformations) with $\mathbb{R}^{n-3}$. We can define a map Ψ from $\mathbb{R}^{n-3}$ to itself as follows. Given $n-3$ real numbers, find a corresponding n-tuple and apply Schwarz-Christoffel to get a polygon. Then apply ι to get a n-tuple on the circle. (The intermediate polygon might be self-intersecting, but this is not a serious problem). It is easy to see this map is well defined and the correct prevertices are a solution of $\Psi(\mathbf{z}) = \iota(\mathbf{v})$. I can show Ψ is onto and it is natural to ask

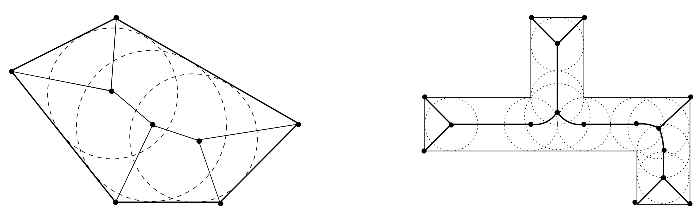
Problem 23. *Is the the map Ψ 1-to-1? Is Ψ a diffeomorphism? Does $F(\mathbf{z}) = \|\Psi(\mathbf{z}) - \iota(\mathbf{v})\|_2^2$ have a unique local minimum? How fast can we solve $\Psi(\mathbf{z}) = \mathbf{w}$ numerically?*

In [49] Driscoll and Vavasis defined a very similar map using cross ratios and observed that in practice the iteration $\mathbf{w}_0 = \iota(\mathbf{v})$, $\mathbf{w}_{n+1} = \mathbf{w}_n - \Psi(\mathbf{w}_n)$ always converged linearly to the correct answer and they asked for an explanation. I can prove $\|\Psi(\mathbf{w}) - \mathbf{w}\| \leq C$, so Ψ looks like the identity on large scales. If this also held on small scales (e.g., its derivative was close to the identity) this would explain the experimental observations of Driscoll and Vavasis.

Problem 24. *Prove the iteration of Driscoll and Vavasis converges to the conformal prevertices.*

More generally, find any algorithm which converges to the true conformal prevertices with an estimate which is independent of the geometry of the polygon. Many more ideas from hyperbolic geometry and geometric function theory could be applied. If we fix all but one coordinate in Ψ, the resulting function need not be monotone [29], but perhaps some variation is. For example, Ψ depended on a choice of triangulation; what happens if we average over many triangulations? In [29] I show how angle scaling (as developed by Epstein, Marden and Markovic) reduces the global convergence problem to a local one.

• **The medial axis:** The medial axis of Ω, $\mathrm{MA}(\Omega)$, consists of the centers of all disks in Ω whose boundaries hit $\partial\Omega$ in two or more points. For polygons the medial axis is a finite tree, and is the boundary of a Voronoi diagram (it bounds regions in P which consist of points closest to certain arcs of P). The medial axis (or skeleton) was introduced by Blum [33] as a way of



describing shapes in biology and there is now a large literature applying it to problems such as mesh generation, computer vision, robotic motion, sphere packing and radiosurgery, e.g., [40], [55], [58], [65], [66], [68], [79], [80], [89], [91]. Other applications are described on David Eppstein's website `www.cs.uci.edu/~eppstein/gina/medial.html`. Many papers deal with algorithms for computing the medial axis (e.g. [37], [38], [45], [53], [82], [83], [92]), but relatively few investigate it as a mathematical object (e.g., [39], [41], [42], [43], [90]) and many of these place strong restrictions on the domain, e.g., piecewise analytic boundary.

The dome of Ω is easily computed from the medial axis and a theorem of Chin, Snoeyink and Wang [38] that the medial axis can be computed in time $O(n)$ is used in my proof that the conformal prevertices can be approximated in time $O(n)$. In [29] I point out that the medial axis can be identified with the dual $\mathbb{R}$-tree of the bending lamination, a well studied measured lamination on the dome of Ω, but the implications of this remain to be investigated. (An $\mathbb{R}$-tree is a metric space in which any two points are connected by a unique arc and this arc is isometric to a segment in $\mathbb{R}$. Any finite tree is a $\mathbb{R}$-tree, but so is $\mathbb{R}^2$ if we define $d(x,y) = |x-y|$ if x, y lie on the same ray from the origin and $= |x| + |y|$ otherwise. See [8], [76].)

There are numerous practical and theoretical problems that are open. Which $\mathbb{R}$-trees occur as medial axes? The medial axis itself is very unstable under perturbations. For example, changing the unit disk to a regular n-gon changes the medial axis from a single point to a union of n

unit length radial segments. Is there a practical alternate version of the medial axis which is more stable under perturbations (what happens if we replace the Euclidean metric by the $\mathbb{R}$-tree metric induced by the bending lamination)? What kind of perturbations? Davis's algorithm used edge lengths to update the prevertex approximations, but perhaps it would be better to compare medial axes.

I would like to use the medial axis to study properties of domains. For example, in [12] I use a Möbius invariant version of Peter Jones' β's based on how closely the boundary approximates a circle. Can we characterize the smoothness of a domain using these and the medial axis? Stephen Vavasis suggested using these ideas to estimate the L^2 norm of harmonic conjugation on a domain (i.e., he wants to find geometric conditions on Ω that assure $|f'|dx$ is an A_2 weight, where $f : \mathbb{D} \to \Omega$ is conformal). Manipulating the medial axis allows us to perturb a domain in a way where we can control both the Euclidean and conformal geometry (approximately), so Peter Jones suggested it might be a good tool for attacking the following well known problem.

Question 25. *Is the space of chord-arc curves connected?*

Recall that a chord-arc curve is a locally rectifiable curve such that the shorter arc connecting two points x, y has length at most $C|x - y|$ for some fixed C. The arc length parameterization has derivative in BMO and this identifies chord-arc curves with an open subset of BMO. This is the topology being considered above.

4. Educational and broader impact of the proposal

One impact of the proposal will be to establish new collaborations between different disciplines within mathematics by finding new connections between them. For example, we have shown Koebe's circle domain theorem is connected with conformal weldings, that Moore's results in planar topology generalized to conformal collapsing would have applications in dynamics, that problems about Kleinian groups (such as the Ahlfors conjecture) can be attacked by heat kernel estimates and probability, and conformal mapping problems (such as Brennan's conjecture) are related to 3-dimensional hyperbolic geometry. The work on heat kernels and Kleinian groups has already stimulated some work by Chang, Qing and Yang on conformally flat manifolds and the connection between hyperbolic geometry and conformal mappings is cited as motivation for some of the recent work of Epstein, Marden and Markovic [50], [52].

The proposal also seeks to establish new collaborations between investigators in pure mathematics and in applied mathematics and computer science. We already have seen that the hyperbolic viewpoint leads to new ideas for numerically computing conformal maps and there are numerous applications of such algorithms (e.g., see the book of Driscoll and Trefethen [48] for examples such as fluid flow, mesh generation and electrostatics). I hope the proposal will lead both to new theorems (rigorous estimates for convergence) and improved algorithms which will have practical benefits (starting from a guaranteed good guess, using known geometric properties of harmonic measure to design better iterative schemes). Investigating the medial axis may also have a broad impact. As noted in the proposal, it already has many applications as a compact description of the shape of a region, but has instability which is undesirable from a computational standpoint. Explaining this instability (or decreasing it using ideas from hyperbolic geometry), might be useful in practice. Conversely, the literature on the medial axis in computational geometry may offer new ideas to low dimensional topology and analysis, such as new conformal invariants and new ideas for how to classify and perturb domains.

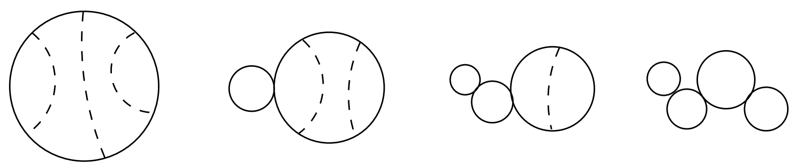
Establishing new connections between disciplines will apply more points of view to important problems, and no doubt a lot of interesting (and unanticipated) work will result on both theoretical and practical problems. My initial results on conformal prevertices seem to be of interest to the applied mathematicians I have discussed them with (e.g., Nick Trefethen called them a "substantial contribution" and Steve Vavasis thought they should have applications beyond conformal mappings). I have been encouraged to submit my paper to one of the SIAM journals and have also been invited to submit the work to the ACM Symposium on Computational Geometry. If nothing else, this indicates that there is interest in the proposed work and real opportunities for new interdisciplinary collaborations.

The proposed work also promotes the integration of research and education at both the graduate and undergraduate levels. Currently I am teaching a graduate course on Kleinian limit sets which is closely related to the proposal. In the near future I plan to teach a graduate course on numerical conformal mappings and hyperbolic geometry. The material is accessible enough that students can understand the problems quickly and are exposed to the interaction of analysis, geometry, topology, probability and some computational aspects as well. I also plan to use this material in MAT 402, an undergraduate seminar which our math majors are required to take. Our department is starting a new masters program in mathematics education; perhaps a topic like the medial axis could be used in secondary education to illustrate how elementary geometric concepts can yield important "real life" applications (e.g., computer recognition of printed characters, optimal placement of medical radiation, ...) which require "abstract" mathematics to understand.

A former graduate student, Zsuzsanna Gonye, finished her thesis on the Hausdorff dimension of escaping geodesics. She showed that for a large class of hyperbolic manifolds the geodesics which escape to infinity at linear speed (or at any specified sub-linear speed) have full dimension. She also shows that a theorem of Steger and myself [32] for finitely generated Fuchsian groups fails for some divergence type groups. She is now at Polytechnic University in Brooklyn, NY. A current student, Karyn Lundberg, is working on problems related to Koebe's theorem and conformal welding. Another, Hrant Hakobyan, will work on the construction of Koebe groups and their properties as described in the proposal. Another student, Anirban Dutta, is still working on his qualifying exams, but has expressed an interest in problems involving Brownian motion or geometric classifiers (such as the medial axis). A pending supplement to my previous grant will be used to support visits to Stony Brook by Melkana Brakalova. She will work with me on geometric properties of solutions of the generalized Beltrami equation.

Many of the problems we have discussed could be understood by undergraduates and can be investigated by numerical means. Here are a couple of ideas that might be suitable for undergraduate research projects.

• **Compute conformal weldings:** The map $(\log(\frac{z-1}{z+1}))^{-1}$ maps the outside of the unit disk to the outside of the two tangent disks $D(\frac{1}{2},\frac{1}{2}) \cup D(-\frac{1}{2},\frac{1}{2})$, identifying the points $\pm i$ to the tangent point at 0. Given an involution h of $\mathbb{T}$ with fixed points ± 1 and n points in the upper half circle, use Möbius conjugates of this map to inductively identity x_k and $h(x_k)$ to a point z_k. Obtain a chain of "blobs" which satisfy the conformal welding relation at the tangent points. Write a program to compute the polygonal arc $z_1, \dots, z_n$. If h is quasisymmetric show it converges to a quasi-arc.



• **Numerically estimate the constant in Sullivan's theorem:** Suppose Ω is a polygon and let $f : \mathbb{D} \to \Omega$ be a conformal map (this can be computed using available software) and let $\iota : \partial\Omega \to \mathbb{T}$ be the restriction of the isometry from S to $\mathbb{D}$. Compute the quasisymmetric constant of $\iota \circ S$ for various examples. Search for the worse case.

• **Iterating Schwartz-Christoffel and ι:** Fix n angles and write a program to iterate the map $\Psi = \iota \circ S$ on n-tuples in the circle. Do the iterates converge? Is there a fixed point? Similarly investigate the closely related map $S \circ \iota$ which maps polygons to polygons. Do they remain bounded under iteration? How do sides degenerate? Near an internal angle of size α the conformal map to the disk acts like $z^{\pi/\alpha}$ and ι behaves like z if $\alpha > \pi$ and $z^{1/\sin(\alpha)}$ if $\alpha \leq \pi$. Thus $\iota \circ S$ will look repelling nears points with angle $< \pi$ and attracting near angles $> \pi$.

• **Dimension of Koebe groups:** Numerically estimate the dimension of limits set (perhaps via the Poincaré series) when R is a once punctured torus (the boundary of this Teichmüller space is well understood and we can write down generators for the groups).

• **Computing Riemann maps:** Use $\iota(\mathbf{v})$ as a starting point for known iterative schemes for finding conformal prevertices. How does this compare to starting from equally spaced points?

• **Bending of hyperbolic convex surfaces:** It is of great interest to estimate the "bending" of the dome. If Ω is a finite union of disks, $B(t)$ is the sum of angles between geodesic faces in S encountered by a geodesic segment of length t on S. In this case the bending angles and the hyperbolic distance can be interpreted in terms of the Delaunay triangulation of the vertices of Ω. Estimate the bending for various examples using available software in computational geometry for computing Delaunay triangulations.

References

[1] L.V. Ahlfors. *Lectures on quasiconformal mappings.* Math. Series, no. 10,. Van Nostrand, 1966.

[2] K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994.

[3] K. Astala and M. Zinsmeister. Mostow rigidity and Fuchsian groups. *Comptes Rendu Acad. Sci., Paris*, 311:301–306, 1990.

[4] K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fuchsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212, 1994.

[5] K. Astala and M. Zinsmeister. Abelian coverings, Poincaré exponent of convergence and holomorphic deformations. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(1):81–86, 1995.

[6] F. Bagemihl. Curvilinear cluster sets of arbitrary functions. *Proc. Nat. Acad. Sci. U. S. A.*, 41:379–382, 1955.

[7] L. Banjai and L.N. Trefethen. A multipole method for Schwarz-Christoffel mapping of polygons with thousands of sides. *SIAM J. Sci. Comp.* to appear.

[8] M. Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.

[9] C. J. Bishop. Bi-Lipschitz homogeneous curves in $\mathbb{R}^2$ are quasicircles. *Trans. Amer. Math. Soc.*, 353(7):2655–2663 (electronic), 2001.

[10] C. J. Bishop. Divergence groups have the Bowen property. *Ann. of Math. (2)*, 154(1):205–217, 2001.

[11] C. J. Bishop. BiLipschitz approximations of quasiconformal maps. *Ann. Acad. Sci. Fenn. Math.*, 27(1):97–108, 2002.

[12] C. J. Bishop. Non-rectifiable limit sets of dimension one. *Rev. Mat. Iberoamericana*, 18(3):653–684, 2002.

[13] C. J. Bishop. Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture. *Ark. Mat.*, 40(1):1–26, 2002.

[14] C. J. Bishop. Quasiconformal mappings of Y-pieces. *Rev. Mat. Iberoamericana*, 18(3):627–652, 2002.

[15] C. J. Bishop. δ-stable Fuchsian groups. *Ann. Acad. Sci. Fenn. Math.*, 28(1):153–167, 2003.

[16] C. J. Bishop, V. Ya. Gutlyanskiĭ, O. Martio, and M. Vuorinen. On conformal dilatation in space. *Int. J. Math. Math. Sci.*, (22):1397–1420, 2003.

[17] C. J. Bishop and P. W. Jones. Compact deformations of Fuchsian groups. *J. Anal. Math.*, 87:5–36, 2002.

[18] C. J. Bishop and J. T. Tyson. Conformal dimension of the antenna set. *Proc. Amer. Math. Soc.*, 129(12):3631–3636 (electronic), 2001.

[19] C. J. Bishop and J. T. Tyson. Locally minimal sets for conformal dimension. *Ann. Acad. Sci. Fenn. Math.*, 26(2):361–373, 2001.

[20] C.J. Bishop. Conformal welding and Koebe's theorem. submitted to Annals of Math.

[21] C.J. Bishop. A criterion for the failure of Ruelle's property. submitted to Erg. Thy. and Dyn. Sys.

[22] C.J. Bishop. An explicit constant for Sullivan's convex hull theorem. In *Proceedings of the Ahlfors-Bers Colloqium, 2001.* Amer. Math. Soc., Providence, RI. to appear.

[23] C.J. Bishop. Interpolating sets for conformal maps. submitted to J. Amer. Math. Soc.

[24] C.J. Bishop. The linear escape limit set. to appear in Proc. Amer. Math. Soc.

[25] C.J. Bishop. Orthogonal functions in H^∞. submitted to Pacific J. Math.

[26] C.J. Bishop. Minkowski dimension and the Poincaré exponent. *Michigan Math. J.*, 43(2):231–246, 1996.

[27] C.J. Bishop. Geometric exponents and Kleinian groups. *Invent. Math.*, 127(1):33–50, 1997.

[28] C.J. Bishop. Big deformations at infinity. 2003. to appear in Illinois J. of Math.

[29] C.J. Bishop. A fast approximation to the Riemann map. 2003. Preprint.

[30] C.J. Bishop and P.W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math*, 179:1–39, 1997.

[31] C.J. Bishop and P.W. Jones. The law of the iterated logarithm for Kleinian groups. In *Lipa's legacy (New York, 1995)*, pages 17–50. Amer. Math. Soc., Providence, RI, 1997.

[32] C.J. Bishop and T. Steger. Representation-theoretic rigidity in PSL(2, **R**). *Acta Math.*, 170(1):121–149, 1993.

[33] H. Blum. A transfomation for extracting new descriptors of shape. In W.W. Dunn, editor, *Proc. Symp. Models for the perception of speech and visual form*, pages 362–380, Cambridge, 1967. MIT Press.

[34] R. Bowen. Hausdorff dimension of quasicircles. *Publ. I. H. E. S.*, 50:11–25, 1979.

[35] R.D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6(1):1–35, 1993.

[36] L. Carleson, P. W. Jones, and J.-C. Yoccoz. Julia and John. *Bol. Soc. Brasil. Mat. (N.S.)*, 25(1):1–30, 1994.

[37] C.-S. Chiang and C.M. Hoffmann. The medial axis transform for 2D regions. *ACM Transactions on graphics*, 1982.

[38] F. Chin, J. Snoeyink, and C. A. Wang. Finding the medial axis of a simple polygon in linear time. *Discrete Comput. Geom.*, 21(3):405–420, 1999.

[39] H. I. Choi, S. W. Choi, and H. P. Moon. Mathematical theory of medial axis transform. *Pacific J. Math.*, 181(1):57–88, 1997.

[40] H.I. Choi, C.Y. Han, and J.-H. Yoon. Medial axis transform distance and its applications 2000, April 2000 Kyung Moon, Seoul, Korea. In *Geometric modeling and computer graphics*, pages 65–69.

[41] S. W. Choi and H.-P. Seidel. Hyperbolic Hausdorff distance for medial axis transformation. *Graphical Models*, 63:369–384, 2001.

[42] S. W. Choi and H.-P. Seidel. Linear one-sided stability of MAT for weakly injective domain. *J. Math. Imaging Vision*, 17(3):237–247, 2002.

[43] S.W. Choi and S.-W. Lee. Stability analysis of medial axis transform. In *Proc. 15th ICPR Barcelona, Spain*, volume 3, pages 139–142, 2000.

[44] E. F. Collingwood. Cluster sets of arbitrary functions. *Proc. Nat. Acad. Sci. U.S.A.*, 46:1236–1242, 1960.

[45] T. Culver, J. Keyser, and D. Manocha. Accurate computation of the medial axis of a polyhedron. In *Proceedings of the fifth ACM symposium on Solid modeling and applications, June 8-11, 1999, Ann Arbor, MI USA*, pages 179–190, 1999.

[46] G. David. Solutions de l'équation de Beltrami avec $\|\mu\|_\infty = 1$. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 13(1):25–70, 1988.

[47] R.T. Davis. Numerical methods for coordinate generation based on Schwarz-Christoffel transformations. In *4th AIAA Comput. Fluid Dynamics Conf., Williamsburg VA*, pages 1–15, 1979.

[48] T. A. Driscoll and L. N. Trefethen. *Schwarz-Christoffel mapping*, volume 8 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, Cambridge, 2002.

[49] T. A. Driscoll and S. A. Vavasis. Numerical conformal mapping using cross-ratios and Delaunay triangulation. *SIAM J. Sci. Comput.*, 19(6):1783–1803 (electronic), 1998.

[50] D. B. A. Epstein, A. Marden, and V. Markovic. Quasiconformal homeomorphisms and the convex hull boundary.

[51] D.B.A. Epstein and A. Marden. Convex hulls in hyperbolic spaces, a theorem of Sullivan and measured pleated surfaces. In *Analaytical and geometric aspects of hyperbolic spaces*, London Math. Soc. Lecture Notes Series 111, pages 113–253. Cambridge University Press, 1987.

[52] D.B.A. Epstein and V. Markovic. The logarithmic spiral: A counterexample to the $K = 2$ conjecture. to appear in Annals of Math.

[53] G. Evans, A. Middleditch, and N. Miles. Stable computation of the 2D medial axis transform. *Internat. J. Comput. Geom. Appl.*, 8(5-6):577–598, 1998.

[54] J. L. Fernández and M. V. Melián. Escaping geodesics of Riemannian surfaces. *Acta Math.*, 187(2):213–236, 2001.

[55] C. Gaudeau, M. Boiron, and J. Thouvenot. Squelettisation et anamorphose dans l'étude de la dynamique des déformations des structures: application à l'analyse de la motricité gastrique. In *Recognition of shapes and artificial intelligence (Second AFCET-IRIA Cong., Toulouse, 1979), Vol. III (French)*, pages 57–63. IRIA, Rocquencourt, 1979.

[56] M. Ghamsari and D.A. Herron. Higher dimensional Ahlfors regular sets and chordarc curves in $\mathbf{R}^n$. *Rocky Mountain J. Math.*, 28(1):191–222, 1998.

[57] M. Ghamsari and D.A. Herron. Bi-Lipschitz homogeneous Jordan curves. *Trans. Amer. Math. Soc.*, 351(8):3197–3216, 1999.

[58] H. N. Gursoy and N. M. Patrikalakis. Automated interrogation and adaptive subdivision of shape using medial axis transform. *Advances in Engineering Software and Workstations*, 13(5/6):287–302, 1991.

[59] D. H. Hamilton. Generalized conformal welding. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 16(2):333–343, 1991.

[60] D. H. Hamilton. Simultaneous uniformisation. *J. Reine Angew. Math.*, 455:105–122, 1994.

[61] D.H. Hamilton. Length of Julia curves. *Pacific J. Math.*, 169(1):75–93, 1995.

[62] Z.-X. He and O. Schramm. Fixed points, Koebe uniformization and circle packings. *Ann. of Math. (2)*, 137(2):369–406, 1993.

[63] Z.-X. He and O. Schramm. Koebe uniformization for "almost circle domains". *Amer. J. Math.*, 117(3):653–667, 1995.

[64] D.A. Herron and V. Mayer. Bi-Lipschitz group actions and homogeneous Jordan curves. *Illinois J. Math.*, 43(4):770–792, 1999.

[65] C.M. Hoffmann. Computer vision, descriptive geometry and classical mechanics. In *Computer Graphics and Mathematics*, pages 229–244. Springer Verlag, Eurographics Series, 1992.

[66] C.M. Hoffmann. *Geometric Approaches to Mesh Generation*, volume 75 of *IMA Volumes in Mathematics and its Applications*, pages 31–52. Springer Verlag, 1995.

[67] L. Howell. *Computation of Conformal Maps by Modified SchwarzChristoffel Transformations.* PhD thesis, 1990.

[68] R. A. Jinkerson, S. L. Abrams, L. Bardis, C. Chryssostomidis, A. Clement, N. M. Patrikalakis, and F. E. Wolter. Inspection and feature extraction of marine propellers. *Journal of Ship Production,*, 9(2):88–106, 1993.

[69] O. Lehto. Homeomorphic solutions of a Beltrami differential equation. In *Festband 70. Geburtstag R. Nevanlinna*, pages 58–65. Springer, Berlin, 1966.

[70] O. Lehto. Remarks on generalized Beltrami equations and conformal mappings. In *Proceedings of the Romanian-Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings (Braşov, 1969)*, pages 203–214. Publ. House of the Acad. of the Socialist Republic of Romania, Bucharest, 1971.

[71] B. Maskit. On the classification of Kleinian groups. I. Koebe groups. *Acta Math.*, 135(3–4):249–270, 1975.

[72] B. Maskit. On the classification of Kleinian groups. II. Signatures. *Acta Math.*, 138(1–2):17–42, 1976.

[73] C. T. McMullen. Kleinian groups and John domains. *Topology*, 37(3):485–496, 1998.

[74] R. L. Moore. Concerning upper semi-continuous collections of continua. *Trans. Amer. Math. Soc.*, 27(4):416–428, 1925.

[75] R. L. Moore. Concerning triods in the plane and the junction points of plane continua. *Proc. Nat. Acad. Sci*, 14:85–88, 1928.

[76] J. W. Morgan. Λ-trees and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 26(1):87–112, 1992.

[77] R. Näkki and J. Väisälä. John disks. *Exposition. Math.*, 9(1):3–43, 1991.

[78] K. Oikawa. Welding of polygons and the type of Riemann surfaces. *Kōdai Math. Sem. Rep.*, 13:37–52, 1961.

[79] N. M. Patrikalakis and T. Maekawa. *Shape Interrogation for Computer Aided Design and Manufacturing.* Springer Verlag, 2002.

[80] G. X. Ritter. Topology of computer vision. In *Proceedings of the 1987 Topology Conference (Birmingham, AL, 1987)*, volume 12, pages 117–158, 1987.

[81] D. Ruelle. Repellers for real analytic maps. *Ergod. Th. & Dym. Sys.*, 2:99–107, 1982.

[82] E.C. Sherbrooke, N. M. Patrikalakis, and E. Brisson. An algorithm for the medial axis transform of 3d polyhedral solids. *IEEE Transactions on Visualization and Computer Graphics*, 2(1):44–61, 1996.

[83] E.C. Sherbrooke, N. M. Patrikalakis, and F.-E. Wolter. Differential and topological properties of medial axis transforms. *CVGIP: Graphical Model and Image Processing*, 58(6):574–592, 1996.

[84] D. Sullivan. Growth of positive harmonic functions and Kleinian group limit sets of planar measure 0 and Hausdorff dimension 2. In *Geometry Symposium (Utrecht 1980)*, Lecture Notes in Math. 894, pages 127–144. Springer-Verlag, 1981.

[85] D. Sullivan. Travaux de Thurston sur les groupes quasi-Fuchsiens et les variétés hyperboliques de dimension 3 fibrées sur S^1. In *Bourbaki Seminar, Vol. 1979/80*, pages 196–214. Springer, Berlin, 1981.

[86] D. Sullivan. Discrete conformal groups and measureable dynamics. *Bull. Amer. Math. Soc.*, 6:57–73, 1982.

[87] D. Sullivan. Related aspects of positivity in Riemannian geometry. *J. Differential Geom.*, 25(3):327–351, 1987.

[88] J. V. Vainio. Conditions for the possibility of conformal sewing. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (53):43, 1985.

[89] J. Wang. Medial axis and optimal locations for min-max sphere packing. *J. Comb. Optim.*, 4(4):487–503, 2000.

[90] E.-F. Wolter. Cut locus and the medial axis in global shape interrogation and representation. 1993. MIT, Dept. of Ocean Engineering, Design Laboratoy Memorandum 92-2.

[91] Q. J. Wu. Sphere packing using morphological analysis. In *Discrete mathematical problems with medical applications (New Brunswick, NJ, 1999)*, volume 55 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 45–54. Amer. Math. Soc., Providence, RI, 2000.

[92] C.-K. Yap. An $O(n \log n)$ algorithm for the Voronoĭ diagram of a set of simple curve segments. *Discrete Comput. Geom.*, 2(4):365–393, 1987.