PROJECT SUMMARY

The PI, Christopher Bishop, will study the geometric properties of conformal mappings in the plane and quasiconformal mappings in space, focusing on the expansion properties of such maps and investigating various applications to geometric function theory, dynamics and topology. There are three main areas that will be considered. First, the connection between three dimensional hyperbolic geometry and the expanding properties of two dimensional conformal maps. The PI has shown that a result of Dennis Sullivan's concerning the geometry of convex bodies in hyperbolic three space implies a factorization theorem for conformal mappings in the plane and this, in turn, implies uniform bounds on the amount of contraction a conformal map in the plane can have. Finding the best constants in the factorization theorem has consequences for well known problems such as dimension distortion, integral means and Brennan's conjecture. Second, the PI will continue his work on limit sets of Kleinian groups, a natural and important class of fractal sets. The questions here are mainly to estimate the fractal dimension of these sets and study the behavior of the dimension as the group is deformed. Again, three dimensional techniques enter naturally, via the connection to hyperbolic 3-manifolds. Third, the PI will work on the metric properties of harmonic measures, particularly results which quantify the idea that harmonic measure cannot be concentrated on a small set. Problems include the lower density conjecture, stability of harmonic measure and the growth rate of diffusion limited aggregation. A few other questions involving quasiconformal and biLipschitz maps are also considered.

RESULTS FROM PRIOR NSF SUPPORT

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Deformations of complex structures

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I worked on several problems involving conformal and quasiconformal mappings. The main results are summarized below and the corresponding papers and preprints are listed at the end. Numbered citations refer to this list; others refer to the bibliography at the end of the proposal.

- **Bowen's dichotomy:** In 1979 Rufus Bowen [Bow79] proved that if G is a cocompact Fuchsian group then the limit set of any quasiconformal deformation G' is either a circle or has Hausdorff dimension strictly greater than one. Later Dennis Sullivan [Sul82] extended this to cofinite groups and in [1] I show that it holds iff G is divergence type. Astala and Zinsmeister proved Bowen's property fails for all convergence groups; there is always a deformation to a group which has a rectifiable, but non-circular, limit set. In [2] I show that Bowen's property can fail in a different way: if G is a convergence group with bounded injectivity radius then there is a deformation whose limit set has dimension one, but is nowhere rectifiable. The idea of the proof is a construction of G-invariant Beltrami coefficients with L^∞ norm ϵ which cause an ϵ sized "wiggle" in the limit set at a specified point and scale.
- **Ruelle's property:** David Ruelle [Rue82] proved that if $\{G_t\}$ analytic family of deformations of a cocompact group G, then the dimension of the limit set is real analytic in t. Astala and Zinsmeister [AZ90], [AZ94], [AZ95] gave examples of convergence groups where this fails and asked about divergence type groups. In [3] I give a condition for Ruelle's property to fail ("big deformations near infinity") which holds for any infinitely generated divergence group with a positive lower bound for its injectivity radius. In [4], I give examples of some infinitely generated divergence groups which do not have this property; for such groups Ruelle's property remains open.
- **Critical exponent and dimension:** For geometrically finite Kleinian groups, it is well known that $\delta = \dim(\Lambda)$, where δ denotes the critical exponent of the Poincaré series and $\dim(\Lambda)$ is the Hausdorff dimension of the limit set. In [5] Peter Jones and I extend this to all quasiFuchisan groups where the Beltrami coefficient of the deformation has compact support modulo G. We also show that the escaping part of the limit set has sigma finite linear measure. This is an application of the non-linear L^2 theory for Schwarzian developed in [BJ94].
- **Factorization and Sullivan's theorem:** In [6], I show there is a $K < \infty$ so that any conformal map $f : \mathbb{D} \to \Omega$ can be factored as $f = g \circ h$ where h is a K-quasiconformal self-map of the disk and $|g'|$ is bounded away from zero uniformly. This reduces many problems about conformal maps to questions about quasiconformal self-maps of the disk. The solution of Bowen's problem is an example where this idea gives a new result. Connections to Brennan's conjecture and other

problems will be explained later in the proposal. Among the geometric consequences is that any bounded quasicircle can be mapped to a circle by a Lipschitz mapping of the plane.

Suppose Ω is a simply connected proper subdomain of the plane and let $C(\partial\Omega)$ denote the hyperbolic convex hull of its boundary in the 3-dimensional hyperbolic upper half-space. There is a boundary component S of the convex hull which faces Ω, and the intrinsic hyperbolic path metric on S makes it isometric to the hyperbolic disk, i.e., there is an isometry $\iota : S \to \mathbb{D}$. Sullivan's convex hull theorem says there is an absolute $K < \infty$ (independent of Ω) and a K-biLipschitz map σ from Ω (with its hyperbolic metric) to S with its path metric which is the identity on $\partial\Omega = \partial S$. This was proven by Epstein and Marden in general with $K \approx 80$. In [7] I show that σ can be taken to have biLipschitz constant ≈ 13.1 and quasiconformal constant ≈ 7.8. In [6] I show that if Sullivan's theorem holds with quasiconformal constant K then the factorization theorem holds with $K + \epsilon$ for any $\epsilon > 0$. This gives a new and unexpected connection between 3-dimensional geometry and conformal maps.

- **Rudin's orthogonality conjecture:** A bounded, holomorphic function f on the unit disk is called orthogonal if the sequence of powers $\{f^n\}$ is orthogonal in the Hardy space H^2. Inner functions (i.e., $|f| = 1$ a.e. on the boundary) with $f(0) = 0$ have this property. Walter Rudin conjectured that these were the only orthogonal functions, but in [7] I construct non-inner examples. Such an example was obtained independently by Carl Sundberg. In addition, I characterize which measures occur as the push forward of Lebesgue measure under an orthogonal function. In particular, normalized area measure occurs. This has the surprising consequence that the Bergman space embeds isometrically as a closed subspace of the Hardy space via a composition operator.
- **BiLipschitz approximations:** If f is a K-quasiconformal self-map of the upper half-plane, it is well known (e.g., [BA56] or [DE86]) that there is a M-quasiconformal map g which agrees with f on the boundary and which is also biLipschitz with respect to the hyperbolic metric. It follows from results on uniquely extremal maps in [BLMM98] that one cannot take $M = K$ in general, but in [8] I show that we can take $M = K + \epsilon$ for any $\epsilon > 0$.
- **Minimal sets for quasiconformal maps:** A set E is minimal if $\dim(E) = \inf_f \dim(f(E))$ where the infimum is overall quasisymmetric images of E. Some sets (such as line segments) are obviously minimal, but examples with fractional dimension α are harder to construct (see [Tys]). In [11] Jeremy Tyson and I obtain examples by constructing "locally minimal" sets, i.e., sets whose dimension can be lowered, but only by maps with large dilatation. In [10] we answer a question of Juha Heinonen by constructing a fractal (the antenna set) where the dimension can be lowered, but the (positive) infimum is never attained.
- **BiLipschitz homogeneous curves:** A set X is called homogeneous for a class of maps F if given any pair $x, y \in X$ there is a map $f \in F$ with $f(x) = y$. In [12] I prove that a closed curve in the plane which is homogeneous with respect to biLipschitz self-maps must be a quasicircle. This

extends results from [GH98], [GH99], [HM99] and implies several different characterizations of such curves in terms of various parameterizations.

• **The local spectrum of the Cauchy integral:** The paper [13] is work with A. Böttcher, Yu. I. Karlovich and I. Spitkovsky. The paper gives a symbol calculus for deciding when the Cauchy integral operator on a Carleson curve with A_p weight is Fredholm.

• **Quasiconformal maps in space:** Lehto [Leh87] showed that if the quasiconformal dilatation of map f of $\mathbb{R}^n$ satisfies $\int |K_f(z) - 1|^2 |z|^{-n} dxdy < \infty$ when $n = 2$, then f is differentiable at the origin. In [14] (joint work with V. Ya. Gutlyanskii, O. Martio and M. Vuorinen) we show a slightly weaker result for $n > 2$. This result is partially motivated by a question of Curt McMullen on the differentiability of quasiconformal conjugations between Kleinian groups at the so called deep points; Peter Jones and I [BJ97b] showed that although such maps do not satisfy Lehto's condition in the plane, they have extensions which do satisfy it in 3-space and so are differentiable.

• **Smallest sets hit by Brownian motion:** In [15] Yuval Peres and I prove that for any analytic set A on the positive axis, the infimum of the Hausdorff dimensions of compact sets in $\mathbb{R}^d$ which are hit by the Brownian image $B_d(A)$ with positive probability, is $d - 2\dim_p(A)$. Here B_d is Brownian motion in $\mathbb{R}^d$, $d \geq 2$, and $\dim_p$ is packing dimension.

Papers written with support of DMS-98-00924

preprints are available at `www.math.sunysb.edu/~bishop/new.html`

[1] Divergence groups have Bowen's property, submitted to *Annals of Math.*

[2] Nonrectifiable limit sets of dimension one, submitted to *Revista Mat. Iberoamericana*

[3] Compact deformations of Fuchsian groups, with Peter Jones, submitted to *J. d'Analyse*

[4] A criterion for the failure of Ruelle's property, submitted to *Inventiones Mat.*

[5] Surfaces which approximate the thrice punctured sphere, submitted to *Michigan Math. J.*

[6] Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture, submitted to *Arkiv för Mat*

[7] An explicit constant for Sullivan's convex hull theorem, submitted to *GAFA*

[8] BiLipschitz approximations to quasiconformal maps, submitted to *AASF*

[9] Orthogonal functions in H^∞, preprint 1998

[10] Conformal dimension of the antenna set, with Jeremy Tyson, to appear in *PAMS*

[11] Locally minimal sets for quasiconformal maps, with Jeremy Tyson, to appear in *AASF*

[12] BiLipschitz homogeneous curves are quasicircles, to appear in *TAMS*

[13] Local spectra of singular integral operators with piecewise continuous coefficients on composed curves, with A. Böttcher, Yu. I. Karlovich and I. Spitkovsky, *Math. Nachr.* 206(1999) 5-83.

[14] On conformal dilatation in space, with V. Ya. Gutlyanskii, O. Martio and M. Vuorinen, preprint 2000.

[15] The smallest sets hit by a Brownian image, with Yuval Peres, preprint 1998.

PROJECT DESCRIPTION

Many problems about conformal maps deal with how much expansion or contraction such a map can have. For example, Makarov's [Mak85] famous theorem says that a conformal map $f : \mathbb{D} \to \Omega$ cannot lower the dimension of a positive measure set on the unit circle, and hence the boundary values of a conformal map do not have too much contraction. The well known Brennan conjecture gives upper bounds for the area of the set where $|f'|$ is small, and so limits the contraction of f in a different way. Bowen's dichotomy (described in the summary of previous work) says a conformal map which deforms a Fuchsian group is either trivial or expands the circle in a uniform way.

In this proposal we consider a new approach to the expansion properties of conformal mappings. We will start with a factorization theorem which bounds the contraction of a conformal map in terms of the contraction of quasiconformal self-maps of the disk. This allows various questions, including Brennan's conjecture, to be reduced to finding the best factorization constant. One way of proving the factorization theorem is by using a theorem of Dennis Sullivan on three dimensional hyperbolic geometry and this leads to a variety of new questions and the introduction of the "convex hull measure" on the boundary of a simply connected domain. The next part of the proposal deals with limit sets of Kleinian groups. We will discuss some well known problems, such as the Ahlfors conjecture, and variety of new problems concerned with understanding the critical exponent of the group and the dimension of the limit set. The third part concerns metric properties of harmonic measure. The overall goal is to understand geometrically the essential support of harmonic measure. Because of the connection between harmonic measure and conformal mappings, results which say that small sets must have small harmonic measure correspond to saying a conformal map cannot contract too much. Thus these results also correspond to "expansion" results for conformal maps.

These three topics clearly have numerous interconnections. For example, my solution of Bowen's problem for limit sets in [1] was an outgrowth of my earlier work on harmonic measure and, in turn, led me to the factorization theorem and its application to problems like Brennan's conjecture.

Factorization, Sullivan's theorem and convex hulls

• **The factorization theorem and Brennan's conjecture:** Recall from the summary of previous work that there is a $K < \infty$, so that if $f : \mathbb{D} \to \Omega$ is a conformal map, then it can be factored as $f = g \circ h$ where h is a K-quasiconformal self-map of the disk and g is expanding in the sense that $|g'|$ is bounded away from zero uniformly on the disk. Thus $|f'| \geq C|h'|$ (for a quasiconformal map $h' = \limsup_{x \to y} |h(x) - h(y)|/|x - y|$). In other words, an arbitrary conformal map $\mathbb{D} \to \Omega$ can only contract as much as a K-quasiconformal self-map of the disk, for some universal $K < \infty$. Currently I can prove this with $K = 7.82$ and it is easy to show $K < 2$ is impossible.

Question 1. *Does the factorization theorem hold for $K = 2$? If not, what is the optimal K?*

Factorization allows various questions about arbitrary conformal maps to be reduced to analogous questions about quasiconformal self-maps of the disk (which should be much easier). For example, if factorization holds for $K = 2$ then Astala's area distortion estimate [Ast94] for quasiconformal maps says that $(h^{-1})'$ is in weak L^4. From this it is easy to deduce that $(f^{-1})'$ is in weak L^4 for any univalent map on $\mathbb{D}$. In particular, this implies

Conjecture 2 (Brennan's conjecture). *If $g : \Omega \to \mathbb{D}$ is conformal then $g' \in L^p$ for all $p < 4$.*

This well known problem from [Bre78] has been studied by many investigators. The best estimate of the optimal p is due to Bertilsson [Ber99], [Ber98] slightly improving a result of Pommerenke [Pom85a], [Pom85b]. To deduce Brennan's conjecture from the factorization theorem, one only needs the estimate near the boundary, i.e., $\lim_{\epsilon \to 0} \sup_{1-\epsilon<|z|<1} K_h(z) \leq 2$. Although this appears weaker than Question 1, can one show (by a "self-similar" construction) that it is equivalent?

• **Sullivan's convex hull theorem:** I was led to the factorization theorem by Sullivan's convex hull theorem: as described in the summary of previous work, this says there is a uniformly K_S-quasiconformal map σ from any simply connected domain Ω to a component S of the boundary of the hyperbolic convex hull of $\partial\Omega$ and this map extends to the identity on $\partial\Omega = \partial S$. Moreover, Thurston showed there is an isometry ι from S to the hyperbolic disk. In [6] I show that if Sullivan's theorem holds with constant K_S then so does the factorization theorem (one can take $g^{-1} = \iota \circ \sigma$). In [7] I show that we can take $K_S = 7.82$ in (the non-equivariant) Sullivan's theorem and this gives the best explicit constant I know for the factorization theorem.

Question 3. *Is $K = K_S$?*

If they are equal, then the convex hull mapping ι somehow picks out the optimal way of mapping $\partial\Omega$ to the circle with bounded derivative. Why should this happen? What is the optimal $K = K(p)$ in the factorization theorem if we require $(g^{-1})' \in L^p$ instead of L^∞?

My proof of $K_S = 7.82$ is an explicit construction of a map $S \to \Omega$, based on a foliation coming from a triangulation of S (following an idea of Epstein and Marden [EM87]) and the map satisfies many additional side conditions. Is there a more natural way to construct it, e.g., using a foliation coming from trajectories of a quadratic differential? Sullivan's theorem is equivalent to asking for the best quasiconformal extension to the disk of the mapping $f \circ \iota$ on the circle (f is a Riemann map $\mathbb{D} \to \Omega$). Can results on optimal quasiconformal extensions be of use here (e.g., [AH95])?

• **The equivariant version:** If Ω is invariant under a group of Möbius transformations, then we can require the map $\sigma : \Omega \to S$ to respect the group action. Define $K_G(\Omega)$ to be the infimum of all K's such that there is a K-quasiconformal map $\Omega \to S$ which commutes with the action of G and is the identity on $\partial\Omega$.

Question 4. *Suppose Ω is G-invariant. Is $K_G(\Omega) = K_S(\Omega)$?*

So far as I know, there is no reason to expect the optimal constants in the equivariant case to be the same as when we allow more general maps. Based on results of McMullen [McM89] concerning extending group invariant quasisymmetric maps, one might suspect they are different.

• **Integral means:** Suppose $f : \mathbb{D} \to \Omega$ is a univalent map and define the integral means

$$I(t,f) = \limsup_{r\to 1} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^t d\theta}{-\log(1-r)}, \qquad B(t) = \sup_f I(t,f),$$

where the supremum is taken over all univalent maps f. Equivalently, $B(t)$ is the smallest number β such that,

$$\int_0^{2\pi} |f'(re^{i\theta})|^t d\theta = O((\frac{1}{1-r})^\beta),$$

for all univalent f. The Brennan conjecture is equivalent to $B(-2) = 1$ (e.g., [Mak98]) and more generally, Kraetzer's conjecture states $B(t) = \max(|t|-1, t^2/4)$ [Kra96]. This has been the subject of much investigation, so I hope that a new approach via factorization might be of interest.

For $1 \leq K < \infty$, define the integral means for quasiconformal self-maps of the disk by

$$I(t,f) = \limsup_{r\to 1} \frac{\log \int_0^{2\pi} (\frac{1-|f(re^{i\theta})|}{1-r})^t d\theta}{-\log(1-r)}, \qquad B(K,t) = \sup_f I(t,f),$$

where the supremum is over all K-quasiconformal maps f of the unit disk to itself. The factorization theorem with constant K implies $B(t) \leq B(K,t)$ for $t \leq 0$, and so it would be interesting to compute $B(2,t)$. Some values are easy, e.g., in [6] I prove that if $t < -\frac{2}{K-1}$ then $B(K,t) = -(K-1)t - 1$. For values of t close to 0, the following is suggested by Kraetzer's conjecture.

Conjecture 5. *For* $-\frac{2}{K-1} \leq t \leq 0$, $B(K,t) = \frac{(K-1)^2}{4}t^2$.

Conjecture 6. *If* $t \leq 0$, *then* $B(t) = B(2,t)$.

Even if the factorization theorem is true for $K = 2$ it is not clear whether we should expect $B(t) = B(2,t)$. First of all, the set of 2-quasiconformal maps which arise from simply connected domains via factorization may only be a "small" set of all 2-quasiconformal self-maps of the disk, with strictly smaller integral means. Second, saying $B(t) = B(2,t)$ says that the "expanding" factor in the factorization theorem can be ignored, which may not be the case.

• **Distortion of dimension:** Carleson and Makarov prove in [CM94] that there is a $M < \infty$ such that for any conformal map f on the disk, $\dim(f(E)) \geq \dim(E)/(2 - 2M^{-1}\dim(E))$, and show that Brennan's conjecture implies this is true with $M = 4$. This would follow directly by using the factorization theorem and Astala's area distortion theorem again; this time in the form of his optimal dimension distortion estimates for K-quasiconformal maps. This states that if f is a K-quasiconformal map then $\dim(f(E)) \geq 2\dim(E)/(2K + (1-K)\dim(E))$. Taking $K = 2$ gives

Conjecture 7. *For any conformal* f *and* $E \subset \mathbb{T}$ *we have* $\dim(f(E)) \geq \dim(E)/(2 - \frac{1}{2}\dim(E))$.

S. Smirnov has shown Astala's result is not sharp for subsets of the circle, and so something even stronger than this conjecture should hold, but even proving this version would be interesting.

• **Dimension of the convex hull measure:** The isometry from $\mathbb{D}$ to the convex hull surface S defines a measure μ_{CH} (the "convex hull measure") by pushing normalized Lebesgue measure on the circle onto $\partial\Omega$. This is analogous to the usual harmonic measure ω defined by pushing Lebesgue measure forward by the Riemann mapping of $\mathbb{D}$ onto Ω. We define the dimension of a measure to be $\dim(\mu) = \inf\{\dim(E) : \mu(E^c) = 0\}$. Makarov's [Mak85] remarkable work shows that $\dim(\omega) = 1$ for harmonic measure on any simply connected domain.

Question 8. *Compute* $\sup_\Omega \dim(\mu_{CH})$, *(the sup is over all simply connected domains).*

I can prove the convex hull measure always has dimension ≥ 1 and the supremum above is strictly between 1 and 2. If Sullivan's theorem holds with constant K then μ_{CH} is the image of harmonic measure on Ω under a K-quasiconformal self-map of Ω. If this map could be extended to a K-quasiconformal map of the whole plane and if we could take $K = 2$ as conjectured, then Astala's dimension distortion estimate and the fact that $\dim(\omega) = 1$ implies $\dim(\mu_{CH}) \leq 4/3$. Although the map need not extend to the whole plane, perhaps it behaves as if it did, at least locally at almost every boundary point. One could make this heuristic rigorous if the following generalization of Makarov's theorem were true.

Conjecture 9. *If* $f : \mathbb{D} \to \Omega$ *is* K*-quasiconformal then there is a* $E \subset \mathbb{T}$ *of full measure such that* $\dim(f(E)) \leq 2K/(K+1)$.

It would suffice to show a K-quasiconformal map $\mathbb{D} \to \Omega$ satisfies a radial growth estimate $f'(re^{i\theta}) \leq r^{-\epsilon-(K-1)/2K}$ for any $\epsilon > 0$ for a.e. θ. Currently I can prove this is true for some exponent depending on K. The number 4/3 occurs in other conjectures related to conformal mappings, e.g., the Carleson-Jones conjecture on conformal dimension [CJ92] and Mandelbrot's conjecture on the frontier of Brownian motion (recently proven by Lawler, Schramm and Werner [GL00]). Does the dimension of the convex hull measure have a direct connection to the Carleson-Jones conformal dimension? Do random domains like the complementary components of Brownian motion maximize $\dim(\mu_{CH})$? Can Astala's theorem be used to explain the "4/3" in these problems?

Limit sets of Kleinian groups

A Kleinian group G is a discrete group of isometries acting on hyperbolic 3-space, $\mathbb{B}$. The quotient $M = \mathbb{B}/G$ is a hyperbolic 3-orbifold. A discrete group acting on the hyperbolic disk is called a Fuchsian group. In either case, the accumulation set of any orbit is called the limit set, Λ, and it splits into two disjoint subsets: the conical limit set Λ_c (corresponding to the radial rays which return to some compact set modulo G infinitely often) and the escaping limit set Λ_e. The group is called elementary if the limit set is finite. The critical exponent of the Poincaré series of G

is $\delta = \inf\{s : \sum_{g\in G} \exp(-s\rho(0, g(0))) < \infty\}$, and it is a theorem of Peter Jones and myself [BJ97a] that $\delta = \dim(\Lambda_c)$ for non-elementary groups. It is well known (e.g., [Sul87]) that if $\delta \geq 1$ then $\lambda_0 = \delta(2-\delta)$, where λ_0 is the base eigenvalue of the Laplacian on M. A Fuchsian group is called divergence type if $\sum_{g\in G} \exp(-\rho(0, g(0))) = \infty$ and is convergence type otherwise. Divergence type is equivalent to many other conditions, e.g., Λ_c having full Lebesgue measure, $R = \mathbb{D}/G$ having no Greens function and the geodesic flow being ergodic (e.g., see [Nic89]). We let $C(\Lambda) \subset \mathbb{B}$ denote the hyperbolic convex hull of the limit set and $C(M) = C(\Lambda)/G \subset M$ is the convex core of M. G is called geometrically finite if the unit neighborhood of $C(M)$ has finite volume. These form a "nice" class of finitely generated groups which are well understood. Finally, G is called topologically tame if M is homeomorphic to the interior of a compact manifold with boundary.

• **Heat kenels and the Ahlfors conjecture:** Some well known problems about finitely generated Kleinian groups state that they have the same nice properties that geometrically finite groups do. Among these problems are

Conjecture 10 (Marden's conjecture). *If G is finitely generated, then it is topologically tame.*

Conjecture 11 (The Ahlfors conjecture). *If G is finitely generated, then $\Lambda = S^2$ or* $\text{area}(\Lambda) = 0$.

Conjecture 12. *If G is finitely generated and non-elementary, then $\delta = \dim(\Lambda)$.*

It is known that $10 \Rightarrow 11$ ([Can93]) and $11 \Rightarrow 12$ ([BJ97a], [Bis96], [Bis97]). Moreover, I can prove that Conjecture 12 is equivalent to

Conjecture 13. *If G is finitely generated, but geometrically infinite then $\delta = 2$.*

The proof of this equivalence uses heat kernel estimates on $M = \mathbb{B}/G$: if $\delta < 2$ then $\lambda_0 > 0$ which implies the heat kernel decays exponentially fast, which is used to show the limit set has positive area if $C(M)$ has infinite volume. More generally, Λ has zero area iff $\lim_{t\to\infty} \int_{C(M)} k(x, y, t)dy = 0$, i.e., iff a Brownian motion eventually exits the convex core of M almost surely. Thus the Ahlfors conjecture can be restated in terms of heat kernel estimates. If G is topologically tame and has a positive lower bound on its injectivity radius then we get a more precise rate of decay, namely $\int_{C(M)} k(x, y, t)dy \leq C/\sqrt{t}$. Does this last estimate characterize topological tameness? What if we allow thin parts? The heat kernel approach gives a new way to attack these conjectures and provides a natural way of comparing them (according to the corresponding estimates of the heat kernel). Using heat kernels, Jones and I proved [BJ97a] that for finitely generated, geometrically infinite groups, the limit set always has dimension 2 (which is a weaker version of Conjecture 13).

Decay of the heat kernel is closely related to volume growth of the convex core near infinity. If G is topologically tame, each infinite end of $C(M)$ has linear volume growth (at least if we assume a positive lower bound for the injectivity radius). If $\delta = 2$, then $\lambda_0 = 0$ which should correspond to merely sub-exponential growth. Based on this we might hope that Conjecture 13 will be much easier to prove, but if the proof comes with concrete estimates of the volume growth or heat kernel decay, it might lead to a progress on the Ahlfors or Marden conjectures as well. Yair Minksy (also at Stony Brook) has recently proven even stronger results [Min99] in the case when G corresponds to a punctured torus group, verifying Thurston's ending lamination conjecture in that case. Perhaps combining our ideas will solve the conjectures.

• **Local connectivity:** If G is finitely generated, it is known in some cases that if Λ is connected then it is locally connected ([AM96], [Min94], [Kla99], [McM00]). A different idea for proving local connectivity is as follows. Suppose γ is a geodesic segment in $\partial C(M)$ which starts and ends at some base point z_0 and let $\tilde{\gamma}$ be the shortest homotopic curve in M also based at z_0. I can show Λ is locally connected if there is an f such that $\ell(\tilde{\gamma}) \geq f(\ell(\gamma))$ and $\int_1^\infty e^{-f} < \infty$. For geometrically finite groups one can take $f(t) = 2 \log t - C$. Is the same true for all finitely generated groups?

• **Critical exponent for quasi-Fuchsian groups:** For geometrically finite groups it is known that $\delta = \dim(\Lambda)$. Conjecture 12 claims the same is true of all finitely generated Kleinian groups. For infinitely generated groups it can certainly fail (even for Fuchsian groups), but we can still seek natural conditions under which it holds. We say that G' is a quasiFuchsian deformation of a Fuchsian group G if it is conjugate to G via a quasiconformal mapping of the plane.

Conjecture 14. *Suppose G' is a quasiFuchsian deformation of a divergence type Fuchsian group G. Then $\delta(G') = \dim(\Lambda(G'))$.*

One could conjecture that this holds merely if $\delta(G) = 1$, but I suspect that a counterexample can be constructed in this case. Since $\Lambda = \Lambda_c \cup \Lambda_e$, saying $\delta = \dim(\Lambda)$ is equivalent to $\dim(\Lambda_e) \leq \delta = \dim(\Lambda_c)$. Thus given a large set of escaping geodesics we want to show we can perturb them to become recurrent without decreasing the dimension of the set by much. For a divergence group almost every ray is recurrent, so this seems quite plausible.

• **The escaping limit set has full dimension:** Fernández and Melián [FM98] proved that in an infinite area Riemann surface with no Green's function, the geodesics rays escaping to ∞ (with a given base point) have dimension 1. What about higher dimensions?

Conjecture 15. *If M is an infinite volume hyperbolic n-manifold with no Greens function then the escaping geodesics have dimension $n-1$.*

The proof of Fernández and Melián is a construction of escaping geodesics based on a decomposition of a Riemann surface into geodesic subdomains, and does not seem to generalize to higher dimensions. A different approach which does generalize is to remove a ball B from M and choose a point $z_0 \in N = M \setminus B$. The resulting manifold (with boundary) has a Greens function. Move the pole of Greens function to infinity, renormalizing so the function always has value 1 at z_0.

Passing to the limit we get a positive harmonic function U on N. If M has a positive lower bound for its injectivity radius, estimates from [BJ97b] show U has bounded gradient. Lifting to the hyperbolic ball $\mathbb{B}$ (via the covering map for M) we get a harmonic function U on $\mathbb{B}$ (minus the lifts of B) with bounded gradient. Restricting to a stopping time region and using a Bloch martingale approximation to U, one shows U has radial limit ∞ on a set of dimension $n-1$ and these radii clearly project to escaping geodesics in M. This proves the conjecture in the "thick" case. If M has thin parts then U need not have bounded gradient and solving the problem requires a better understanding of harmonic functions in these parts.

- **The law of the iterated logarithm (LIL) for Kleinian groups:** Another problem which requires a better understanding of harmonic functions in thin parts is to determine the exact Hausdorff guage function for limit sets of geometrically infinite, but topologically tame Kleinian groups (the dimension is 2 by a result in [BJ97a]). Jones and I proved in [BJ97b] that if G also has injectivity radius bounded away from zero, then the limit set Λ has positive, finite Hausdorff measure for the function $\varphi(t) = t^2\sqrt{\log\frac{1}{t}\log\log\log\frac{1}{t}}$ extending work of Sullivan [Sul81]. The machinery we have developed shows that the size of the limit set can be computed if we know the rate of decay of the heat kernel $k(x,y,t)$ on the quotient manifold $M = B/G$. The next step is to uncover the correct result when the injectivity assumption fails, e.g., when M looks like $S\times\mathbb{R}$ for some surface S with a puncture. Then M has a thin part which can transport Brownian motion a long way; roughly speaking, there is a n^{-3} chance of exiting the thin part at distance n from where you enter. Thus the behavior of Greens function on the manifold should be approximated by the Greens function on the integers which has these transition probabilities. I plan to compute a Green's function for the walk and this will provide a good guess for the Green's function on M.
- **Ruelle's property:** We say a Fuchsian group G has Ruelle's property if for any analytic family of deformations $\{G_t\}$ of G, the dimension of the limit set is an analytic function of t.

Conjecture 16. *A Fuchsian group G has Ruelle's property iff it is finitely generated.*

Ruelle [Rue82] proved cocompact groups have the property, and Astala and Zinsmeister [AZ94], [AZ95] gave some infinitely generated examples where Ruelle's property fails. I can prove the conjecture if $R = \mathbb{D}/G$ has injectivity radius which is bounded and bounded away from zero. In particular, I show Ruelle's property fails if there is a sequence of G-invariant dilatations $\{\mu_n\}$ whose supports leave every compact set (modulo G), with $\sup_n \|\mu_n\|_\infty < 1$ and which all increase the dimension uniformly above 1 (I call this condition having "big deformations near infinity").

Question 17. *Which Fuchsian groups G have big deformations near infinity?*

I am looking for an answer in terms of the geometry of $R = \mathbb{D}/G$. For example, if G is divergence type and the injectivity radius of R has a positive lower bound, then there are such deformations

[4]. On the other hand, if R is an infinite union of Y-pieces and only a finite number of the boundary geodesics have length $\geq \epsilon$ for any ϵ, then the condition fails [5]. This gives a concrete class of examples to test Conjecture 16 with. Moreover, it shows an answer to Question 17 should be in terms of "how often" R "looks like" a thrice punctured sphere near infinity. This will involve defining a boundary for G in terms of limits of renormalizations and a notion of capacity on this boundary in terms of how much time a random geodesic spends near a given boundary group.

The support of harmonic measure

For a simply connected domain Ω in the plane, the harmonic measure is the image of normalized Lebesgue measure on the unit circle under a conformal map $f : \mathbb{D} \to \Omega$ (for more general domains one uses Greens function or Brownian motion, but many of the same questions remain interesting). Fix $x \in \partial\Omega$ and define a continuous branch of $\arg(z - x)$ on Ω. We say x is a twist point of Ω ($x \in \mathrm{Tw}(\Omega)$) if both

$$\liminf_{z\to x, z\in\Omega} \arg(z - x) = -\infty, \qquad \limsup_{z\to x, z\in\Omega} \arg(z - x) = +\infty.$$

On the other hand, we say x is an inner tangent ($x \in \mathrm{Tan}(\Omega)$) if there is a unique $\theta_0 \in [0, 2\pi)$ such that for every $0 < \epsilon < \pi/2$ there is a $\delta > 0$ such that :

$$\{x + re^{i\theta} : 0 < r < \delta, |\theta - \theta_0| < \pi/2 - \epsilon\} \subset \Omega$$

McMillan's twist point theorem [McM69] states that (with respect to harmonic measure) almost every boundary point of Ω is of one of these two types. Two results are of interest here. First there is Makarov's theorem [Mak85] that there is a subset $A \subset \mathrm{Tw}(\Omega)$ such that $\omega(A) = \omega(\mathrm{Tw}(\Omega))$ but such that A has zero 1-dimensional measure. The second is the theorem in [BJ90], that $\Gamma \cap \mathrm{Tw}(\Omega)$ has zero harmonic measure whenever Γ is a rectifiable curve. This says that even though harmonic measure on the twist points is concentrated on a set A of zero length, this set is so dispersed in the plane that it cannot lie on any rectifiable curve.

• **The lower density conjecture:** Makarov's proof implies that $\limsup_{r\to 0} \frac{\omega(D(x,r))}{r} = \infty$ at almost every $[\omega]$ twist point. (At almost every tangent point this ratio has a finite, nonzero limit.)

Conjecture 18. *At almost every (ω) twist point* $\liminf_{r\to 0} \frac{\omega(D(x,r))}{r} = 0$.

Thus at twist points the ratio should oscillate between 0 and ∞ as $r \to 0$. I can prove this if Ω is a quasidisk; the difficulty lies at boundary points with "many" directions of approach through Ω. This conjecture easily implies the result of [BJ90] about twist points and rectifiable curves, so should be at least as difficult to prove. The factorization theorem may be helpful here since it limits the contraction of conformal maps (and hence limits the concentration of harmonic measure).

• **Instability of harmonic measure:** The following problem arose in discussions with Lennart Carleson. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism. Let ω denote harmonic measure on a simply connected domain Ω, $\tilde{\omega}$ harmonic measure on the image domain $\tilde{\Omega} = f(\Omega)$, and let ω_f be the image of ω under f. One might expect the measures $\tilde{\omega}$ and ω_f to be mutually absolutely continuous if f was sufficiently smooth, but this is false if f is an affine mapping of a snowflake. More generally,

Conjecture 19. *If f is a non-conformal affine mapping, then $\omega_f \perp \tilde{\omega}$ on $Tw(\tilde{\Omega})$.*

Suppose f is affine and $g : \mathbb{D} \to \Omega$ and $h : \mathbb{D} \to f(\Omega)$ are conformal. Then $\varphi = h^{-1} \circ f \circ g$ is a quasiconformal map of the disk to itself and the Beltrami coefficient μ has constant modulus and argument equal to $\arg(g')$ (which is a harmonic Bloch function).

Conjecture 20. *Suppose u is a real, harmonic Bloch function, $0 < \epsilon < 1$ and $\mu = \epsilon \exp(iu(z))$. Then the corresponding quasiconformal self-map f_μ of the disk is singular on the set $E \subset \mathbb{T}$ where u fails to have non-tangential limits (i.e., there is a $F \subset E$ such that $|F| = |E|$ and $|f(F)| = 0$).*

One way to view the problem is that although there are Carleson type conditions on a Beltrami coefficient which insure the corresponding map is absolutely continuous (e.g. [FKP91]), there are not corresponding results which insure it is singular. To get a singular map on the boundary, we need is not just that the dilatation has large absolute value, but some type of independence of the argument at different scales (provided by harmonicity in the problem above). A possibly easier version is to let μ be constant on dyadic Whitney squares and let u be a dyadic martingale.

• **A Brownian motion proof of Makarov's theorem:** Makarov [Mak85] proved (in a much more precise form) that any set of dimension < 1 has zero harmonic measure on the boundary on any simply connected domain. Makarov's theorem is proved by using growth estimates for the Riemann mapping, but it might be possible to prove using Brownian motion. Consider a compact set E and a Brownian motion started in its complement, run until the first time it hits E. We say the path separates E if there are points of E in different complementary components of the path. If E is on the boundary of a simply connected domain Ω and is separated by a path γ, then γ must hit $\partial\Omega \setminus E$ before it hits E. Thus if E is almost surely separated by Brownian paths, it must have harmonic measure zero in any simply connected domain. I proved in [Bis91] that the following is true if $E \subset \mathbb{R}$ or if we assume $\dim(E) < 1/2$.

Conjecture 21. *If $E \subset \mathbb{R}^2$ is compact and $\dim(E) < 1$ then Brownian paths separate it a.s..*

• **Diffusion Limited Aggregation (DLA):** To define (continuous) DLA, fix a unit disk at the origin. Start another disk near infinity and move it along a Brownian path until the first time it hits the first disk and then stop it. Successively add new disks in the same way (i.e., according to harmonic measure) and try to describe what the resulting collection looks like. Let N be the number of disks and R the radius of the set. Trivially, $\sqrt{N} \lesssim R \lesssim N$. I proved $R \lesssim N^{4/5}$ (unpublished) and Kesten [Kes87] showed $R \lesssim N^{2/3}$ almost surely. This bound depends on the fact that harmonic measure cannot be too concentrated (if it is concentrated on the "tips" of the

DLA then the radius increases quickly; if it is spread across the "fjords" then DLA grows more slowly). Obtaining a non-trivial lower bound seems much harder.

Question 22. *Is there an $s > 1/2$ such that $R \geq CN^s$ almost surely?*

It is also interesting to consider a DLA based on the convex hull measure discussed earlier. This measure is less concentrated at "tips" than harmonic measure, so the corresponding DLA should grow more slowly. Indeed, the convex hull measure of any unit ball on the boundary of DLA will be $\leq C/R$ and hence the probability of increasing the radius from R to $R+1$ at an step is $\leq C/R$. Thus we expect $R \leq CN^{1/2}$, and so the convex hull DLA should grow at the slowest possible rate and have a scaling limit with positive area, which would be an interesting result in itself [Mar92] (especially if we can show it is "fractal"). On the other hand, if the convex hull DLA does grow faster than $N^{1/2}$, then a comparison with harmonic measure should lead to a nontrivial lower bound for the growth of the usual DLA clusters.

Miscellaneous problems

• **Minimal sets:** We shall say a set $E \subset \mathbb{T}$ is minimal for conformal maps if $\dim(f(E)) \geq \dim(E)$ for every conformal $f : \mathbb{D} \to \Omega$. Obviously intervals are minimal and Makarov [Mak87] proved that every set of dimension 1 is minimal, but it is not obvious whether there are minimal sets with dimension $0 < \alpha < 1$. A subset E of the unit circle $\mathbb{T}$ is called minimal for quasisymmetric mappings if $\dim(f(E)) \geq \dim(E)$ for every quasisymmetric map $f : \mathbb{T} \to \mathbb{T}$. Jeremy Tyson (currently a NSF postdoc at Stony Brook) has conjectured that there are no such sets with dimension $0 < \alpha < 1$. Using the factorization theorem one can show that if $E \subset \mathbb{T}$ is minimal for quasisymmetric maps it is also is minimal for conformal maps.

Question 23. *Are there minimal sets of dimension $0 < \alpha < 1$ for quasisymmetric maps on the circle? For conformal maps?*

• **BiLipschitz homogeneous sets:** As noted in the summary of previous work, I proved a biLipschitz homogeneous (BLH) curve in the plane must be bounded turning (BT), i.e., given $x, y \in \Gamma$, the shorter arc connecting these points has diameter $\leq M|x-y|$; in the plane this exactly being a quasicircle. The implication BLH $\Rightarrow$ BT is false in higher dimensions, but

Conjecture 24. *Suppose Γ is a Jordan curve in Euclidean 3-space and there is a continuous one parameter group of bilipschitz maps acting $\mathbb{R}^3$ whose restriction acts transitively on Γ. Then Γ is bounded turning.*

An idea for the proof is that if Γ is BLH but not BT then there should be scales where the curve "looks like" a tight helix. If we take a rectifiable arc in the complement with end points on Γ and push it by the group action, we want to show these arcs must "wrap around" the helixes at different scales and this forces the length to increase unboundedly and we get a contradiction. Of course, making this precise will call for an argument that combines some algebraic topology (to account for the "wrapping around") with metric arguments (to explain the "looks like a helix").

A tougher problem is to understand what happens if we do not assume Γ is a Jordan curve. Planar continua which are homogeneous with respect to self homeomorphisms have not been completely classified, but besides points and closed curves there are known exotic examples: the pseudo-arc and a circle of pseudo-arcs (see e.g., [Rog83], [Rog92]). Can these be BLH?

Question 25. *If X is a BLH continua in $\mathbb{R}^2$, must it be a closed Jordan curve?*

I expect that a careful study of the "crookedness" of the pseudo-arc will show that there must be points surrounded in all directions by other points of the set, with metric estimates. There are certainly points that are not surrounded (e.g., points on the boundary of the convex hull) and from this one should be able to show a pseudo-arc cannot be BLH.

Educational and broader impact of the proposal

Recent work by a variety of people has stimulated more interaction between analysis and topology and the proposed work contributes to this trend by giving new connections between these areas, e.g., using convex hulls to study Brennan's conjecture or using heat kernel estimates to investigate the Ahlfors conjecture. Moreover, funds from the grant will be used to support the Complex Analysis and Geometry seminar at Stony Brook which seeks to encourage just this sort of interaction.

Part of the support I am requesting is for travel to Mittag-Leffler during a special year in complex analysis (organized by L. Carleson, P. Jones and N. Makarov) which will emphasize the interactions of complex analysis with other fields. My proposal is an excellent fit to this program and I expect to expand my research interests to other problems which will be considered there (e.g., conformal invariance of critical percolation). It would also be an ideal place to discuss applications of the factorization approach to complex dynamics. One test problem is the analog of Bowen's property for Julia sets: if J is a connected Julia set of a rational map R then is it either the whole plane, an analytic curve or dimension > 1? There are several related results (e.g., [Zdu90], [Urb91], [Ham95], [Ham96]) but I believe the general problem is still open.

Currently I am teaching a graduate course on limit sets and deformations which is closely related to my recent results and the questions in this proposal. One distinct advantage of this material is that recent and interesting results can be proven right away and that the students are exposed to the interaction of analysis, geometry, topology and probability. I hope eventually to produce a text based on this material and viewpoint.

I have a graduate student, Zsuzsanna Gonye, who is finishing her thesis on the Hausdorff dimension of escaping geodesics. She shows that for a large class of hyperbolic manifolds the geodesics which escape to infinity at linear speed (or at any specified sub-linear speed) have full dimension.

This corrects an erroneous comment in [BJ97b]. She also shows that a theorem of Steger and myself [BS93] for finitely generated Fuchsian groups fails for some divergence type groups. I expect to soon have new students working on problems related to the proposal.

Many of the problems discussed in the proposal could be modified for graduate work and several others suggest numerical experiments appropriate for undergraduate research. In particular, it should be possible to numerically estimate the constant in Sullivan's theorem as follows. Let $f : \mathbb{D} \to \Omega$ be a conformal map and let $\iota : \partial\Omega \to \mathbb{T}$ be the restriction of the isometry from S to $\mathbb{D}$. We want to compute the composition $h = \iota \circ f$ on the unit circle and use cross ratios of quadruples to give a lower bound for the optimal quasiconformal extension [Leh87]. The conformal map f can be computed by existing programs, e.g., Don Marshall's program `unzip`. The boundary map ι is actually quite easy to compute explicitly for certain domains. If Ω is finitely bent (i.e., the corresponding surface S is a finite union of geodesic faces), then ι is Möbius on each face of S and once we have mapped one face into the disk, the maps on adjacent faces are forced by continuity.

Another computation concerns an estimate of Carleson and Jones [CJ92] for harmonic measure on simply connected domains,

$$\omega(B(x,r)) \leq C \exp(-a \int_r^1 \frac{1}{\delta(t)} dt), \qquad \delta(t) = \max_{|x-y|=t} \mathrm{dist}(y, \partial\Omega).$$

They show that any $a < 1/2$ will work, but it is easy to see by considering wedges that this is sharp only when the angle is 2π. The convex hull measure satisfies a similar estimate which is sharp for all wedges when $a = 1$. Thus, the estimate of Carleson and Jones seems to give a better result for the convex hull measure than for harmonic measure. This could be numerically tested for a variety of other domains. Other possible undergraduate projects related to the proposal include

- numerically investigate the integral means corresponding to quasiconformal self-maps of the disk
- compute deformations Fuchsian groups corresponding to the sphere minus three holes. Do any of these have a deformation with dimension > 1? This is related to Ruelle's property.
- simulate the growth of convex hull DLA.
- simulate Brownian motion in order to study its winding around sets.
- numerically estimate the dimension of the convex hull measure for various fractal examples.
- use computer graphics to "draw" exotic homogeneous continua.

References

[AH95] J.M. Anderson and A. Hinkkanen. Quadrilaterals and extremal quasiconformal extensions. *Comment. Math. Helv.*, 70(3):455–474, 1995.

[AM96] J.W. Anderson and B. Maskit. On the local connectivity of limit set of Kleinian groups. *Complex Variables Theory Appl.*, 31(2):177–183, 1996.

[Ast94] K. Astala. Area distortion of quasiconformal mappings. *Acta Math.*, 173(1):37–60, 1994.

[AZ90] K. Astala and M. Zinsmeister. Mostow rigidity and Fuchsian groups. *Comptes Rendu Acad. Sci., Paris*, 311:301–306, 1990.

[AZ94] K. Astala and M. Zinsmeister. Holomorphic families of quasi-Fuchsian groups. *Ergod. Th. & Dynam. Sys.*, 14:207–212, 1994.

[AZ95] K. Astala and M. Zinsmeister. Abelian coverings, Poincaré exponent of convergence and holomorphic deformations. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 20(1):81–86, 1995.

[BA56] A. Beurling and L. Ahlfors. The boundary correspondence under quasiconformal mappings. *Acta Math.*, 96:125–142, 1956.

[Ber98] D. Bertilsson. Coefficient estimates for negative powers of the derivative of univalent functions. *Ark. Mat.*, 36(2):255–273, 1998.

[Ber99] D. Bertilsson. *On Brennan's conjecture in comformal mapping.* PhD thesis, Royal Institute of Technology, Stockholm, Sweden, October 1999.

[Bis91] C.J. Bishop. A characterization of Poissonian domains. *Ark. Mat.*, 29(1):1–24, 1991.

[Bis96] C.J. Bishop. Minkowski dimension and the Poincaré exponent. *Michigan Math. J.*, 43(2):231–246, 1996.

[Bis97] C.J. Bishop. Geometric exponents and Kleinian groups. *Invent. Math.*, 127(1):33–50, 1997.

[BJ90] C.J. Bishop and P.W. Jones. Harmonic measure and arclength. *Ann. Math.*, 132:511–547, 1990.

[BJ94] C.J. Bishop and P.W. Jones. Harmonic measure, L^2 estimates and the Schwarzian derivative. *J. Anal. Math.*, 62:77–113, 1994.

[BJ97a] C.J. Bishop and P.W. Jones. Hausdorff dimension and Kleinian groups. *Acta. Math*, 179:1–39, 1997.

[BJ97b] C.J. Bishop and P.W. Jones. The law of the iterated logarithm for Kleinian groups. In *Lipa's legacy (New York, 1995)*, pages 17–50. Amer. Math. Soc., Providence, RI, 1997.

[BLMM98] V. Božin, N. Lakic, V. Marković, and M. Mateljević. Unique extremality. *J. Anal. Math.*, 75:299–338, 1998.

[Bow79] R. Bowen. Hausdorff dimension of quasicircles. *Publ. I. H. E. S.*, 50:11–25, 1979.

[Bre78] J.E. Brennan. The integrability of the derivative in conformal mapping. *J. London Math. Soc. (2)*, 18(2):261–272, 1978.

[BS93] C.J. Bishop and T. Steger. Representation-theoretic rigidity in $\mathrm{PSL}(2, \mathbf{R})$. *Acta Math.*, 170(1):121–149, 1993.

[Can93] R.D. Canary. Ends of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 6(1):1–35, 1993.

[CJ92] L. Carleson and P. W. Jones. On coefficient problems for univalent functions and conformal dimension. *Duke Math. J.*, 66(2):169–206, 1992.

[CM94] L. Carleson and N.G. Makarov. Some results connected with Brennan's conjecture. *Ark. Mat.*, 32(1):33–62, 1994.

[DE86] A. Douady and C.J. Earle. Conformally natural extension of homeomorphisms of the circle. *Acta Math.*, 157(1-2):23–48, 1986.

[EM87] D.B.A. Epstein and A. Marden. Convex hulls in hyperbolic spaces, a theorem of Sullivan and measured pleated surfaces. In *Analaytical and geometric aspects of hyperbolic spaces*, London Math. Soc. Lecture Notes Series 111, pages 113–253. Cambridge University Press, 1987.

[FKP91] R.A. Fefferman, C.E. Kenig, and J. Pipher. The theory of weights and the Dirichlet problem for elliptic equations. *Ann. of Math. (2)*, 134(1):65–124, 1991.

[FM98] J.L. Fernández and M.V. Melián. Escaping geodesics of Riemannian surfaces. 1998. Preprint.

[GH98] M. Ghamsari and D.A. Herron. Higher dimensional Ahlfors regular sets and chordarc curves in $\mathbf{R}^n$. *Rocky Mountain J. Math.*, 28(1):191–222, 1998.

[GH99] M. Ghamsari and D.A. Herron. Bi-Lipschitz homogeneous Jordan curves. *Trans. Amer. Math. Soc.*, 351(8):3197–3216, 1999.

[GL00] W. Werner G.F. Lawler, O. Schramm. Analyticity of intersection exponents for planar Brownian motion. 2000. preprint.

[Ham95] D. Hamilton. Length of Julia curves. *Pacific J. Math.*, 169(1):75–93, 1995.

[Ham96] D. Hamilton. Rectifiable Julia curves. *J. London Math. Soc. (2)*, 54(3), 1996.

[HM99] D.A. Herron and V. Mayer. Bi-Lipschitz group actions and homogeneous Jordan curves. *Illinois J. Math.*, 43(4):770–792, 1999.

[Kes87] H. Kesten. How long are the arms in DLA? *J. Phys. A*, 20(1):L29–L33, 1987.

[Kla99] E. Klarreich. Semiconjugacies between Kleinian group actions on the Riemann sphere. *Amer. J. Math.*, 121(5):1031–1078, 1999.

[Kra96] P. Kraetzer. Experimental bounds for the universal integral means spectrum of conformal maps. *Complex Varibles Theory App.*, 31:305–309, 1996.

[Leh87] O. Lehto. *Univalent functions and Teichmüller spaces*. Springer-Verlag, New York, 1987.

[Mak85] N.G. Makarov. On the distortion of boundary sets under conformal mappings. *Proc. London Math. Soc.*, 51:369–384, 1985.

[Mak87] N.G. Makarov. Conformal mapping and Hausdorff measures. *Ark. Mat.*, 25(1):41–89, 1987.

[Mak98] N.G. Makarov. Fine structure of harmonic measure. *Algebra i Analiz*, 10(2):1–62, 1998. English translation in St. Petersburg J. Math, 1999.

[Mar92] P. March. Remarks on scaling a model of Witten-Sander type. *J. Statist. Phys.*, 67(5-6):1117–1149, 1992.

[McM69] J.E. McMillan. Boundary behavior of a conformal mapping. *Acta. Math.*, 123:43–67, 1969.

[McM89] C. McMullen. Amenability, Poincaré series and quasiconformal maps. *Invent. Math.*, 97(1):95–127, 1989.

[McM00] C. McMullen. Local connectivity, Kleinian groups and geodesics on the blowup of the torus. 2000.

[Min94] Y.N. Minsky. On rigidity, limit sets, and end invariants of hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 7(3):539–588, 1994.

[Min99] Y.N. Minsky. The classification of punctured-torus groups. *Ann. of Math. (2)*, 149(2):559–626, 1999.

[Nic89] P.J. Nicholls. *The ergodic theory of discrete groups*. LMS Lecture Notes 143. Cambridge University Press, 1989.

[Pom85a] Ch. Pommerenke. On the integral means of the derivative of a univalent function. *J. London Math. Soc. (2)*, 32(2):254–258, 1985.

[Pom85b] Ch. Pommerenke. On the integral means of the derivative of a univalent function. II. *Bull. London Math. Soc.*, 17(6):565–570, 1985.

[Rog83] J.T. Rogers. Homogeneous continuua. *Top. Proc.*, 8:213–233, 1983.

[Rog92] J.T. Rogers. Classifying homogeneous continuua. *Top. and Appl.*, 44:341–352, 1992.

[Rue82] D. Ruelle. repellers for real analytic maps. *Ergod. Th. & Dym. Sys.*, 2:99–107, 1982.

[Sul81] D. Sullivan. Growth of positive harmonic functions and Kleinian group limit sets of planar measure 0 and Hausdorff dimension 2. In *Geometry Symposium (Utrecht 1980)*, Lecture Notes in Math. 894, pages 127–144. Springer-Verlag, 1981.

[Sul82] D. Sullivan. Discrete conformal groups and measureable dynamics. *Bull. Amer. Math. Soc.*, 6:57–73, 1982.

[Sul87] D. Sullivan. Related aspects of positivity in Riemannian geometry. *J. Differential Geom.*, 25(3):327–351, 1987.

[Tys] J.T. Tyson. Sets of minimal Hausdorff dimension for quasiconformal maps. *Proc. Amer. Math. Soc.*

[Urb91] M. Urbański. On the Hausdorff dimension of a Julia set with a rationally indifferent periodic point. *Studia Math.*, 97(3):167–188, 1991.

[Zdu90] A. Zdunik. Parabolic orbifolds and the dimension of the maximal measure for rational maps. *Invent. Math.*, 99(3):627–649, 1990.

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[4] The law of the iterated logarithm for Kleinian groups, with P.W. Jones, in *Lipa's Legacy*, pages 17-50, Amer. Math. Soc., 1997.

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