FUNCTION THEORETIC CHARACTERIZATIONS OF WEIL-PETERSSON CURVES

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ABSTRACT. The Weil-Petersson class is the closure of the smooth closed curves in the Weil-Petersson metric on universal Teichmüller space defined by Takhtajan and Teo. We give some new characterizations of this class of curves and some new proofs of previously known characterizations. In particular, we give a new, more geometric characterization of the conformal weldings of such curves and characterize the curves themselves in terms of Peter Jones's β-numbers.

Date: March 25, 2021; revised January 17, 2022.

1991 *Mathematics Subject Classification.* Primary: 30C62, 30F60 Secondary: 53A10, 46E35, 60J67 .

Key words and phrases. universal Teichmüller space, Weil-Petersson class, chord-arc curves, beta numbers, Schwarzian derivatives, Dirichlet class, conformal weldings, Sobolev spaces.

The author is partially supported by NSF Grant DMS 1906259.

1. Introduction

This paper is a companion to [11] which gives twenty equivalent definitions of the Weil-Petersson class of closed Jordan curves. The definitions given in that paper mostly involve smoothness properties of the curve Γ or curvature properties of a surface in hyperbolic space that has Γ as its asymptotic boundary. In both these cases, the definitions extend to curves in $\mathbb{R}^n$ and are proven equivalent in that setting. In this paper, we deal with the definitions that are purely 2-dimensional such as those involving properties of the conformal map onto the domain bounded by Γ, the conformal welding corresponding to Γ and the complex dilatation of quasiconformal reflections across Γ. To state our results precisely, we need to recall a few definitions.

A quasicircle is the image of the unit circle $\mathbb{T} = \{|z| = 1\}$ under a quasiconformal mapping f of the plane, e.g., a homeomorphism of the plane that is conformal outside the unit disk $\mathbb{D}$, and whose dilatation $\mu = f_{\overline{z}}/f_z$ belongs to $\mathbb{B}_1^\infty$, the open unit ball in $L^\infty(\mathbb{D})$. The collection of planar quasicircles corresponds to universal Teichmüller space $T(1)$ and the usual metric is defined in terms of $\|\mu\|_\infty$. Motivated by problems arising in string theory (e.g. [14], [15]), Takhtajan and Teo [51] defined a Weil-Petersson metric on universal Teichmüler space $T(1)$ that makes it into a Hilbert manifold. With this topology, $T(1)$ has uncountably many connected components, but one of these components is exactly the closure of the smooth curves; this component is called the Weil-Petersson class and is denoted $T_0(1)$.

Suppose Γ is a closed curve in the plane and let f be a conformal map from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to Ω, the bounded complementary component of Γ. If f is conformal on $\mathbb{D}$, then f' is never zero, so $\Phi = \log f'$ is a well defined holomorphic function on $\mathbb{D}$. Recall that the Dirichlet class is the Hilbert space of holomorphic functions F on the unit disk such that $|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$. We will define a curve Γ to be in the Weil-Petersson class if and only if the conformal map $f : \mathbb{D} \to \Omega$ is such that $\log f'$ is in the Dirichlet class. Theorem 1.12 of the Takhtajan and Teo paper [51] shows that this is equivalent to the definition described above.

Saying $\log f'$ is in the Dirichlet class means that

$$\int_{\mathbb{D}} |(\log f')'|^2 dxdy = \int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dxdy < \infty. \tag{1.1}$$

This can also be written as

$$\int_{\mathbb{D}} |(\log f'(z))'|^2 (1-|z|^2)^2 dA_\rho(z) < \infty \tag{1.2}$$

where dA_ρ is the hyperbolic area on $\mathbb{D}$ and the integrand is now invariant under pre-compositions by Möbius transformations of the disk.

The space $H^{1/2}(\mathbb{T}) \subset L^2(\mathbb{T})$ is defined by the finiteness of the seminorm

$$\begin{aligned} D(f) &= \iint_{\mathbb{D}} |\nabla u(z)|^2 dxdy = \frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi} \left|\frac{f(e^{is})-f(e^{it})}{\sin\frac{1}{2}(s-t)}\right|^2 dsdt \\ &\simeq \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|f(z)-f(w)|^2}{|z-w|^2}|dz||dw|. \end{aligned}$$

where u is the harmonic extension of f to $\mathbb{D}$. The equality of the first and second integrals is called the Douglas formula, after Jesse Douglas who introduced it in his solution of the Plateau problem [20]. See also Theorem 2.5 of [4] (for a proof of the Douglas formula) and [44] (for more information about the Dirichlet space). See [1] and [19] for additional background on fractional Sobolev spaces.

It follows from these definitions that for a conformal map f, $\log f'(z)$ is in the Dirichlet class on $\mathbb{D}$ if and only if the radial limits $\log|f'|$ and $\arg(f')$ are in $H^{1/2}(\mathbb{T})$. Since $\arg(f')$ can be unbounded, it is, perhaps, surprising that this is equivalent to $f'/|f'| \in H^{1/2}$:

Theorem 1.1. $\Gamma = f(\mathbb{T})$ *is Weil-Petersson if and only if it is chord-arc and* $f'/|f'| \in H^{1/2}(\mathbb{T})$.

This is essentially Theorem 5 in [25], by Gallardo-Gutiérrez, González, Pérez-González, Pommerenke and Rättyä. One direction is easy. It is well known that Weil-Petersson curves must be chord-arc (e.g., see Section 2 of [11]) and if $\log f' = \log|f'| + i\arg f'$ is in the Dirichlet class, then $\arg f' \in H^{1/2}(\mathbb{T})$. Using $|e^{ix}-e^{iy}| \leq |x-y|$ and the Douglas formula we get

$$\int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{e^{i\arg f'(x)}-e^{i\arg f'(y)}}{x-y}\right|^2 dxdy \leq \int_{\mathbb{T}}\int_{\mathbb{T}} \left|\frac{\arg f'(x)-\arg f'(y)}{x-y}\right|^2 dxdy < \infty.$$

Thus $\exp(i\arg f') \in H^{1/2}(\mathbb{T})$. The converse direction seems harder. As noted above, one proof is given in [25]; we give another in Section 8 of this paper; a much less direct proof is given in [11].

If is standard fact (e.g., see Lemma 10.2) that if F is holomorphic function on $\mathbb{D}$ then

$$|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$$

if and only if

$$|F(0)|^2 + |F'(0)|^2 + \int_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy < \infty.$$

Applying this to $F = \log f'$, we see that (1.1) could be replaced by the condition

$$\int_{\mathbb{D}} \left| \left(\frac{f'''}{f'}\right) - \left(\frac{f''}{f'}\right)^2 \right|^2 (1-|z|^2)^2 dxdy < \infty. \tag{1.3}$$

This integrand is reminiscent of the Schwarzian derivative of f given by

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{1.4}$$

For a Möbius transformation sending $\mathbb{D}$ to a half-plane, the Schwarzian is constant zero, but the expression in (1.3) blows up to infinity at a boundary point. However, for conformal maps into bounded quasidisks, the integrals of these two quantities are simultaneously finite or infinite. Cui [18] proved:

Theorem 1.2. Γ *is Weil-Petersson iff it is a quasicircle and* $\Gamma = f(\mathbb{T})$*, where* f *is conformal on* $\mathbb{D}$ *and satisfies*

$$\int_{\mathbb{D}} |S(f)(z)|^2 (1-|z|^2)^2 dxdy < \infty. \tag{1.5}$$

See also Theorem II.1.12 of Takhtajan and Teo's book [51] and Theorem 1 of [39] by Pérez-González and Rättyä. As with (1.2), we can rewrite (1.5) as

$$\int_{\mathbb{D}} |S(f)(z)|^2 (1-|z|^2)^4 d\mathrm{A}_\rho(z) < \infty. \tag{1.6}$$

If f is univalent on $\mathbb{D}$ then

$$\sup_{z\in\mathbb{D}} |S(f)(z)|(1-|z|^2)^2 \le 6. \tag{1.7}$$

See Chapter II of [30] for this and other properties of the Schwarzian. If f is holomorphic on the disk and satisfies (1.7) with 6 replaced by 2, then f is injective, i.e., a conformal map. If 2 is replaced by a value $t < 2$, then f also has a K-quasiconformal extension to the plane, where K depends only on t. This is due to Ahlfors and Weill [2], who gave a formula for the extension and its dilatation

$$f(w) = f(z) + \frac{(1-|z|^2)f'(z)}{\overline{z} - \frac{1}{2}(1-|z|^2)(f''(z)/f'(z))} \tag{1.8}$$

$$\mu(w) = -\frac{1}{2}(1-|z|^2)^2 S(f)(z) \tag{1.9}$$

where $w \in \mathbb{D}^*$ and $z = 1/\overline{w} \in \mathbb{D}$. See also Section 4 of [17], Formula (3.33) of [36] and or Equation (9) of [42].

Theorem 1.3. *Γ is Weil-Petersson if and only if $\Gamma = f(\mathbb{T})$ where f is a quasiconformal map of the plane that is conformal on $\mathbb{D}^*$ and whose dilatation μ on $\mathbb{D}$ satisfies*

$$\int_{\mathbb{D}} \frac{|\mu(z)|^2}{(1-|z|^2)^2} dxdy = \int_{\mathbb{D}} |\mu(z)|^2 dA_\rho < \infty. \tag{1.10}$$

This is due to Guizhen Cui; see Theorem 2 of [18].

Another variation on this theme is to consider the map $R(z) = f(1/\overline{f^{-1}(z)})$. This is an orientation reversing quasiconformal map of the sphere to itself that fixes Γ pointwise, swaps the complementary components of Γ, and whose dilatation satisfies

$$\int_{\Omega \cup \Omega^*} |\mu(z)|^2 d\mathrm{A}_\rho(z) < \infty, \tag{1.11}$$

where $d\mathrm{A}_\rho$ is hyperbolic area on each of the domains Ω, Ω^*.

Corollary 1.4. *Γ is Weil-Petersson if and only if it is the fixed point set of a quasiconformal involution of the sphere whose complex dilatation μ is in L^2 with respect to the hyperbolic metric on the complement of Γ.*

A circle homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ is called a conformal welding if $\varphi = f^{-1} \circ g$ where f, g are conformal maps from the two sides of the unit circle to the two sides of a closed Jordan curve Γ. There are many weldings associated to each Γ, but they all differ from each other by compositions with Möbius transformations of $\mathbb{T}$. Not every circle homeomorphism is a conformal welding, but weldings are dense in the homeomorphisms in various senses; see [10].

A circle homeomorphism is called M-quasisymmetric if it maps adjacent arcs in $\mathbb{T}$ of the same length to arcs whose length differ by a factor of at most M; we call φ quasisymmetric if it is M-quasisymmetric for some M. The quasisymmetric maps are exactly the circle homeomorphisms that can be continuously extended to quasiconformal self-maps of the disk, and are also exactly the conformal weldings of quasicircles. See [3], [10]. Weil-Petersson weldings were first characterized by Yuliang Shen [48]. He proved:

Theorem 1.5. Γ *is Weil-Petersson if and only if* $\log \varphi' \in H^{1/2}(\mathbb{T})$.

To see necessity, observe that $\log f'$ is in the Dirichlet class on $\mathbb{D}$ if and only if its radial boundary values satisfy $\log f' \in H^{1/2}(\mathbb{T})$. Thus $\log f'$ in the Dirichlet space implies $\log f' \in H^{1/2}(\mathbb{T})$. Similarly for the conformal map g from $\{|z| > 1\}$ to the region outside Γ. There are several ways to see this, e.g., Theorem 1.7 below clearly implies Weil-Petersson curves are invariant under inversion through points not on the curve, so g can be written as f for the inverted curve followed by the inversion. Then we have $\log \varphi'(x) = -\log f'(\varphi(x)) + \log g'(x)$. Beurling and Ahlfors [8] proved $H^{1/2}(\mathbb{T})$ is invariant under pre-compositions with quasisymmetric circle homeomorphisms, so $\log f' \circ \varphi \in H^{1/2}(\mathbb{T})$ and hence $\log \varphi' \in H^{1/2}(\mathbb{T})$. Thus Shen's condition is necessary for Γ to be Weil-Peterson.

We will give a new proof of sufficiency by showing that Shen's condition implies the following more geometric condition on φ. If $I \subset \mathbb{T}$ is an arc, let $m(I)$ denote its midpoint. For a homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ define

$$\operatorname{qs}(\varphi, I) = \frac{|\varphi(m(I)) - m(\varphi(I))|}{\ell(\varphi(I))}.$$

Theorem 1.6. Γ *is Weil-Petersson if and only if* φ *satisfies*

$$\sum_I \operatorname{qs}^2(\varphi, I) \leq C < \infty, \tag{1.12}$$

where the sum is over any dyadic decomposition of $\mathbb{T}$*, and* C *is independent of the choice of decomposition.*

A dyadic interval I in $\mathbb{R}$ is one of the form $(2^{-n}j, 2^{-n}(j+1)]$. Dyadic intervals on $\mathbb{T}$ are defined in an analogous manner by repeated bisection, starting from some base point, usually chosen to be $1 \in \mathbb{T}$. A dyadic square in $\mathbb{R}^2$ is the product of 2 dyadic intervals of the same length. This length is called the side length of Q and is denoted $\ell(Q)$. Note that $\operatorname{diam}(Q) = \sqrt{2}\ell(Q)$. For a positive number $\lambda > 0$, we let λQ denote the cube concentric with Q but with diameter $\lambda \text{diam}(Q)$, e.g., $3Q$ is the "triple" of Q, a union of Q and 8 adjacent copies of itself. We let $Q^\uparrow$ denote the parent of Q; the unique dyadic cube containing Q and having twice the side length. Q is one of the 4 children of $Q^\uparrow$.

Given a set $E \subset \mathbb{R}^n$ and a dyadic cube Q, define Peter Jones's β-number as

$$\beta(Q) = \beta_E(Q) = \frac{1}{\text{diam}(Q)} \inf_L \sup\{\text{dist}(z, L) : z \in 3Q \cap E\},$$

where the infimum is over all lines L that hit $3Q$. Peter Jones invented the β-numbers as part of his traveling salesman theorem [28]. One consequence of his theorem is that for a Jordan curve Γ,

$$\ell(\Gamma) \simeq \text{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \text{diam}(Q), \tag{1.13}$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$. The following result shows that Weil-Petersson curves satisfy a strong form of rectifiability.

Theorem 1.7. Γ *is Weil-Petersson if and only if*

$$\sum_Q \beta_\Gamma(Q)^2 < \infty, \tag{1.14}$$

where the sum is over all dyadic cubes.

All the results described so far can be summarized as follows:

Theorem 1.8. *With notation as above, the following are equivalent for a closed Jordan curve* Γ *in the plane:*

(1) $\mu(z) \in L^2(dA_\rho)$,
(2) $S(f)(z)(1-|z|^2)^2 \in L^2(dA_\rho)$,
(3) $(\log f'(z))'(1-|z|^2) \in L^2(dA_\rho)$,
(4) $f'/|f'| \in H^{1/2}(\mathbb{T})$,
(5) $\log \varphi' \in H^{1/2}(\mathbb{T})$,
(6) $\sum_I \text{qs}(\varphi, I)^2 < \infty$,
(7) $\sum_Q \beta_\Gamma^2(Q) < \infty$.

Conditions (1)-(3), (5) were previously known to be equivalent; Conditions (4), (6) and (7) are new. The implications (3) $\Rightarrow$ (5) and (3) $\Rightarrow$ (4) were already discussed above. In the remainder of the paper we will prove the new implications (5) $\Rightarrow$ (6)

$\Rightarrow$ (1), (3) $\Rightarrow$ (7) $\Rightarrow$ (1), (4) $\Rightarrow$ (3), and sketch new proofs of the known implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3), Together, these prove the equivalence of Conditions (1)-(7).

The definition of the Weil-Petersson metric by Takhtajan and Teo [51] was motivated by problems coming from string theory. Furthermore, one of the characterizations given in [11] involves minimal surfaces in hyperbolic 3-space that have Γ as an asymptotic boundary: Γ is Weil-Petersson iff such a surface S has finite total curvature. An equivalent formulation is that S has finite renormalized area, a concept with has strong motivations arising from string theory and quantum entanglement, e.g., [34], [45], [50]. See the introduction of [5] for further details and references. Also see [41], where the authors argue that Weil-Petersson curves are the correct setting for 2-dimensional conformal field theory.

Other results in [11] describe the Weil-Petersson class as the curves with arclength parameterization in $H^{3/2}(\mathbb{T})$, finite Möbius energy (a concept arising in knot theory), and also characterize them as curves whose length is well approximated by inscribed polygons in a precise sense. The Weil-Petersson class also arises in computer vision: see the papers of Sharon and Mumford [46], Feiszli, Kushnarev and Leonard [23], and Feiszli and Narayan [24]. Indeed, the problem of geometrically characterizing Weil-Petersson curves was originally suggested to me by David Mumford in December of 2017. Further connections of the Weil-Petersson class to Brownian motion, Loewner energy and Schramm-Loewner Evolutions (SLE) are described in [43], [47], [52], [53], [54], [55]. In fact, my initial results on the Weil-Petersson class were directly motivated by a lecture on these connections given by Yilin Wang at an IPAM workshop in January of 2019.

Dragomir Šarić pointed out that the condition in Theorem 1.6 can be rewritten as $\sum_K \mathrm{qs}(\varphi, K)^2 < \infty$ where the sum is over all intervals K that are unions of all pairs adjacent dyadic intervals of the same length. This sum contains the sum over the usual dyadic family, and so it suffices to bound the sum over any dyadic family (regardless of base point). I also thank Kari Astala, Martin Chuaqui Tim Mesikepp, Raanan Schul, Leon Takhtajan, Dror Varolin, Rongwei Yang and Michel Zinsmeister for helpful comments on this paper. I especially thank Jack Burkart and María Jose González for detailed readings or various early drafts and providing many helpful comments and corrections. Two anonymous referees carefully read the manuscript and provided numerous corrections and suggestions, both mathematical and grammatical. I greatly appreciate the effort they put into improving the paper.

Finally, we recall some standard notation. Given two quantities A, B that both depend on a parameter, we write $A \lesssim B$ if there is a constant C so that $A \leq CB$ holds independent of the parameter. We write $A \gtrsim B$ if $B \lesssim A$, and we write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold. The notation $A \lesssim B$ means the same as the "big-Oh" notation $A = O(B)$.

2. Analogous results

Before starting the proof of Theorem 1.8, I would like to point out that it is completely analogous to other known results when the Dirichlet space is replaced by other spaces of analytic functions on the unit disk. See Table 1. Each column (after the first) of the table represents a theorem: the top entry is the name of a function space X so that $\log f' \in X$, and the lower rows give various conditions related to f that are equivalent to $\log f; \in X$. The description of these conditions is given in the first column: $\log f'(z)(1-|z|^2)$, $S(f)(1-|z|^2)^2$, the dilatation μ of a QC extension of f, the conformal welding homeomorphism φ, and $\Gamma = f(\mathbb{T})$. Theorem 1.8 of this paper is the rightmost column of the table, and the main new contribution of this paper is the bottom entry of this column.

Here are the definitions needed to interpret Table 1. We let $\mathcal{B}_0$ denote the little Bloch class of holomorphic functions on $\mathbb{D}$ such that

$$\mathcal{B}_0 = \{f : \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) \to 0 \text{ as } |z| \to 1\}$$

or, more concisely, $|f'(z)|(1-|z|^2) \in C_0(\mathbb{D})$ (continuous functions that tend to zero at the boundary). A Carleson measure on $\mathbb{D}$ is a non-negative measure μ so that

$$\mu(D(x,r)) \leq Cr,$$

for some fixed $C < \infty$ and all $x \in \mathbb{T}$, $r > 0$. We define a certain class of functions that give Carleson measures as

$$\text{CM}(\mathbb{D}) = \{f : |f|^2(1-|z|^2)^{-1}dxdy \text{ is a Carleson measure }\},$$

and $\text{CM}_0(\mathbb{D}) \subset \text{CM}(\mathbb{D})$ are the functions so that

$$\frac{1}{r}\int_{\mathbb{D}\cap D(x,r)} |f|^2(1-|z|^2)^{-1}dxdy \to 0,$$

$\log f'$	$\mathcal{B}_0$	BMOA	VMOA	Dirichlet
$(\log f')'(1-\|z\|^2)$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$S(z)(1-\|z\|^2)^2$	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
μ	$C_0(\mathbb{D})$	$\mathrm{CM}(\mathbb{D})$	$\mathrm{CM}_0(\mathbb{D})$	$L^2(dA_\rho)$
$\varphi = g^{-1} \circ f$	symmetric	strongly quasisymmetric	$\log \varphi' \in \mathrm{VMO}$	$\log \varphi' \in H^{1/2}$
$\Gamma = f(\mathbb{T})$	asymptotically conformal	Bishop-Jones condition	asymptotically smooth	$\sum \beta^2 < \infty$

TABLE 1. Each column represents a theorem: each row consists of different conditions that can be placed on a quantity associated to a conformal map $f : \mathbb{D} \to \Omega$ that is equivalent to $\log f;' \in X$, where X is the space named at the top of the column. The conditions and they grow more stringent as we move left-to-right in the table. The proofs of the equivalences in the center three columns come from many sources including: [6], [7], [13], [18], [21], [22], [33], [40], [49].

as $r \to 0$. The space BMOA (bounded mean oscillation) on the unit circle has several equivalent definitions (see Chapter VI of Garnett's book [26]); a convenient one to use here is that $f \in \mathrm{BMO}$ if its harmonic extension u to $\mathbb{D}$ satisfies $|\nabla u(z)|(1-|z|^2) \in \mathrm{CM}(\mathbb{D})$. BMOA is the subspace of functions such that the harmonic extension u is holomorphic. We say a function $f \in \mathrm{BMOA}$ is in VMOA if $|f'|(1-|z|^2) \in \mathrm{CM}_0(\mathbb{D})$. The space $L^2(dA_\rho)$ is defined as

$$A_2(\mathbb{D}) = \{f : \int_{\mathbb{D}} |f(z)|^2 d\mathrm{A}_\rho(z) = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{-2} dxdy < \infty\}.$$

For a circle homeomorphism ϕ, symmetric means $\phi(I)/\phi(J) \to 1$, for adjacent intervals with $|I| = |J| \to 0$, and ϕ is strongly quasisymmetric if for there are $\delta, \epsilon > 0$ such that $|E| \le \delta |I|$ implies $|\phi(E)| \le \epsilon |I|$ for every measurable $E \subset I$, and arc $I \subset \mathbb{T}$.

The geometric conditions in the bottom row of Table 1 are perhaps the hardest to state and understand. Given an arc γ with endpoints z, w, let L be the line through z and w and let $\beta(\gamma) = \max_{z \in \gamma} \mathrm{dist}(z, L)/|z-w|$. A curve Γ is called asymptotically conformal if $\beta(\gamma) \to 0$ as $\mathrm{diam}(\gamma) \to 0$, and a rectifiable curve is called asymptotically smooth if $\Delta(\gamma) = \ell(\gamma) - \mathrm{crd}(\gamma) = o(\mathrm{crd}(\gamma))$ as $\mathrm{diam}(\gamma) \to 0$. Asymptotically smooth curves are rectifiable by definition, but asymptotically conformal curves need not be, e.g., one can construct a variant of the the usual snowflake curve in which the triangle heights tend to zero at smaller generations, but do not give a rectifiable curve. The most awkward condition to state is the Bishop-Jones condition [13]: this says that there is a $M < \infty$ so that for each $z \in \Omega = \mathbb{R}^2 \setminus \Gamma$ there is a subdomain U_z bounded by a M-chord-arc curve, so that $z \in U_z \subset \Omega$ and such that

$$\mathrm{dist}(z, \partial U_z) \simeq \ell(\partial U_z) \simeq \ell(\partial U_z \cap \Gamma).$$

The paper [37] by Pau and Peláez describes another possible column of Table 1 corresponding to Q_p spaces and, no doubt, further such results remain to be discovered.

3. β-numbers and multi-resolution families

Before starting the proof of Theorem 1.8, we recall some facts about β-numbers and multi-resolution families. Recall that a standard dyadic interval in $\mathbb{R}$ is an interval of the form $[k2^{-n}, (k+1)2^{-n})$ for integers k, n. Below we will also consider translates of the standard dyadic family by a constant, usually $\pm 1/3$. The standard dyadic intervals on the circle $\mathbb{T}$ are images of the dyadic intervals on $\mathbb{R}$ under the map $x \to \exp(2\pi i x)$. This family can be rotated to form other dyadic families on the circle.

A multi-resolution family in a metric space X is a collection of bounded sets $\{X_j\}$ in X such that there is are $N, M < \infty$ so that

(1) For each $r > 0$, the sets with diameter between r and Mr cover X,
(2) each bounded subset of X hits at most N of the sets X_k with $\mathrm{diam}(X)/M \leq \mathrm{diam}(X_k) \leq M\mathrm{diam}(X)$.
(3) any subset of X with positive, finite diameter is contained in at least one X_j with $\mathrm{diam}(X_j) \leq M\mathrm{diam}(X)$.

Dyadic intervals on $\mathbb{T}$ or $\mathbb{R}$ are not a multi-resolution family, e.g., $X = [-1, 1] \subset \mathbb{R}$ is not contained in any dyadic interval, violating (3). However, the family of triples of all dyadic intervals (or cubes) do form a multi-resolution family. Similarly, if we add all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family (this is sometimes called the "$\frac{1}{3}$-trick", [35]). The analogous construction for dyadic squares in $\mathbb{R}^2$ is to take all translates by elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\}^2$.

The following simple lemma shows that in many computations, the choice of a particular multi-resolution family is not important.

Lemma 3.1. *Suppose $\{X_j\}$, $\{Y_k\}$ are two multi-resolution families on a space X and that α is a function mapping subsets of X to $[0,\infty)$ that satisfies $\alpha(E) \lesssim \alpha(F)$, whenever $E \subset F$ and $\operatorname{diam}(F) \lesssim \operatorname{diam}(E)$. Then*

$$\sum_j \alpha(X_j) \simeq \sum_k \alpha(Y_k).$$

Proof. By Condition (3) above, each X_j is contained in some set $Y_{k(j)}$ of comparable diameter. Hence $\alpha(X_j) \lesssim \alpha(Y_{k(j)})$ by assumption. Each Y_k is contained in a comparably sized X_m, and X_m can contain at most a bounded number of comparably sized subsets X_j. Thus each Y_k is only chosen boundedly often as a $Y_{k(j)}$. Thus $\sum_j \alpha(X_j) \lesssim \sum_k \alpha(Y_k)$. The opposite direction follows by reversing the roles of the two families. □

If we have a multi-resolution family on the unit circle then its image under a conformal map of the unit disk to quasidisk gives a multi-resolution family on the bounding quasicircle. This fact is Lemma 9.9 in Tim Mesikepp's thesis [32].

It will be convenient to consider several equivalent formulations of condition (1.14) involving Jones's β-numbers. For $x \in \mathbb{R}^2$ and $t > 0$, define

$$\beta_\Gamma(x,t) = \frac{1}{t} \inf_L \max\{\operatorname{dist}(z,L) : z \in \Gamma, |x-z| \le t\},$$

where the infimum is over all lines hitting the disk $D = D(x,t)$ and let $\widetilde{\beta}_\Gamma(x,t)$ be the same, but where the infimum is only taken over lines L hitting x. Although β (and its variations) depend on Γ, this set is usually clear from context and we will often drop it from our notation, i.e., we simply write β in place of β_Γ.

Since $\widetilde{\beta}$ is defined as the infimum over a smaller collection of lines than β, clearly $\beta(x,t) \le \widetilde{\beta}(x,t)$ and it is not hard to prove that $\widetilde{\beta}(x,t) \le 2\beta(x,t)$ if $x \in \Gamma$. Given a Jordan arc γ with endpoints z,w we let

$$\beta(\gamma) = \frac{\max\{\operatorname{dist}(z,L) : z \in \gamma\}}{|z-w|},$$

where L is the line passing through z and w. The following "well known" fact says that the various formulations of the β-numbers are equivalent.

Lemma 3.2. *[Lemma B.2, [12]] If Γ is a closed Jordan curve or a Jordan arc in $\mathbb{R}^n$ such that (1.14) holds, then Γ is a chord-arc curve. Moreover, (1.14) holds if and* *only if any of the following conditions holds:*

$$\int_0^\infty \iint_{\mathbb{R}^n} \beta^2(x,t)\frac{dxdt}{t^{n+1}} < \infty, \tag{3.1}$$

$$\int_0^\infty \int_\Gamma \widetilde{\beta}^2(x,t)\frac{dsdt}{t^2} < \infty, \tag{3.2}$$

$$\sum_j \beta^2(\Gamma_j) < \infty, \tag{3.3}$$

where dx is volume measure on $\mathbb{R}^n$, ds is arclength measure on Γ, and the sum in (3.3) is over a multi-resolution family $\{\Gamma_j\}$ for Γ. Convergence or divergence in (3.1) and (3.2) is not changed if $\int_0^\infty$ is replaced by $\int_0^M$ for any $M > 0$.

4. Theorem 1.8: (3) ⇒ (7)

We start with some standard definitions that we will use throughout the renainder of this paper. Given an arc $I \subset \mathbb{T} = \partial\mathbb{D}$ of length $|I|$ less than 1, we define the corresponding Carleson square Q_I as

$$Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| < |I|\}.$$

We define the "top half" of Q_I as

$$W_I = T(Q_I) = \{z \in \mathbb{D} : z/|z| \in I, |I|/2 < 1 - |z| < |I|\}.$$

When I ranges over the dyadic subintervals of $\mathbb{T}$ these sets (together with a single disk around the origin) form a Whitney decomposition $\mathcal{W}$ of $\mathbb{D}$, i.e., collection of closed sets W with disjoint interiors that cover $\mathbb{D}$ and satisfy $\operatorname{diam}(W) \simeq \operatorname{dist}(W, \partial\mathbb{D})$.

For each element $W \in \mathcal{W}$, let

$$\eta(W) = \max_{z\in W} |(\log f')'|(1 - |z|^2) = \max_{z\in W} |\frac{f''}{f'}|(1 - |z|^2).$$

Standard results, e.g., Theorem VII.2.1 of [27], imply that $\eta(W) \leq 6$ for all $W \in \mathcal{W}$ and any conformal f. Each element $W \in \mathcal{W}$ has hyperbolic area comparable to 1, and Euclidean area comparable to $(1 - |z|^2)^2$ for any $z \in W$. Hence

$$\int \left|\frac{f''}{f'}\right|^2 dxdy = \sum_{\mathcal{W}} \int_W \left|\frac{f''}{f'}\right|^2 dxdy \lesssim \sum_{\mathcal{W}} \eta(W)^2.$$

Let $u(z) = f''(z)/f'(z)$ and let $B_z = D(z,(1-|z|)/2)$. Since u is a holomorphic function, the mean value property and the Cauchy-Schwarz inequality imply

$$|u(z)|^2 \leq \left(\frac{1}{\mathrm{area}(B_z)}\left|\int_{B_z} u(z)dxdy\right|\right)^2 \leq \frac{1}{\mathrm{area}(B_z)}\int_{B_z}|u(z)|^2dxdy.$$

Thus the maximum of $|u|^2$ over W is bounded above by the integral of $|u|^2$ over the union of W and a bounded number of adjacent Whitney boxes (enough to cover the set $\cup_{z\in W}B_z$). Each Whitney box is only used a bounded number of times, so we see that (3) of Theorem 1.8 is equivalent to

$$\sum_{W\in\mathcal{W}}\eta(W)^2<\infty. \tag{4.1}$$

The next step closely follows a calculation from Pommerenke's paper [40]. Pommerenke's result implies that if $\eta(W)\leq\epsilon$ for all Whitney boxes inside $2Q_I$, then $f(I)$ is a quasi-arc with small constant. Clearly (4.1) implies that this holds for all sufficiently small Carleson boxes, say smaller than some $r>0$. The conformal map f restricted to such a Carleson square Q_{2I} has a K-quasiconformal extension to the reflection across $2I$, with K close to 1. Hence a simple consequence of Mori's theorem (Theorem III.C of [3]) is that if $J\subset I$ is a sub-arc, then

$$\frac{\mathrm{diam}(f(J))}{\mathrm{diam}(f(I))}\leq C\left(\frac{\mathrm{diam}(J)}{\mathrm{diam}(I)}\right)^\alpha, \tag{4.2}$$

for some $C=C(\alpha)<\infty$, and where we may take $\alpha<1$ as close to 1 as we wish, if r is small enough. For our purposes, it will suffice to take $\alpha=3/4$ (any value $>1/2$ would work). Henceforth, we assume r has been chosen so that (4.2) holds for all arcs of length less than r.

For $z,w\in\mathbb{D}$, let $\rho(z,w)$ denote the hyperbolic distance between z and w. Note that Whitney squares are approximately unit size in the hyperbolic metric: each has uniformly bounded hyperbolic diameter and contains a hyperbolic ball of radius bounded uniformly away from zero. Suppose $z_0\in W=T(Q_I)\in\mathcal{W}$, and suppose $z\in 2Q_I$. Let $W=W_0,\dots,W_N$ be the list of Whitney squares hit by the hyperbolic geodesic γ from z_0 to z; note that $N=N(z)\simeq 1+\rho(z_0,z)$ and that the boxes can be ordered so that $W_0=W$, $k\simeq 1+\rho(W_0,W_k)$, and (taking Euclidean diameters)

$$\mathrm{diam}(W_k)\simeq\mathrm{diam}(W_0)\exp(-\rho(W_0,W_k)). \tag{4.3}$$

Moreover, we can choose points $z_k \in \gamma \cap W_k$ that are ordered by increasing distance from z_0. See Figure 1.

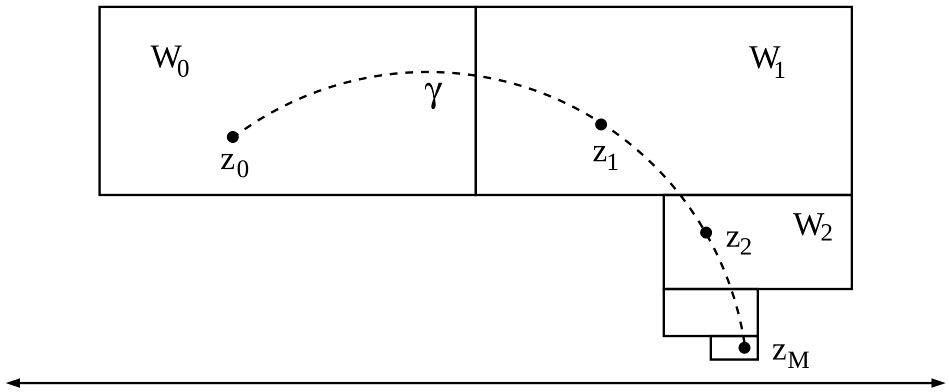

FIGURE 1. The chain of Whitney boxes connecting z_0 to z_M. The hyperbolic geodesic from z_0 to z is denoted γ

By a linear rescaling of f we may assume $f'(z_0) = 1$, without changing the values of $(\log f')'$ and hence without changing the η's. It is convenient to truncate the sequence of Whitney boxes in some cases. Define $M \leq N$ to be the first index where

$$\sum_{k=0}^{M} \eta(W_k) \geq 1, \tag{4.4}$$

or set $M = N$ if there is no such index. If $M = N$, let $z_M = z$; otherwise let z_M be some point in $\gamma \cap W_M$. Note that by our assumptions either $z_M = z$ or, as a consequence of (4.2,

$$\begin{aligned} |f(z_M) - f(z)| &\lesssim \operatorname{diam}(f(W_0)) \left(\frac{\operatorname{diam}(W_M)}{\operatorname{diam}(W_0)} \right)^{\alpha} \\ &\lesssim \operatorname{diam}(f(W_0)) \cdot \exp(-\alpha\rho(W_M, W_0)). \end{aligned}$$

If $z_M \neq z$ then (4.4) holds, so

$$\begin{aligned} |f(z_M) - f(z)| &\lesssim \operatorname{diam}(f(W_0)) \left(\sum_{k=0}^{M} \eta(W_k) \right) \exp(-\alpha\rho(W_M, W_0)) \\ &\lesssim \operatorname{diam}(f(W_0)) \left(\sum_{k=0}^{M} \eta(W_k) \exp(-\alpha\rho(W_k, W_0)) \right). \end{aligned}$$

Similarly, either $z_M = z$ or

$$\begin{aligned}
|z_M - z| &\lesssim \operatorname{diam}(W_M) \\
&\lesssim \operatorname{diam}(W_0)\exp(-\rho(W_M, W_0)) \\
&\lesssim \operatorname{diam}(W_0)\left(\sum_{k=0}^{M}\eta(W_k)\right)\exp(-\alpha\rho(W_M, W_0)) \\
&\lesssim \operatorname{diam}(W_0)\left(\sum_{k=0}^{M}\eta(W_k)\exp(-\alpha\rho(W_k, W_0))\right).
\end{aligned}$$

Pommerenke's estimate (see his proof of (i) $\Rightarrow$ (ii) in Theorem 1 on page 201 of [40]), says that if $z_n \in \gamma \cap W_n$,

$$|\log f'(z_n)| \le \int_{z_0}^{z}\left|\frac{f''(t)}{f'(t)}\right||dz| \le \sum_{k=0}^{n-1}\int_{z_k}^{z_{k+1}}\frac{\eta(W_k)}{\operatorname{diam}(W_k)}|dz| \lesssim \sum_{k=0}^{n-1}\eta(W_k).$$

Thus

$$|f'(z_n) - 1| \le \exp\left(C\sum_{k=0}^{n-1}\eta(W_k)\right) - 1.$$

Integrating, and using $e^x - 1 \lesssim x$ for $x \in [0,1]$,

$$\begin{aligned}
|[f(z_M) - f(z_0))] - [z_M - z_0]| &\lesssim \int_{z_0}^{z_M}\left[\exp(C\sum_{k=0}^{M}\eta(W_k)) - 1\right]|dz| \\
&\lesssim \sum_{k=0}^{M-1}\int_{z_k}^{z_{k+1}}\left[\sum_{j=0}^{k-1}\eta(W_j)\right]|dz| \\
&\lesssim \sum_{k=0}^{M-1}\operatorname{diam}(W_k)\left[\sum_{j=0}^{k-1}\eta(W_j)\right] \\
&\lesssim \sum_{j=0}^{M-1}\eta(W_j)\sum_{k=j+1}^{M-1}\operatorname{diam}(W_k) \\
&\lesssim \sum_{j=0}^{M-1}\eta(W_j)\operatorname{diam}(W_j).
\end{aligned}$$

The last inequality uses that fact that the $\{\operatorname{diam}(W_k)\}$ decreases approximately geometrically (the geodesic Γ can hit more than one Whitney box of the same size, but only boundedly many, so the sequence of sizes is $O(\lambda^k)$ for some $\lambda < 1$). Hence the sum above is dominated by a multiple of its largest term. Using (4.3) and the fact that $\alpha < 1$, we see that

$$(f(z_M) - f(z_0)) - (z_M - z_0) = O\left(\operatorname{diam}(W_0) \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha\rho(W_0, W_k))\right).$$

Thus by adding $f(z) - f(z)$ and $z - z$ to both sides and rearranging we get

$$\begin{aligned} f(z) - f(z_0) &= (z - z_0) + O\left(\sum_{k=0}^{M} \eta(W_k)\operatorname{diam}(W_k)\right) + (f(z) - f(z_M)) + (z - z_M) \\ &= (z - z_0) + O\left(\operatorname{diam}(W_0) \sum_{k=0}^{M} \eta(W_k) \exp(-\alpha\rho(W_0, W_k))\right). \end{aligned}$$

The last equation holds since we have previously shown that each of the last three terms in the first line is dominated by the sum in the second line above, and that $|z - z_M| \simeq |z - z_0| \simeq \operatorname{diam}(W_0)$. This equation says that f restricted to I is linear with a small error, hence $f(I)$ will be close to a line segment with small error. More precisely, since f has a quasiconformal extension to the plane, and we have normalized so $f'(z_0) = 1$, we have $\operatorname{diam}(f(I)) \simeq |I|$ and

$$\beta(\gamma) \lesssim \sup_{z \in I} \sum_{k=0}^{M(z)} \eta(W_k) \exp(-\alpha\rho(W, W_k))).$$

In particular, for each $W = T(Q_I) \in \mathcal{W}$ that is small enough, we can choose a point $z_0 \in W$, a boundary point $z \in 2I$, the geodesic segment γ_W connecting z_0 to z, and the chain of adjacent Whitney boxes $\mathcal{C}(W)$ hitting γ_W, so that the corresponding sum is within a factor of 2 of the supremum over all points $z \in 2I$. Next choose s so that $1 < s < 2\alpha = 3/2$. Then summing over all sufficiently small Whitney boxes in $\mathbb{D}$ and using the Cauchy-Schwarz inequality gives

$$\begin{aligned} \sum_{W:\operatorname{diam}(W)<r} \beta^2(\gamma_W) &\lesssim \sum_{W} \left(\sum_{W' \in \mathcal{C}(W)} \eta(W') \exp(-\alpha\rho(W, W')))\right)^2 \\ &\lesssim \sum_{W} \left(\sum_{W' \in \mathcal{C}(W)} \eta^2(W') \exp(-s\rho(W, W')))\right) \times \\ &\qquad \left(\sum_{W' \in \mathcal{C}(W)} \exp((s - 2\alpha)\rho(W, W')))\right). \end{aligned}$$

Since $s - 2\alpha < 0$ and the distances between W and W' grow linearly, the sum in the second term of the product is a geometric sum, and hence converges to a number bounded independent of W. Thus if $I_W \subset \mathbb{T}$ denotes the base interval of W and $\sigma_W = f(I_W)$,

$$\sum_{W:\mathrm{diam}(W)<r} \beta^2(\sigma_W) \lesssim \sum_{W'} \eta^2(W') \sum_{W:W'\in\mathcal{C}(W)} \exp(-s\rho(W,W'))$$

Since $s > 1$, the second sum above is bounded. Hence we obtain

$$\sum_{W} \beta^2(\sigma_W) \lesssim \sum_{W'} \eta^2(W').$$

The collection $\{I_W\}$ is not itself a multi-resolution family for $\mathbb{T}$, but a finite family of rotations of $\mathbb{D}$ gives such a family (the $\frac{1}{3}$-trick) and the proof above applies equally well to each family. Thus the β^2-sum is finite over some multi-resolution family of arcs for Γ, and hence (3) implies (1.7) using Lemma 3.2.

5. Theorem 1.8: (7) $\Rightarrow$ (1)

Suppose $\Omega \subset \mathbb{R}^2$ is open and $x \in \Omega$. Then there is a maximal closed dyadic square Q containing x so that $\mathrm{diam}(Q) \leq \frac{1}{2}\mathrm{dist}(Q, \partial\Omega)$. The collection of such squares covers Ω and have pairwise disjoint interiors and is called Whitney decomposition of Ω using dyadic squares.

Assume (7) holds in Theorem 1.8. Since $\sum_Q \beta_\Gamma^2(Q) < \infty$, only finitely many of the β's can be larger than 1/1000. Let $U(\epsilon)$ denote the ϵ-neighborhood of Γ, and choose ϵ_0 so small that $U(\epsilon_0)$ only contains dyadic Whitney squares Q with $\beta_\Gamma(Q) < 1/1000$. Form a triangulation of Ω by connecting the center of each square to the vertices on its boundary. Note that neighboring triangles have comparable diameters and that all angles are bounded uniformly above 0 and below π.

We will define a reflection across Γ that is defined on a neighborhood of Γ and is piecewise affine on the above triangles. Let $\mathcal{S}_k$ be the collection of squares Q in the Whitney decomposition so that $\ell(Q) = 2^{-k}$ and let $\mathcal{S} = \cup_{k>k_0}\mathcal{S}_k$ where k_0 is chosen so that the elements of $\mathcal{S}$ are all contained in $U(\epsilon_0/100)$. Order the elements of $\{Q_j\}_1^\infty = \mathcal{S}$ so that side lengths are non-increasing. For each Q_j choose a dyadic square Q'_j of comparable size that hits Γ and so that $3Q'_j$ contains Q_j. Note that $Q'_j \subset U$, so $\beta_\Gamma(Q'_j)$ is small. To begin, choose a line L_j that minimizes the definition of $\beta_\Gamma(Q'_j)$. Reflect all four vertices of Q_1 across L_1. In general, reflect each vertex v of Q_j across L_j to a point v^* in Ω^*, if it was not already reflected by belonging to some Q_k with $k < j$. See Figure 2.

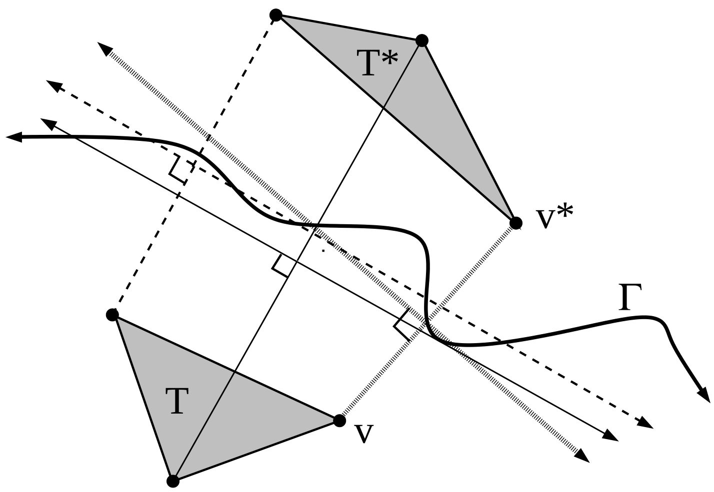

FIGURE 2. Each vertex on one side of Γ is reflected across a line approximating Γ. Since different vertices may use different lines the corresponding triangles become distorted, but since the lines come within $O(\beta)$ of each other and have angles differing by $O(\beta)$, the triangles are related by an affine map that is quasiconformal with dilatation $O(\beta)$.

The main point is that each vertex v belongs a uniformly bounded number of Q_J's and the different possible reflections v^* of v corresponding to these different squares all lie within distance $\beta_j \cdot \text{dist}(v, \Gamma)$ of each other, where Q_j is any of the Whitney squares having v as a corner and $\beta_j = \beta_\Gamma(Q'_j)$. This occurs because all the lines we might use have directions that differ by at most $O(\beta_j)$, and they all pass within $O(\beta_j \ell(Q'_j))$ of some point in Q'_j. We now define affine maps on each element of our triangulation that lies inside $U(\epsilon_0/1000)$ by sending each vertex to its reflection v^*. Suppose T is a triangle associated to Q_j. Then $\text{diam}(T) \simeq \text{dist}(T, \Gamma)$. The reflected vertices of T form a triangle T^* that is within $O(\beta_j)$ of being congruent to T. Thus the following simple result applies:

Lemma 5.1. *Suppose $T = (v_1, v_2, v_3)$ is a planar triangle of diameter 1 and that the three interior angles are bounded uniformly away from 0 and π. Suppose T' is another triangle whose vertices (v'_1, v'_2, v'_3) are each within ϵ of the corresponding* *vertex of T the affine map that extends $v_j \to v_j'$, $j = 1, 2, 3$ is quasiconformal with complex dilatation bounded by $O(\epsilon)$.*

Proof. One way to check this is by an explicit computation: the affine map that sends $\{0, 1, a\}$ to $\{0, 1, b\}$ with a, b in the upper half-plane is $z \to \alpha z + \beta \bar{z}$ where $\alpha = (b - \bar{a})/(\bar{a} - a)$ and $\beta = 1 - \alpha$. From this one sees that the complex dilatation is the constant $\mu = \beta/\alpha = (a - b))/(b - \bar{a}) = \tanh(\rho(a, b))$, where ρ denotes the hyperbolic metric on $\mathbb{H}^2$. □

Extending the map between vertices linearly, we get an affine map from T to T^* that is quasiconformal with dilatation bounded $O(\beta_j)$. Putting together the different triangles we get a piecewise affine map.

The homeomorphism R constructed above on a neighborhood of U of $\Gamma \subset \mathbb{R}^2$ is clearly quasiconformal on $U \setminus \Gamma$. Since Γ is a quasicircle, it is removable for quasiconformal homeomorphisms and hence our map is quasiconformal on all of U, i.e., we have defined a quasiconformal reflection across Γ on a neighborhood U of Γ. Each triangle T has hyperbolic area $\simeq 1$, so

$$\int_T |\mu(z)|^2 d\mathrm{A}_\rho(z) = O(\beta_\Gamma^2(Q)),$$

for some dyadic square Q with $\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, T) \simeq \mathrm{diam}(T)$. Therefore

$$\int_U |\mu(z)|^2 d\mathrm{A}_\rho(z) = O\left(\sum_Q \beta_\Gamma^2(Q)\right)$$

since each Q occurs for only boundedly many T. We now extend this map diffeomorphically (and hence quasiconformally) to the rest of Ω to obtain a reflection satisfying (1.11).

6. Theorem 1.8: (5) $\Rightarrow$ (6)

We start with some notation and prove a series of preliminary lemmas. Suppose φ is a circle homeomorphism that is absolutely continuous and such that both φ' and $\log|\varphi'|$ are in $L^1(\mathbb{T})$. Let U be the harmonic extension of $|\varphi'|$ from $\mathbb{T}$ to $\mathbb{D}$. Let u be the harmonic extension of $\log|\varphi'|$ from $\mathbb{T}$ to $\mathbb{D}$. As before, let $I \subset \mathbb{T}$ be an arc, $Q = Q_I = \{z : z/|z| \in I, 1 - \ell(I) \leq z < 1\} \subset \mathbb{D}$ be the Carleson square with base I,

$$T(I) = T(Q_I) = \{z \in Q_I : z \leq 1 - \ell(I)/2\}$$

is the top half of Q, z_I is the center of the top edge of $T(I)$. Let $T^*(Q)$ be the union of $T(Q')$ for all Q' so that $T(Q')$ touches $T(Q)$, i.e., this is approximately a unit hyperbolic neighborhood of $T(Q)$.

Let $u_I = \int_I u ds/\ell(I)$ be the average of u over I. For u harmonic on $\mathbb{D}$, Q a dyadic Carleson square and $0 \leq \alpha \leq 1$ define (the sum is over dyadic Carleson subsquares of $3Q$)

$$\epsilon(u, Q, \alpha) = \left(\sum_{Q' \subset 3Q} \int_{T(Q')} |\nabla u|^2 \left(\frac{\ell(Q')}{\ell(Q)} \right)^{1-\alpha} dxdy. \right)^{1/2}$$

Note that $\alpha < \alpha'$ implies $\epsilon(u, Q, \alpha) \leq \epsilon(u, Q, \alpha')$. In what follows, the constant in inequalities of the form $\lesssim \epsilon(u, Q, \alpha)$, may depend on α (but for our proof, we will only need one value, say $\alpha = 1/2$).

Suppose $W \subset W' \subset \mathbb{D}$ are domains so that if Q, Q' are Carleson squares so that $Q \subset W$ and $Q' \subset 3Q$, then $T(Q') \subset W'$. For our application, we will simply take $W = \{z : \frac{7}{8} < |z| < 1\}$ and $W' = \mathbb{D}$.

Lemma 6.1. *$\sum_{Q \subset W} \epsilon^2(u, Q, \alpha) \lesssim \int_{W'} |\nabla u|^2 dxdy < \infty$.*

Proof. Note that

$$\begin{aligned}
\sum_{Q \subset W} \epsilon^2(u, Q, \alpha) &= \sum_{Q \subset W} \sum_{Q' : Q' \subset 3Q} \int_{T(Q')} |\nabla u|^2 \left(\frac{\ell(Q')}{\ell(Q)} \right)^{1-\alpha} dxdy \\
&\leq \left(\sum_{Q' \subset W'} \int_{T(Q')} |\nabla u|^2 dxdy \right) \left(\sum_{Q : Q' \subset 3Q_I} \left(\frac{\ell(Q')}{\ell(I)} \right)^{1-\alpha} \right) \\
&\lesssim \int_{W'} |\nabla u|^2 dxdy,
\end{aligned}$$

since the second sum (the one over Q) is dominated by a geometric series whose largest term is $\simeq 1$. □

Recall that u is the Poisson integral of $\log |\varphi'|$. If Q is a Carleson square with base I and top edge E, then the averages of u over I and E will be denoted u_I and u_I^* respectively. Our first goal is to prove

$$|u(z_I) - u_I|^2 \lesssim \epsilon(u, Q, 0)^2 \leq \epsilon(u, Q, \alpha)^2. \tag{6.1}$$

We will do this in two steps.

Lemma 6.2. *With notation as above,*

$$|u(z_I) - u_I^*|^2 \lesssim \int_{T^*(Q)} |\nabla u|^2 \frac{\ell(Q')}{\ell(I)} dxdy. \tag{6.2}$$

Proof. If u is harmonic, then the vector field $|\nabla u$ has a natural interpretation as a holomorphic function, and thus it satisfies the mean value property. So for $z \in T(Q)$, the Cauchy-Schwarz inequality implies

$$|\nabla u(z)| \leq \int_B |\nabla u| \frac{dzdy}{\text{area}(B)} \leq \left(\int_B |\nabla u|^2 \frac{dzdy}{\text{area}(B)} \right)^{1/2} \leq \frac{1}{\ell(I)} \left(\int_B |\nabla u|^2 dzdy \right)^{1/2},$$

where B is a ball of radius $(1 - |z|)/2$ around z. Thus

$$\max_{z,w \in T(Q)} |u(z) - u(w)| \leq \max_{z \in T(Q)} |\nabla u(z)| \text{diam}(T(Q)) \lesssim \left(\int_{T^*(Q)} |\nabla u|^2 dzdy \right)^{1/2}, \tag{6.3}$$

where $T^*(Q)$ is the union of all top halves of dyadic Carleson cubes touching $T(Q)$. Hence the difference between $u(z_I)$ and u_I^*, is bounded by the right side of (6.2). □

Since $Q' \subset T^*(Q)$ implies $\ell(Q) \simeq \ell(I)$, and since $T^*(Q) \subset 3Q$, we see that (6.2) implies

$$|u(z_I) - u_I^*|^2 \lesssim \int_{T^*(Q)} |\nabla u|^2 \frac{\ell(Q')}{\ell(I)} dxdy \leq \epsilon(u, Q, 0)^2. \tag{6.4}$$

Lemma 6.3. *With notation as above,*

$$|u_I - u_I^*|^2 \lesssim \epsilon(u, Q, 0)^2. \tag{6.5}$$

Proof. The difference of averages $u_I - u_I^*$ can computed by applying Green's theorem

$$\iint_Q u\Delta v - v\Delta u dxdy = \int_{\partial Q} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} ds$$

on Q with the functions u and $v = \log 1/|z|$. Both interior terms vanish by harmonicity. The boundary integral along I equals $u_I \ell(I)$, the boundary integral along the top edge E of Q gives $-u_I^* \ell(I)$ plus a term bounded by

$$\ell(I) \int_J |\nabla u| ds \lesssim \ell(I) \left(\int_{T^*(Q)} |\nabla u|^2 dzdy \right)^{1/2},$$

which is less than the right side of (6.5) times $\ell(I)$.

The normal derivative of v over the radial sides of Q is zero, so the integrals over these sides of Q can be bounded using (6.3) and the Cauchy-Schwarz inequality (break each radial side of Q into intervals by intersecting with top halfs of dyadic sub-squares):

$$\begin{aligned}\int_0^{1-|z_I|} |\nabla u| r dr &\lesssim \sum_{Q':Q'\cap\partial Q\neq\emptyset} \ell(Q') \left(\int_{T^*(Q')} |\nabla u|^2 dz dy\right)^{1/2} \\ &\lesssim \left(\sum_{Q'} \ell(Q')\right)^{1/2} \left(\sum_{Q'} \int_{T^*(Q')} |\nabla u|^2 \ell(Q') dz dy\right)^{1/2} \\ &\lesssim \sqrt{\ell(I)} \left(\sum_{Q'} \int_{T^*(Q')} |\nabla u|^2 \ell(Q') dz dy\right)^{1/2}\end{aligned}$$

where the sum is over cubes $Q' \subset 3Q_I$ that hit the sides of Q_I; this sum is dominated by the sum over all $Q' \subset 3Q_I$, so this proves (6.5). □

Lemma 6.4. *Assume* $u(z_I) = 0$. *With notation as above*

$$\int_I (\exp((\alpha/2)u^2) - 1) \frac{ds}{\ell(I)} \lesssim \epsilon^2(u, Q_I, \alpha).$$

Proof. This estimate is essentially Theorem 1 of [38], which itself is a simpler version of results in [9], [16] and [31]. First note that for u in the Dirichlet class and $z \in Q$, the Cauchy-Schwarz inequality implies

$$|u(z) - u(z_I)| \lesssim \left(\int_Q |\nabla u|^2 dx dy\right)^{1/2} \left(1 + \sqrt{\log \frac{|I|}{1-|z|}}\right).$$

One can prove this by integrating $|\nabla u|$ over the line segment from z_I to I, applying the Cauchy-Schwarz inequality and the fact that ∇u satisfies the mean value inequality.

Hence if the Dirichlet integral of u over Q is small enough, and $z \in Q = Q_I$ then

$$|u(z)| = |u(z) - u(z_I)| \leq 1 + \sqrt{\log \frac{1-|z_I|}{1-|z|}}.$$

This holds for all small enough squares. Set $W(z) = \exp(\frac{\alpha}{2}u(z)^2) - 1$. Then explicit calculation shows

$$\begin{aligned}|\nabla W|^2 &\lesssim |\nabla u(z)|^2 \cdot |u(z)|^2 \exp(\alpha |u(z)|^2) \\ &\lesssim |\nabla u(z)|^2 \cdot \left(1 + \log \frac{|I|}{1-|z|}\right) \cdot (1-|z|)^{-\alpha/2} \\ &\lesssim |\nabla u(z)|^2 \cdot (1-|z|)^{-\alpha},\end{aligned}$$

and

$$\Delta W = 2\alpha |u'|^2 (1 + (\alpha/2)|u|^2) \exp(\frac{\alpha}{2}|u|^2).$$

satisfies a similar estimate.

Now apply Green's theorem with $u = \log 1/|z|$ and $v = W$. The boundary integral along I, the base of Q, is what we want to bound. The estimate given above for ΔW shows the interior integral is bounded by a multiple of

$$\sum_{Q' \subset Q} \int_{T(Q')} |\nabla u(z)|^2 \left(\frac{\ell(Q')}{\ell(Q)} \right)^{1-\alpha} dxdy \lesssim \epsilon^2(u, Q, \alpha),$$

as desired. The boundary integrals are similarly handled by the Cauchy-Schwarz inequality and the estimate for $|\nabla W|$ given above. □

Corollary 6.5. *Assume $u(z_I) = 0$. With notation as above,*

$$\int_I [\exp(u(s)) - 1] \frac{ds}{\ell(I)} \lesssim \epsilon(u, Q, \alpha).$$

Proof. By the Cauchy-Schwarz inequality

$$\int_I [\exp(u(s)) - 1] \frac{ds}{\ell(I)} \leq \left(\int_I \frac{(\exp(u(s)) - 1)^2}{\exp(\alpha u^2(s)/2) - 1} \frac{ds}{\ell(I)} \right)^{1/2} \times \left(\int_I (\exp(\alpha u^2(s)/2) - 1) \frac{ds}{\ell(I)} \right)^{1/2}$$

Note that $(e^x - 1)^2/(e^{\alpha x^2/2} - 1)$ is bounded on $\mathbb{R}$ by some constant $C(\alpha)$: the numerator and denominator of the fraction are both $\simeq x^2$ near $= 0$ and the ratio tends to zero as $x \nearrow \infty$. Thus the first integral is bounded depending only on α. Lemma 6.4 implies second integral is bounded by $\epsilon(u, Q, \alpha)$. □

Recall that U is the harmonic extension of $|\varphi'|$. Thus, very roughly, we expect $\log U \approx u$. We make this precise as follows. Let U_I be the average of U over I (hence also the average of $|\varphi'|$ over I, i.e., it equals $\ell(\varphi(I))/\ell(I)$)).

Lemma 6.6. *For any $0 < \alpha < 1$, $0 \leq \log(U_I) - u_I \lesssim \epsilon(u, Q, \alpha)$.*

Proof. The first inequality is Jensen's inequality: since $\log x$ is concave down, there is an affine function such that $\log x \leq L(x)$ for all $x > 0$, with equality at $x = U_I$, and hence

$$\log U_I = L(\int_I |\varphi'| ds/\ell(I)) = \int_I L(|\varphi'|) \frac{ds}{\ell(I)} \geq \int_I \log |\varphi'| \frac{ds}{\ell(I)} = u_I.$$

To prove the second inequality, note that multiplying φ' by a constant C changes both $\log U_I$ and u_I by the same additive constant $\log C$, so their difference is unchanged. Hence we may assume $u_I = 0$ and $U_I \geq 0$. Then using $\log x \leq x - 1$,

$$\log(U_I) \quad = \quad \log \int_I |\varphi'(s)| \frac{ds}{\ell(I)} = \log \int_I \exp(u(s)) \frac{ds}{\ell(I)} \leq \int_I [\exp(u(s)) - 1] \frac{ds}{\ell(I)}.$$

The last term is bounded by $\epsilon(u, Q, \alpha)$ by Corollary 6.5. □

Lemma 6.7. *With notation as above,*

$$|\ell(\varphi(J)) - \exp(u(z_I))\ell(J)| \lesssim \ell(\varphi(I)) \cdot \epsilon(u, Q, \alpha)$$

where J is either the left or right half of I.

Proof. Note that since φ' is absolutely continuous,

$$\ell(\varphi(J)) = \int_J |\varphi'| ds = \int_J \exp(u(s)) ds,$$

$$\left|\frac{\ell(\varphi(J))}{\exp(u(z_I))\ell(J)} - 1\right| \quad = \quad \left|\int_J [\exp(u(s) - u(z_I)) - 1] \frac{ds}{\ell(J)}\right| \lesssim \epsilon(u, Q, \alpha)$$

by Corollary 6.5. Thus

$$|\ell(\varphi(J)) - \exp(u(z_I))\ell(J)| \quad \lesssim \quad \exp(u(z_I))\ell(J)\epsilon(u, Q, \alpha)$$

Note that by Jensen's inequality (as in Lemma 6.6) $u(z_I) \leq \log U_I$, so

$$\exp(u(z_I))\ell(J) \leq \frac{1}{2} U_I \cdot \ell(I) = \frac{\ell(\varphi(I))}{2\ell(I)} \ell(I) = \frac{1}{2} \ell(\varphi(I)). \tag{6.6}$$ □

Lemma 6.8. *If J is either the left or right half of I, then*

$$|\ell(\varphi(J)) - \frac{1}{2}\ell(\varphi(I))| \quad \lesssim \quad \ell(\varphi(I)) \cdot \epsilon(u, Q, \alpha).$$

Proof. Observe that

$$\begin{aligned} |\ell(\varphi(J)) - \frac{1}{2}\ell(\varphi(I))| \quad &\leq \quad |\ell(\varphi(J)) - \exp(u(z_I))\ell(J)| + |\exp(u(z_I))\ell(J) - \frac{1}{2}\ell(\varphi(I))| \\ &= \quad |\ell(\varphi(J)) - \exp(u(z_I))\ell(J)| + \ell(J)|\exp(u(z_I)) - U_I| \end{aligned}$$

The first term in the last line is bounded using Lemma 6.7. The second term is bounded using the fact that $x \leq y$ implies $e^y - e^x \leq e^y(y - x)$ (since the derivative of e^t is less than e^y on $[x, y]$):

$$\begin{aligned} |\exp(u(z_I)) - U_I| &= |\exp(u(z_I)) - \exp(\log(U_I))| \\ &= U_I|u(z_I) - \log(U_I)| \\ &\lesssim U_I \cdot \epsilon(u, Q, \alpha). \end{aligned}$$

Since $U_I = \ell(\varphi(I))/\ell(I)$ by definition, we get

$$\ell(J)|\exp(u(z_I)) - U_I| \leq \ell(J)\frac{\ell(\varphi(I))}{\ell(I)}|u(z_I) - \log(U_I)| \leq \frac{1}{2}\ell(\varphi(I))\epsilon(u, Q, \alpha). \quad \square$$

Proof of (5) ⇒ (6). Assume $\log|\varphi'| \in H^{1/2}(\mathbb{T}) \subset \text{VMO} \subset L^1$; hence φ' is also in L^1 by the John-Nirenberg theorem (or one can use sharper results for $H^{1/2}$ in [9], [16], [29], [31]). Hence the harmonic extension u of this function satisfies $\int_{\mathbb{D}} |\nabla u|^2 dxdy < \infty$. Note that

$$|\varphi(m(I)) - m(\varphi(I))| \leq \max_J |\ell(\varphi(J)) - \frac{1}{2}\ell(\varphi(I))|,$$

where the maximum is taken over J being the left or right half of I. Thus by the preceding lemmas,

$$\text{qs}^2(\varphi, I) = \left|\frac{\varphi(m(I)) - m(\varphi(I))}{\ell(\varphi(I))}\right|^2 \leq \epsilon^2(u, Q_I, \alpha),$$

where Q is the Carleson square with base I. Summing over all dyadic intervals I is finite by Lemma 6.1. $\square$

7. Theorem 1.8: (6) ⇒ (1)

For each dyadic interval $I \subset \mathbb{T}$ with $\ell(I) \leq \pi/4$, triangulate $T(Q)$ using its four corners and the center of its lower edge as vertices. This gives a triangulation of an annulus $\{r < |z| < 1\}$ as shown in Figure 3. Given a circle homeomorphism φ and an interval I with left endpoint$x$, we can map each vertex $x(1-\ell(I))$ to $\varphi(x)(1-\ell(\varphi(I))$ and we extend this to a piecewise linear map of the triangulation, just as we did in Section 5. If the distortion is small enough (less than some fixed constant), the image triangle has the same orientation as the original, and the affine map of each triangle onto its corresponding image defines a homeomorphism of the annulus.

We want to show the quasiconformal distortion μ of these affine maps is bounded in terms of the quasisymmetric distortion of the boundary map ϕ. See Figure 4. Let $\mathcal{D}$ be a dyadic decomposition of $\mathbb{T}$, $I \in \mathcal{D}$ a dyadic interval, and J the adjacent dyadic

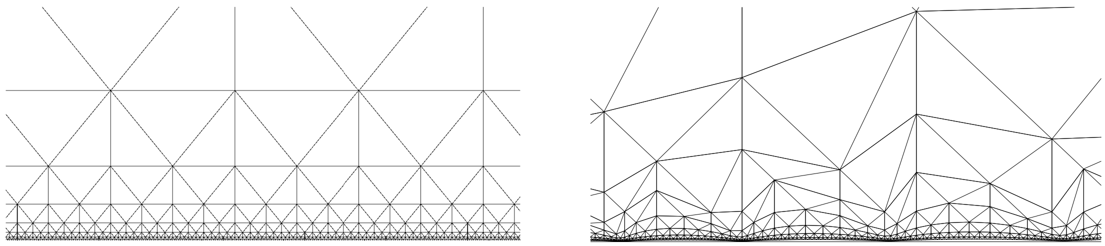

FIGURE 3. Triangulating the top half of each dyadic Carleson square gives a triangulation of the upper half-plane. The vertices can be mapped based on the boundary map, and the resulting image triangulation gives a piecewise linear quasiconformal map with the given boundary values (at least near the boundary where the quasisymmetric constant is small).

interval of the same length counterclockwise from I. From elementary geometry, it is easy to see that the dilations of the three affine maps corresponding to the dyadic interval I are bounded by a constant times the maximum of the terms $\mathrm{qs}(\varphi, K)$ where K is ranges over I, J and $I \cup J$. By assumption I and J come from the same dyadic decomposition $\mathcal{D}$ of $\mathbb{T}$. If $I \cup J$ were also an interval from this decomposition, then the $|\mu|$ would be bounded in terms of three terms for the form $\mathrm{qs}(\varphi, K)$ summed over this decomposition, and we would be done: the integral of $|\mu|^2$ with respect to hyperbolic area could be bounded by $O(\sum \mathrm{qs}(\varphi, K)^2)$.

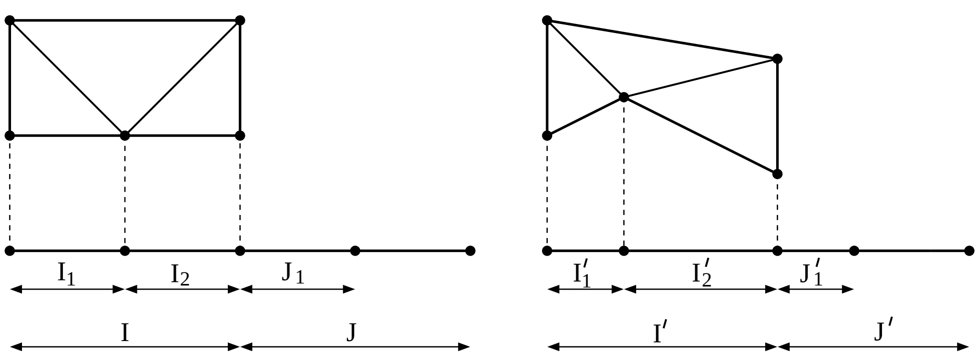

FIGURE 4. Here $I' = \varphi(I)$, and similarly for the other intervals. If $K = I \cup J$ is a dyadic interval, then μ can be bounded directly in terms of $\mathrm{qs}(\varphi, I), \mathrm{qs}(\varphi, J), \mathrm{qs}(\varphi(I \cup J))$. In general, $I \cup J$ is not in the same dyadic family as I and J, so we have to estimate the last term using a sum over the multi-resolution family obtained by the "$\frac{1}{3}$-trick".

In general, however, $I \cup J$ need not be an interval from $\mathcal{D}$, the dyadic decomposition containing I and J (i.e., I and J are adjacent, but don't have a common parent). However, by considering all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family $\mathcal{T}$ (recall the "$\frac{1}{3}$-trick"), so there is element of this family K that contains $I \cup J$ and has comparable size. We claim that $\mathrm{qs}(\varphi, I \cup J)$ can be bounded using a sum over intervals in $\mathcal{T}$.

Lemma 7.1. *Suppose $\varphi : [0,1] \to [0,1]$ is an increasing homeomorphism and $0 \le x \le 1$. Then*

$$|x - \varphi(x)| \le \sum_{n=0}^{\infty} \mathrm{qs}(\varphi, I_n)|\varphi(I_n)|,$$

where I_n is the dyadic interval of length 2^{-n} containing x. If x is the endpoint of a dyadic interval I, we need only sum over intervals strictly larger than I.

Proof. This is obvious if $x \in \{0, 1\}$ since these points map to themselves. The lemma is also clear for $x = 1/2$, and we only need to take the $n = 0$ term of the sum. In general, assume the inequality holds for both endpoints of a dyadic interval I_N of length 2^{-N}, where the bounding sum is only over larger dyadic intervals $I_0 \supset I_2 \dots$ containing I_N. If y is the midpoint of I and z the midpoint of $\varphi(I)$, then by definition

$$|\varphi(y) - z| \le \mathrm{qs}(\varphi, I)|\varphi(I)|.$$

Moreover, by the induction hypothesis, the endpoints of $\varphi(I)$ are within

$$\sum_{n=0}^{N-1} \mathrm{qs}(\varphi, I_n)|\varphi(I_n)|,$$

of the endpoints of I. Thus z is within the same distance of y. Thus

$$|\varphi(y) - y| \le |\varphi(y) - z| + |z - y| \le \sum_{n=0}^{N} \mathrm{qs}(\varphi, I_n)|\varphi(I_n)|.$$

This proves the lemma for dyadic rationals and the general case follows since φ is continuous. □

This lemma implies that $\mathrm{qs}(\varphi, I \cup J)\ell(I \cup J)$ can be bounded in terms of a sum of the form $\sum_{K \in \mathcal{C}(I)} \mathrm{qs}(\varphi, K)|K|$, where $\mathcal{C}(I)$ is the collection of intervals from the multi-resolution family $\mathcal{T}$ that contain an endpoint of I or J and have length $\lesssim |I \cup J|$. Since $\varphi : \mathbb{T} \to \mathbb{T}$ is quasisymmetric, it is also Hölder continuous, so there is a $0 < \lambda < 1$

so that every dyadic interval I of length 2^{-n} maps to an interval of length at most λ^n. The collection $\mathcal{C}(I)$ has only a bounded number of intervals of each length $2^{-n}|I|$ and the image lengths also decay geometrically so

$$\sum_{K\in\mathcal{C}(I)} |\varphi(K)| \lesssim |\varphi(I)|$$

Thus, by Cauchy-Schwarz, the sum of the dilatations over all triangles in our infinite triangulation of the disk is bounded by

$$\begin{aligned}
\sum |\mu(T)|^2 &\lesssim \sum_{I\in\mathcal{D}} \left(\sum_{K\in\mathcal{C}(I)} \mathrm{qs}(\varphi, K) \frac{|\varphi(K)|}{|\varphi(I)|} \right)^2 \\
&\lesssim \sum_{I\in\mathcal{D}} \sum_{K\in\mathcal{C}(I)} \mathrm{qs}(\varphi, K)^2 \frac{|\varphi(K)|}{|\varphi(I)|} \\
&= \sum_{K\in\mathcal{T}} \mathrm{qs}(\varphi, K)^2 \sum_{I\in\mathcal{D}, K\in\mathcal{C}(I)} \frac{|\varphi(K)|}{|\varphi(I)|} \\
&\lesssim \sum_{K\in\mathcal{T}} \mathrm{qs}(\varphi, K)^2.
\end{aligned}$$

The last line holds because for a fixed K the sum over I decays geometrical (there are a bounded number of I's in each generation associated to K, and their image lengths increase geometrically with the generation). Thus the dilatations are square summable over the set of triangles. Since each triangle has uniformly bounded hyperbolic area, we see that (6) implies (1) in Theorem 1.8.

8. Theorem 1.8: (4) $\Rightarrow$ (3)

First, some notation. For a point $z \in \mathbb{D}$, $|z| > 1/2$, let $I_z \subset \mathbb{T}$ be the arc centered at $z/|z|$ with length $2(1-|z|)$. For $M > 0$, $M \cdot I$ denotes the concentric arc with length $M|I|$.

Let $u(z) = \exp(i \arg f'(z))$ and let $g = u|_{\mathbb{T}}$ denote its boundary values. The function u is a mapping of the unit disk into the unit circle. Let $v(z)$ be the harmonic function on $\mathbb{D}$ with the same boundary values, i.e., v is the Poisson extension of g. Then v maps $\mathbb{D}$ into itself, and we claim this map is proper, i.e., points near $\mathbb{T}$ map to points near $\mathbb{T}$. To prove this, note that by the John-Nirenberg theorem (e.g., Theorem VI.2.1 of [26]) since $g \in H^{1/2} \subset \mathrm{VMO}$, for any $M < \infty$ and $\delta, \epsilon > 0$ we have

$$|\{x \in MI_z : |g(x) - v(z)| > \delta\} \leq \epsilon |I_z|$$

for all sufficiently short arcs $I \subset \mathbb{T}$. Since $g(x)$ only takes values in the unit circle, this implies $|v(z)| > 1-\delta$ if $|z|$ is close enough to 1. In particular, there is an annulus $A_r = \{r < |z| < 1\}$ where $|v(z)| > 3/4$.

Next we claim that $|u(z)-v(z)| \leq 1/4$ for z close enough to $\mathbb{T}$. Since $g(x)$ is close to constant on most of MI_z, the image of MI_z under f is a sub-arc γ of $\Gamma = f(\mathbb{T})$, and on this sub-arc the tangent directions are close to the constant value $\tau = izg(z)/|zg(z)|$ except on a small fraction of its length. Here we are using the fact that for chord-arc curves, harmonic measure and length are A^∞-equivalent, e.g., small harmonic measure implies small length in a quantitative way. See Section VII.4 of [27]. This implies γ is $o(\mathrm{diam}(\gamma))$ close to a line segment in direction τ in the Hausdorff metric. Since Γ is chord-arc and $\mathrm{diam}(\gamma) \gg \mathrm{dist}(f(z), \Gamma)$ if M is large, this implies

$$\mathrm{dist}(f(z), \Gamma \setminus \gamma) \gtrsim \mathrm{diam}(\gamma) \gg \mathrm{dist}(f(z), \Gamma).$$

Thus near z, f approximates a linear map and $u(z)$ is as close to $g(z)$ as we wish, if z is close enough to $\mathbb{T}$. Thus $|u - v| \leq 1/4$ on A_r if r is close enough to 1.

Thus v is a continuous map from A_r into $\{|z| > 3/4\}$ that stays within $1/4$ of u. Along any simple path in A_r to a boundary point on $\mathbb{T}$ we can lift v to a map V into the vertical strip $\{x + iy : -\log 2 < x < 0\}$ via the logarithm, and so that $V(z)$ stays close to $i \arg f'(z)$. This implies the lift is well defined on the whole annulus, because different possible lifts differ by at least $2\pi i$ and two different values can't both be close to the single value of $i \arg f'$.

V has a radial limit wherever g does, and this limit L satisfies $\exp(iL) = g(x)$. Since $|L - \arg f'(x)| \leq 1/4$, we deduce that $L = \arg f'(x)$. Since g has radial limits almost everywhere, so does V. Moreover, the gradient of V at z is comparable to the gradient of v at z (locally V is the composition of v with a branch of the logarithm, and the latter map has derivative close to 1 in the annulus). Thus

$$\int_{A_r} |\nabla V|^2 dxdy \lesssim \int_{\mathbb{D}} |\nabla v|^2 dxdy < \infty$$

since v is in the Dirichlet class (recall that it's boundary values $g = \exp(i \arg f') \in H^{1/2}(\mathbb{T})$ by assumption). By smoothing off V along the inner boundary of A_r we can create a smooth function $\widetilde{V}$ on $\mathbb{D}$ that has bounded Dirichlet integral and boundary values $\arg f'$ almost everywhere. This proves $\arg f' \in H^{1/2}$ by the Sobolev trace theorem. (One can also use Dirichlet's principle that the existence of such a V implies the harmonic extension of $\arg f'$ also has finite Dirichlet integral; everything can be justified in an elementary manner by replacing $f(z)$ by $f(rz)$, $0 < r < 1$ and taking limits as $r \nearrow 1$.) This completes the proof of (4) $\Rightarrow$ (3).

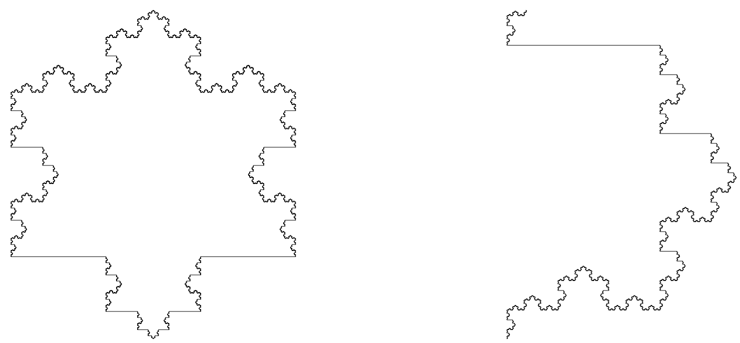

FIGURE 5. The snowflake (and an enlargement) stopped on certain horizontal segments. Here $\exp(i \arg f') = 1$ a.e., but $\arg f' \notin H^{1/2}$.

We note that the chord-arc assumption in Theorem 1.1 is necessary; assuming Γ is a quasicircle is not enough. Suppose Γ is given by the usual construction of the von Koch snowflake, but stopping on every segment that is horizontal and points to the right (in the usual counterclockwise orientation of the curve). See Figure 5. In the limiting curve these horizontal segments have full harmonic measure, so if f is a conformal map from $\mathbb{H}^2$, then $\arg f'$ is a multiple of 2π on each, so $\exp(i \arg f') = 1$ almost everywhere. It is not hard to prove Γ has infinite length, so it is not WP.

9. Theorem 1.8: (1) $\Rightarrow$ (2)

As noted in the introduction, this implication of Theorem 1.8 is known; we merely sketch the proof Astala and Zinsmeister for completeness. Assume $\Gamma = f(\mathbb{T})$ where f is quasiconformal on the plane, conformal on $\mathbb{D}^*$ and has dilatation μ supported in $\mathbb{D}$. In [6], Astala and Zinsmeister use the identity

$$\iint_{\mathbb{D}} f_{\overline{z}}(z)dxdy = \frac{\pi}{6} \lim_{w \to \infty} w^4 S(f)(w),$$

a change of variables, and the Cauchy-Schwarz inequality to deduce

$$|S(f)(z)|^2(|z|^2-1)^2 \lesssim \iint_{\mathbb{D}} |w-z|^{-4}|\mu(w)|^2 dudv.$$

Integrating this with respect z, and reversing the order of integration gives

$$\iint_{\mathbb{D}} |S(f)(z)|^2(1-|z|^2)^2 dxdy \lesssim \iint_{\mathbb{D}} |\mu(w)|^2(1-|w|^2)^{-2} dudv.$$

10. Theorem 1.8: (2) $\Rightarrow$ (3)

This implication is also known, but we give an alternate proof using ideas from [13]. The following is Lemma 3.4 of [13].

Lemma 10.1. *Given $\epsilon > 0$, $n \in \mathbb{N}$, there are $C = C(\epsilon) > 0$ and $\delta = \delta(\epsilon, n) > 0$ so that the following holds. Suppose Q is a Carleson square and $|S(f)(z)| \leq \delta(1-|z|^2)^{-2}$, for all $z \in T(Q)$. Then there is a hyperbolic geodesic γ so that every $z \in Q$ with $1-|z| \geq 2^{-n}\ell(Q)$ and $|(\log f'(z))'| = |f''z()/f'(z)| \geq \varepsilon/(1-|z|^2)$ satisfies $d^3_{\mathbb{H}}(z,\gamma) \leq C$. Moreover, for every $\eta > 0$ there is a $\delta > 0$ so that if γ_0 is a sub-arc of γ connecting $z_0 \in T(Q)$ to $z_1 \in Q \cap \{z : 1-|z| = 2^{-n}\ell(Q)\}$, then $|f'(z_1)| \geq 2^{2n(1-\eta)}|f'(z_0)|$. (In other words, f' grows almost as fast as $(1-|z|)^{-2}$ along γ.)*

The first step is to use this lemma to show that if Γ is a quasicircle we may assume that

$$|f''(z)/f'(z)| \leq \epsilon/(1-|z|^2) \tag{10.1}$$

for $z \in Q$. Note that if (1.5) holds, we must have $|S(f)(z)|(1-|z|^2)^2 \to 0$, as $|z| \nearrow 1$ (we can apply the mean value theorem to $S(f)$). Hence for any $\delta > 0$, the hypothesis of Lemma 10.1 holds for all sufficiently small Carleson squares. Suppose z_0, z_1 are as in the lemma. The Koebe $\frac{1}{4}$-theorem implies that there is an arc $J \subset \mathbb{T}$ containing the base I of Q and whose image has chord length comparable to $|f'(z_0)|(1-|z_0|)$ (choose one endpoint of J from each component of $3I \setminus I$). Similarly, there is a subarc $\gamma_1 \subset \gamma_0$ with chord length (and hence diameter) comparable to

$$\begin{aligned} |f'(z_1)|(1-|z_1|) &\geq |f'(z_0)| \left(\frac{1-|z_0|}{1-|z_1|}\right)^{2(1-\eta)} (1-|z_1|) \\ &= |f'(z_0)|(1-|z_0|) \left(\frac{1-|z_0|}{1-|z_1|}\right)^{1-2\eta} \\ &= |f'(z_0)|(1-|z_0|)2^{n(1-2\eta)}. \end{aligned}$$

If $\eta < 1/2$ then this tends to infinity with n, and this implies that Γ is not a quasicircle, contrary to assumption. Therefore, if Q is small enough (10.1) holds. Given this, we can now follow the proof of Lemma 1 in the paper of Astala and Zinsmeister [6]. First we recall another standard estimate. Set $F = \log f'$.

Lemma 10.2. *If F is holomorphic on $\mathbb{D}$ then $\iint_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$, if and only if*

$$I := \iint_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy < \infty.$$

Proof. Assume F has the power series expansion $F(z) = \sum_{n=0}^{\infty} b_n z^n$. Then a simple computation in polar coordinates leads to

$$\iint_{\mathbb{D}} |F'(z)|^2 dxdy = 2\pi \sum_{n=1}^{\infty} n^2 |b_n|^2 \int_0^1 r^{2n-1} dr = \sum_{n=1}^{\infty} (\pi n)|b_n|^2,$$

and hence

$$\begin{aligned} I &= \iint_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy \\ &= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \int_0^1 r^{2n-4}(1-2r^2+r^4) r dr \\ &= 2\pi \sum_{n=1}^{\infty} n^2 (n-1)^2 |b_n|^2 \left(\frac{1}{2n-2} - \frac{2}{2n} + \frac{1}{2n+2}\right) \\ &= \sum_{n=1}^{\infty} 2\pi \frac{n(n-1)}{n+1} |b_n|^2 \simeq \sum_{n=2}^{\infty} \pi n |b_n|^2 \end{aligned}$$

Thus both infinite series (and hence both integrals) diverge or converge together. □

Next we must show I is finite. Using (1.3) and (1.4) we see that

$$F''(z) = S(f)(z) - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = S(f)(z) - \frac{1}{2}(F'(z))^2,$$

and hence

$$\begin{aligned} I = \iint_{\mathbb{D}} |F''(z)|^2 (1-|z|^2)^2 dxdy &\leq \iint_{\mathbb{D}} |S(f)(z)|^2 (1-|z|^2)^2 dxdy \\ &\quad + \frac{1}{4} \iint_{\mathbb{D}} |F'(z)|^4 (1-|z|^2)^2 dxdy \\ &\quad + \iint_{\mathbb{D}} |S(f)(z)|^2 |F'(z)|^2 (1-|z|^2)^2 dxdy \\ &= I_1 + \frac{1}{4} I_2 + I_3. \end{aligned}$$

Note that since $|xy| \leq (x^2+y^2)/2$, we get

$$|S(f)(z)||F'(z)|^2 \leq \frac{1}{2}|\epsilon F'(z)|^4 + \frac{1}{2}|S(f)(z)|^2$$

and hence $I_3 \leq \frac{1}{2}I_1 + \frac{1}{2}I_2$. Therefore

$$I \leq \frac{3}{2}I_1 + \frac{3}{4}I_2. \tag{10.2}$$

Now we use our assumption that $|F'(z)|(1-|z|^2) \leq \epsilon$ is small on the annulus $A = \{z : r < |z| < 1\}$. This implies

$$I_2 = \int_A |F'(z)|^4(1-|z|^2)^2 dxdy \leq \epsilon^2 \int_A |F'(z)|^2 dxdy \leq \epsilon^2 MI.$$

If ϵ is small enough, then $I_2 \leq I$, and then (10.2) becomes $I \leq \frac{3}{2}I_1 + \frac{3}{4}I$, which implies $I \leq 6I_1 < \infty$. This proves $F = \log f'$ is in the Dirichlet class.

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