Annals of Mathematics **202** (2025), 1–78

Weil-Petersson curves, β-numbers, and minimal surfaces

By Christopher J. Bishop

Abstract

This paper gives geometric characterizations of the Weil-Petersson class of rectifiable quasicircles, i.e., the closure of the smooth planar curves in the Weil-Petersson metric on universal Teichmüller space defined by Takhtajan and Teo. Although motivated by the planar case, many of our characterizations make sense for curves in $\mathbb{R}^n$ and remain equivalent in all dimensions. We prove that Γ is Weil-Petersson if and only if it is well approximated by polygons in a precise sense, has finite Möbius energy or has arclength parametrization in $H^{3/2}(\mathbb{T})$. Other results say that a curve is Weil-Petersson if and only if local curvature is square integrable over all locations and scales, where local curvature is measured using various quantities such as Jones's β-numbers, nonlinearity of conformal weldings, Menger curvature, the "thickness" of the hyperbolic convex hull of Γ, and the total curvature of minimal surfaces in hyperbolic space. Finally, we prove that planar Weil-Petersson curves are exactly the asymptotic boundaries of minimal surfaces in $\mathbb{H}^3$ with finite renormalized area.

1. Introduction

This paper gives several geometric characterizations of the Weil-Petersson class of rectifiable quasicircles. This collection of planar closed curves has previously known connections to geometric function theory, operator theory and certain random processes such as Schramm-Loewner evolutions (SLE). Our new characterizations will also link it to various ideas in harmonic analysis and geometric measure theory (e.g., Sobolev spaces, knot energies, β-numbers, biLipschitz involutions, Menger curvature) and in hyperbolic geometry (e.g., convex hulls, minimal surfaces, isoperimetric inequalities, renormalized area).

Keywords: universal Teichmüller space, Weil-Petersson class, chord-arc curves, traveling salesman theorem, Schwarzian derivatives, Dirichlet class, minimal surfaces, renormalized area, finite total curvature, Möbius energy, Loewner energy

AMS Classification: Primary: 30C62; Secondary: 28A75, 53A10, 30F60, 46E35.

Moreover, most of our characterizations extend to curves in $\mathbb{R}^n$ and remain equivalent there, defining new classes of curves that may be of interest in analysis and geometry. The name "Weil-Petersson class" comes from work of Leon Takhtajan and Lee-Peng Teo [123] defining a Weil-Petersson metric on universal Teichmüller space, analogous to the well known metric on finite dimensional Teichmüller spaces of Riemann surfaces defined by André Weil and Hans Petersson. The same collection of curves was earlier studied by Hui Guo [65] and Guizhen Cui [37] using the terms "integrable Teichmüller space of degree 2" and "integrably asymptotic affine maps", respectively.

A quasicircle is the image of the unit circle $\mathbb{T}$ under a quasiconformal mapping f of the plane, e.g., a homeomorphism of the plane that is absolutely continuous on almost all lines, conformal outside the unit disk $\mathbb{D}$, and whose dilatation $\mu = f_{\overline{z}}/f_z$ belongs to $\mathbb{B}_1^\infty$, the open unit ball in $L^\infty(\mathbb{D})$. The collection of planar quasicircles (modulo similarities) corresponds to universal Teichmüller space $T(1)$, and the usual metric is defined in terms of $\|\mu\|_\infty$. Motivated by ideas in string theory to apply Hilbert space methods to spaces of loops (e.g. [23], [24]), Takhtajan and Teo [123] defined a Weil-Petersson metric on universal Teichmüler space $T(1)$ that makes it into a Hilbert manifold. This topology on $T(1)$ has uncountably many connected components, but one of these components, denoted $T_0(1)$, is exactly the closure of the smooth curves; this is the Weil-Petersson class. Takhtajan and Teo proved these curves are precisely the images of $\mathbb{T}$ under quasiconformal maps with dilatation $\mu \in L^2(d\mathrm{A}_\rho) \cap \mathbb{B}_1^\infty$, where A_ρ is hyperbolic area on $\mathbb{D}$. Thus, roughly speaking, Weil-Petersson quasicircles are to L^2 as general quasicircles are to L^∞.

Takhtajan and Teo give an alternate characterization in terms of the conformal mapping $f : \mathbb{D} \to \Omega$, where Ω is the domain bounded by Γ. They show Γ is Weil-Petersson if and only if $\log f' \in W^{1,2}$, i.e., $(\log f')' = f''/f' \in L^2(\mathbb{D}, dxdy)$. By the Sobolev trace theorem, the boundary values of $\log f'$ are in the Sobolev space $H^{1/2}(\mathbb{T})$. We will prove that this implies the arclength parametrization of Γ is in the space $H^{3/2}(\mathbb{T})$ and that this characterizes Weil-Petersson curves.

THEOREM 1.1. *Γ is Weil-Petersson if and only if it is chord-arc and the arclength parametrization is in the Sobolev space $H^{3/2}(\mathbb{T})$.*

A rectifiable curve Γ is called chord-arc if, for all $x, y \in \Gamma$, we have $\ell(\gamma) = O(|x-y|)$, where $\gamma \subset \Gamma$ is the shortest sub-arc with endpoints x, y. The definition of $H^{3/2}(\mathbb{T})$ will be given in Section 3, as will the simple proof of necessity. Sufficiency of $H^{3/2}$ follows from other characterizations of the Weil-Petersson class given in this paper. This was first observed by David Mumford, who conjectured Theorem 1.1 based on an earlier draft of this paper. Takhtajan and Teo had proven that $T_0(1)$ is a topological group, and Mumford also pointed out that the above theorem makes Jordan curves in $H^{3/2}$ into a topological group, extending known results for $H^s(\mathbb{T})$, $s > 3/2$ (the group structure is obtained by identifying closed curves with circle homeomorphisms via conformal welding, as described in Sections 2 and 3). See also [31], [71] and the remarks following Definition 6.

It has been an open problem to give a "geometric" characterization of the Weil-Petersson class, as opposed to the known "function theoretic" characterizations. See Remark II.1.2 of [123]. Theorem 1.1 is our first step in this direction, and it will lead to a variety of more purely geometric characterizations. Like being an $H^{3/2}$ curve, most of these conditions also make sense for curves in $\mathbb{R}^n$, $n \geq 2$, and we will prove that they remain equivalent in higher dimensions. For example, one immediate consequence of Theorem 1.1 is:

THEOREM 1.2. Γ *is Weil-Petersson if and only if it has finite Möbius energy, i.e.,*

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^2} - \frac{1}{\ell(x,y)^2} \right) dxdy < \infty, \tag{1.1}$$

where $\ell(x,y)$ *denotes the length of the shorter sub-arc of* Γ *connecting* x *and* y, *and both integrals are with respect to arclength on* Γ.

Möbius energy is one of several "knot energies" introduced by Jun O'Hara, see [95], [96], and [97]. It blows up when the curve is close to self-intersecting, so in the special case of curves in $\mathbb{R}^3$, continuously deforming a curve to minimize the Möbius energy should lead to a canonical "nice" representative of each knot type. This was proven for irreducible knots by Michael Freedman, Zheng-Xu He and Zhenghan Wang [53], who also showed that Möb(Γ) is Möbius invariant (hence the name), that finite energy curves are chord-arc, and in $\mathbb{R}^3$ they are topologically tame (there is an ambient isotopy to a smooth embedding). Theorem 1.2 follows from Theorem 1.1 by a result of Simon Blatt [21] (we sketch a proof in Section 3). The connection to Weil-Petersson curves indicates that Möbius energy may also be interesting in dimensions other than 3.

Theorem 1.2 has several different reformulations. For example, it is essentially the same as the "Jones Conjecture" stated independently in [56]. We will explain the connection in Section 3. Another variation is rather elementary to state, using only the definition of arclength. If a closed Jordan curve Γ has finite length $\ell(\Gamma)$, choose a base point $z_1^0 \in \Gamma$ and for each $n \geq 1$, let $\{z_j^n\}$, $j = 1, \dots, 2^n$ be the unique set of ordered points with $z_1^n = z_1^0$ that divides Γ into 2^n equal length arcs (called the nth generation dyadic subarcs of Γ). Let Γ_n be the inscribed 2^n-gon with these vertices. See Figure 1. Clearly, $\ell(\Gamma_n) \nearrow \ell(\Gamma)$, and the Weil-Petersson class is characterized by the rate of this convergence.

Theorem 1.3. *With notation as above, a curve* Γ *is Weil-Petersson if and only if*

$$\sum_{n=1}^{\infty} 2^n \left[\ell(\Gamma) - \ell(\Gamma_n)\right] < \infty \tag{1.2}$$

with a bound that is independent of the choice of the base point.

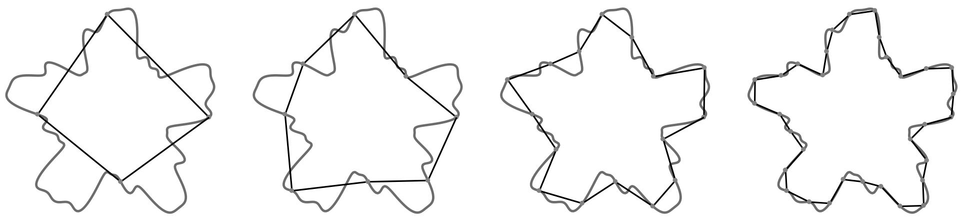

Figure 1. Inscribed dyadic polygons. Γ is Weil-Petersson if and only if its length is rapidly approximated using such polygons.

Another consequence of Theorem 1.1 involves Peter Jones's β-numbers: given a curve $\Gamma \subset \mathbb{R}^2$, and a square Q in the plane let

$$\beta_\Gamma(Q) = (2\sqrt{2}/3) \cdot \inf_L \sup_{z \in \Gamma \cap 3Q} \frac{\operatorname{dist}(z, L)}{\operatorname{diam}(Q)},$$

where the infimum is over all lines hitting $3Q$, the square concentric with Q and with three times the side length ($\beta = 0$ if $\Gamma \cap 3Q = \emptyset$). See Figure 2. The factor of $2\sqrt{2}/3$ is not essential, and it is only included to normalize β so that it always lies in $[0, 1]$.

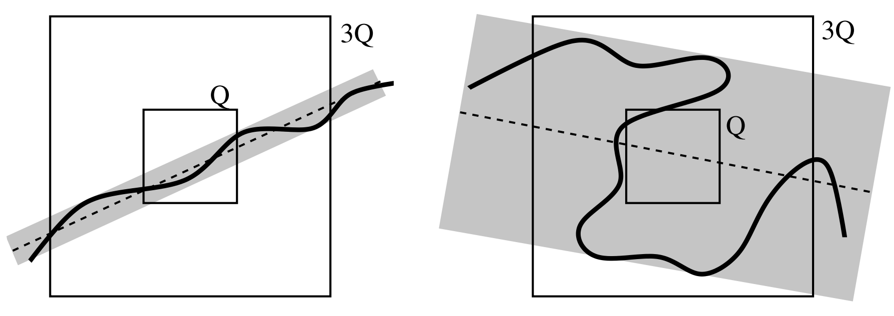

Figure 2. β-numbers measure how close Γ is to a line near Q. On the left β is small; on the right is it close to 1.

Theorem 1.4. *A Jordan curve* Γ *is Weil-Petersson if and only if*

$$\sum_Q \beta_\Gamma^2(Q) < \infty, \tag{1.3}$$

where the sum is over all dyadic squares in the plane.

Dyadic squares and cubes will be defined in Section 4. This theorem is our fundamental result of the form: "Γ is Weil-Petersson if and only if curvature is square integrable over all locations and scales". All of our other criteria can be formulated in an analogous way, using different measures of local curvature (even Theorems 1.2 and 1.3, although they do not immediately look like L^2 curvature conditions). Other versions involve Schwarzian derivatives, Menger curvature, and the Gauss curvatures of minimal surfaces; these all measure deviation from flatness in different, but closely related, ways.

Jones [74] introduced the β-numbers in his famous "traveling salesman theorem" that characterizes subsets of rectifiable curves in the plane. In the special case of a Jordan curve Γ, his result says that

$$\ell(\Gamma) \simeq \mathrm{diam}(\Gamma) + \sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q), \tag{1.4}$$

where the sum is over all dyadic squares in $\mathbb{R}^2$. Thus (1.3) is a strengthening of Jones's condition (1.4). Analogs of Jones's theorem have been proven in $\mathbb{R}^n$ by Kate Okikiolu [98], and in Hilbert space by Raanan Schul [114] (with some corrections in [9] and [10] by Matthew Badger and Sean McCurdy, who also consider curves in more general Banach spaces). Analogous results in some other metric spaces are given in [38], [51], [81], and [82]. Several of these papers have been superseded by Sean Li [80] using a new "stratified" version of the β-numbers to prove a traveling salesman theorem on arbitrary Carnot groups.

For curves in $\mathbb{R}^n$, $n \geq 3$, we will prove that (1.3) is equivalent to the conditions in Theorems 1.1, 1.2 and 1.3, using the following refinement of Jones's theorem that is proven in [18].

THEOREM 1.5. *If* $\Gamma \subset \mathbb{R}^n$, $n \geq 2$, *is a Jordan arc, then*

$$\ell(\Gamma) = \mathrm{crd}(\Gamma) + O\left(\sum_Q \beta_\Gamma^2(Q)\mathrm{diam}(Q)\right), \tag{1.5}$$

where the sum is over all dyadic cubes in $\mathbb{R}^n$.

Here $\mathrm{crd}(\Gamma) = |z-w|$ denotes the "chord" distance between the endpoints z, w of Γ. The point of Theorem 1.5 is that the $\mathrm{diam}(\Gamma)$ term in (1.4) can be replaced by $\mathrm{crd}(\Gamma) \leq \mathrm{diam}(\Gamma)$, and that this term is only multiplied by "1" in (1.5).

For the rest of the introduction, we return to the planar case $n = 2$, where the statements are simplest; in higher dimensions some changes are needed due to technicalities that arise, e.g., a minimal surface in $\mathbb{H}^3$ must be replaced by a minimal current or flat chain in $\mathbb{H}^{n+1}$. These changes will be discussed in Section 6.

The hyperbolic upper half-space is defined as $\mathbb{H}^3 = \mathbb{R}^3_+ = \{(x,t) : x \in \mathbb{R}^2, t > 0\}$, with the hyperbolic metric $d\rho = ds/t$. The hyperbolic convex hull of $\Gamma \subset \mathbb{R}^2$, denoted $\mathrm{CH}(\Gamma)$, is the smallest convex set in $\mathbb{H}^3$ that contains all (infinite) hyperbolic geodesics with both endpoints in Γ. Except when Γ is a circle, $\mathrm{CH}(\Gamma)$ has a non-empty interior and has two boundary surfaces (both with asymptotic boundary Γ), called the "domes" of either side of Γ. See Figure 3 for the domes of a square and its complement. For $z \in \mathrm{CH}(\Gamma)$, we define $\delta(z)$ to be the maximum of the hyperbolic distances from z to the two boundary components of $\mathrm{CH}(\Gamma)$. See Figure 9. This function serves as a Möbius invariant version of the β-numbers.

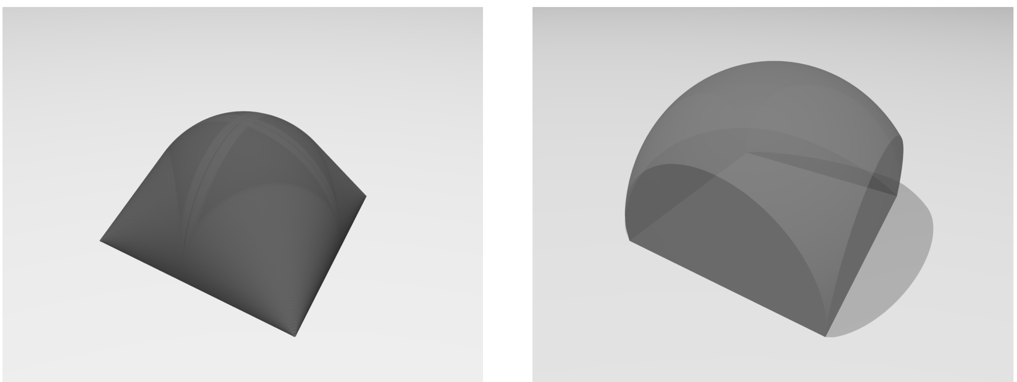

Figure 3. The domes of a square and its complement. The convex hull of the boundary is the region between these two surfaces. The dome on the left is a hemisphere over the inscribed disk with four half-cones attached, each with a vertex at a corner of the square. The dome on the right is the hemisphere over the circumscribed disk, cut by four vertical planes over the sides of the square.

Our hyperbolic Weil-Petersson criteria will involve integrating some quantity such as δ over points (x,t) on some surface $S \subset \mathbb{H}^3$ that has $\Gamma \subset \mathbb{R}^2$ as its asymptotic boundary; usually S will be one of the two connected components of $\partial\mathrm{CH}(\Gamma)$, the cylinder $\Gamma \times (0,1]$, or a minimal surface contained in $\mathrm{CH}(\Gamma)$. Suppose $S \subset \mathbb{H}^3$ is a 2-dimensional, properly embedded sub-manifold that has an asymptotic boundary that is a closed Jordan curve in $\mathbb{R}^2$. The Euler characteristic of S will be denoted $\chi(S)$, i.e., $\chi(S) = 2 - 2g - h$ if S is a surface of genus g with h holes. We let $K(z)$ denote the Gauss curvature of S at z. The hyperbolic metric $d\rho = ds/2t$ was chosen so that $\mathbb{H}^3$ has constant Gauss curvature -1. If the principle curvatures of S at z are $\kappa_1(z)$ and $\kappa_2(z)$, then $K(z) = -1 + \kappa_1(z)\kappa_2(z)$ (this is the Gauss equation). The norm of the second fundamental form is given by $|\mathcal{K}(z)|^2 = \kappa_1(z)^2 + \kappa_2(z)^2$. The surface S is called a minimal surface if $\kappa_1 = -\kappa_2$ (i.e., the mean curvature $H = (\kappa_1 + \kappa_2)/2$ is zero). In this case we will write $\kappa = |\kappa_j|$, $j = 1, 2$, and so $K(z) = -1 - \kappa^2(z)$. These definitions, and other basic results from differential geometry, such as the Gauss-Bonnet theorem, can be found in various textbooks, e.g., [69].

Michael Anderson [7] has shown that every closed Jordan curve on $\mathbb{R}^2$ bounds a simply connected minimal surface in $\mathbb{H}^3$, but there may be other minimal surfaces with boundary Γ that are not disks. See Figure 4.

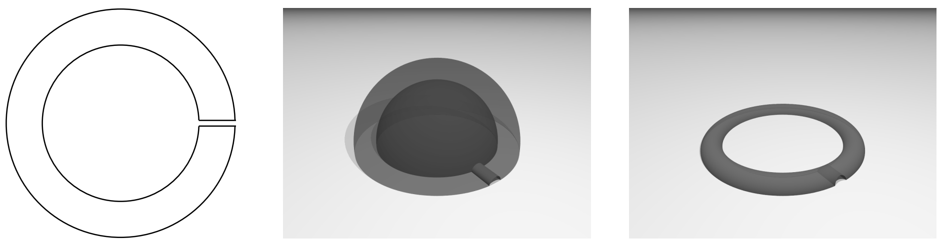

Figure 4. A planar curve from Anderson's paper [7] illustrating that a curve $\Gamma \subset \mathbb{R}^2$ can be the boundary of multiple minimal surfaces. The first is topologically a disk; the second is topologically a torus with a hole removed.

However, every minimal surface S with asymptotic boundary Γ is contained in $\mathrm{CH}(\Gamma)$ and the principle curvatures of S at a point z satisfy $|\kappa_j(z)| = O(\delta(z))$, (see Lemma 19.1). Let A_ρ denote hyperbolic area and L_ρ hyperbolic length.

Theorem 1.6. *For a closed curve $\Gamma \subset \mathbb{R}^2$, the following are equivalent*:

(1) $\Gamma \subset \mathbb{R}^2$ *is a Weil-Petersson curve.*

(2) Γ *asymptotically bounds a surface $S \subset \mathbb{H}^3$ so that*
$$\int_S |\delta(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(3) Γ *asymptotically bounds a surface $S \subset \mathbb{H}^3$ so that $|\mathcal{K}(z)| \to 0$ as $z \to \mathbb{R}^2 = \partial\mathbb{H}^3$ through S and*
$$\int_S |\mathcal{K}(z)|^2 d\mathrm{A}_\rho(z) < \infty.$$

(4) *Every minimal surface S asymptotically bounded by Γ has finite Euler characteristic and finite total curvature, i.e.,*
$$\int_S \big|\kappa(z)\big|^2 d\mathrm{A}_\rho(z) = \int_S \big|K(z) + 1\big| d\mathrm{A}_\rho(z) < \infty.$$

(5) *There is some minimal surface S with finite Euler characteristic and asymptotic boundary Γ so that S is the union of a nested sequence of compact Jordan subdomains $\Omega_1 \subset \Omega_2 \subset \dots$ with*
$$\limsup_{n\to\infty} \left[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)\right] < \infty.$$

In 1993 Geraldo de Oliveira Filho ([99, Theorem B]) showed that a complete, immersed minimal disk in $\mathbb{H}^n$ having finite total curvature has an asymptotic boundary Γ that is rectifiable, and he asked if Γ must be C^1. By Part (4) above, the answer is no, since Theorem 1.4 implies that Weil-Petersson curves need not be C^1. The curve $\gamma(t) = t \cdot \exp(i/|\log 1/t|)$ satisfies $\beta(Q) \simeq 1/n$ if $0 \in Q$ and $\operatorname{diam}(Q) = 2^{-n}$, and one can check that (1.3) is satisfied even though γ has an infinite spiral at 0. However, Weil-Petersson curves are "almost" C^1 in the sense that $H^s \subset C^{1,s-3/2} \subset C^1$ for all $s > 3/2$, e.g., Lemma 8.2 of [39].

The isoperimetric difference in Part (5) of Theorem 1.6 is also known as the renormalized area of S, at least in the special case that Ω is the truncation of $S \subset \mathbb{H}^3$ at a fixed height above the boundary. More precisely, set

$$S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s > t\}, \quad \partial S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s = t\}$$

and define the renormalized area of S to be

$$\mathcal{RA}(S) = \lim_{t \searrow 0} [\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t)]$$

when this limit exists and is finite. We will show that these truncations satisfy the last part of Theorem 1.6, and hence the following holds.

Corollary 1.7. *For any closed curve $\Gamma \subset \mathbb{R}^2$ and for any minimal surface $S \subset \mathbb{H}^3$ with finite Euler characteristic and asymptotic boundary Γ, we have*

$$\mathcal{RA}(S) = -2\pi\chi(S) - \int_S \kappa^2(z) d\mathrm{A}_\rho. \tag{1.6}$$

In other words, either Γ is Weil-Petersson and both sides are finite and equal, or Γ is not Weil-Petersson and both sides are $-\infty$.

Proposition 3.1 of [5] by Spyridon Alexakis and Rafe Mazzeo gives a version of (1.6) for surfaces in the setting of n-dimensional Poincaré-Einstein manifolds (that formula also involves the Weyl curvature, which vanishes in $\mathbb{H}^3$), but they use the additional assumption that Γ is $C^{3,\alpha}$. However, as noted earlier, Weil-Petersson curves need not be even C^1. Corollary 1.7 shows that the Alexakis-Mazzeo result holds without any conditions on Γ, at least in the case of $\mathbb{H}^3$.

Our proof of Corollary 1.7 will show that the exact method of truncation in the definition of renormalized area is not important.

Corollary 1.8. *Suppose $S = \cup_n K_n \subset \mathbb{H}^3$ is a minimal surface, where $K_1 \subset K_2 \subset \dots$ are nested compact sets such that $S \setminus K_n$ is a topological annulus for all n. Then*

$$\mathcal{RA}(S) = \lim_{n\to\infty} \sup_{\Omega \supset K_n} \big[\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)\big],$$

where the supremum is over compact domains $K_n \subset \Omega \subset S$ bounded by a single Jordan curve. As above, either both terms are finite and equal, or both are $-\infty$.

Renormalized area has strong motivations arising from string theory. Maldacena [84] proposed that the expectation value of the Wilson loop operator (a precursor of string theory) should be the area of a minimal surface with asymptotic boundary Γ. It was pointed out by Mans Hennington and Kostas Skenderis [68], and by Robin Graham and Edward Witten [64], that area should be renormalized area. More recently, it has been suggested that renormalized area be used to measure the entanglement entropy of regions in conformal field theory, in a way that is analogous to how the entropy of a black hole is measured by the area of its event horizon, e.g., [94], [111], and [122]. See the introduction of [5] for further details and references. Also see [106] by David Radnell, Eric Schippers, and Wolfgang Staubach, where they argue that Weil-Petersson curves are the correct setting for 2-dimensional conformal field theory.

The Weil-Petersson class also arises in computer vision: see the papers of Eitan Sharon and Mumford [116], Matt Feiszli, Sergey Kushnarev and Katheryn Leonard [49], and Feiszli and Akil Narayan [50]. The latter paper computes geodesics for the Weil-Petersson metric as optimal "morphing" paths between different shapes, and such calculations lead naturally to the question of identifying the closure of the smooth curves in this metric.

Indeed, the problem of geometrically characterizing Weil-Petersson curves was originally suggested to me by Mumford in December of 2017. However, I did not work seriously on Mumford's question until attending an IPAM workshop on the geometry of random sets a year later. Motivated by SLE, Yilin Wang and Steffen Rohde [109] had previously defined the Loewner energy of a closed loop, and Wang subsequently proved that it is finite if and only if the curve is Weil-Petersson. See [127], [129] and Definition 24 in Appendix A. Her lecture at IPAM contained a summary of results from the Takhtajan-Teo paper [123], including the characterization of the Weil-Petersson curves in terms of $\log f' \in W^{1,2}$. In [19], Jones and I had characterized curves for which $\log f' \in \mathrm{BMO}$ (bounded mean oscillation), so this alternative definition provided a useful starting point for me.

I thankfully acknowledge discussions with Kari Astala, Martin Chuaqui, Blaine Lawson, Pekka Koskela, Dragomir Saric, Raanan Schul, Leon Takhtajan, Dror Varolin, Fredrik Viklund, Rongwei Yang, Yilin Wang, and Michel Zinsmeister. I am grateful to Atul Shekhar for pointing me to the paper [56]. I thank Jack Burkart and María González for reading various early drafts and providing many helpful comments and corrections. I am deeply appreciative to Mike Anderson, Claude LeBrun, Rafe Mazzeo and Andrea Seppi for very enlightening discussions of curvature, minimal surfaces, renormalized area and Willmore energy, to John Morgan for explaining the Smith Conjecture, and especially to David Mumford for sharing his perspective on these problems and for his thoughtful and continued encouragement of this work. Finally, many thanks to the anonymous referee for a careful reading of a difficult manuscript and for several recommendations that improved it.

The remainder of the paper is organized as follows. The next five sections state various definitions of the Weil-Petersson class: known function theoretic ones, new criteria involving Sobolev smoothness, new conditions involving the β-numbers, and finally, new characterizations using hyperbolic geometry, first in $\mathbb{H}^3$ and then in higher dimensions. These definitions and some of the easier implications between them are proven in the first half of the paper, and more involved proofs are left for later sections. For the convenience of the reader, Table 1 in Section 7 summarizes all the definitions, and Figure 11 shows a directed graph indicating the implications that are proven in this paper and shows where to find the corresponding proofs. An appendix describes some other known characterizations of the Weil-Petersson class, giving 26 equivalent definitions in all. A 27th was recently announced by Viklund and Wang in [125]. Indeed, since the first draft of this manuscript was posted, several papers and preprints have appeared that solve problems stated here or give extensions of our results. For example, Jared Krandel [75] has extended Theorem 1.5 to curves in Hilbert space. In [26], Martin Bridgeman, Kenneth Bromberg, Franco Vargas Pallete, and Yilin Wang answer a question related to this paper by proving the Loewner energy of a planar curve equals the renormalized volume between two certain surfaces in hyperbolic 3-space (under certain smoothness assumptions). Zhiyuan Geng and Fang-Hua Lin [63] show that curves with $H^{3/2}$ smoothness arise naturally in an energy minimization problem related to liquid crystals, and Huaying Wei and Katsuhiko Matsuzaki [130] consider L^p analogs of Weil-Petersson curves for $p \neq 2$.

2. **Function theoretic characterizations**

A quasiconformal (QC) map h is a homeomorphism of a planar domain Ω to another domain Ω', so that h is absolutely continuous on almost all lines and whose dilatation $\mu = h_{\overline{z}}/h_z$ is in $\mathbb{B}_1^\infty$ (the open unit ball of L^∞). See [3] or [78]. We say that h is a planar quasiconformal map if $\Omega = \Omega' = \mathbb{R}^2$. The measurable Riemann mapping theorem says that given a $\mu \in \mathbb{B}_1^\infty$, there is a planar quasiconformal map h with this dilatation. If μ is supported on the unit disk, $\mathbb{D}$, then there is a quasiconformal $h : \mathbb{D} \to \mathbb{D}$ with this dilatation. A quasiconformal map h is called K-quasiconformal if its dilatation satisfies $\|\mu\|_\infty \leq k = (K-1)/(K+1)$. More geometrically, at almost every point, h

is differentiable and its derivative (which is a real linear map) send circles to ellipses of eccentricity at most K.

A quasicircle $\Gamma = f(\mathbb{T})$ is the image of the unit circle $\mathbb{T}$ under a planar quasiconformal map. Such curves have a well known geometric characterization: Γ is a quasicircle if and only if for all subarcs $\gamma \subset \Gamma$ with $\operatorname{diam}(\gamma) \leq \operatorname{diam}(\Gamma)/2$ we have $\operatorname{diam}(\gamma) = O(|z-w|)$, where z, w are the endpoints of γ (this is one of about thirty equivalent conditions given in [62]). Weil-Petersson curves are quasicircles by definition, but they are also rectifiable and satisfy even more stringent conditions.

Suppose Γ is a closed curve in the plane, and let f be a conformal map from the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to Ω, the bounded complementary component of Γ. If f is conformal on $\mathbb{D}$, then f' is never zero, so $\Phi = \log f'$ is a well defined holomorphic function on $\mathbb{D}$. Recall that the Dirichlet class is the Hilbert space of holomorphic functions F on the unit disk such that $|F(0)|^2 + \int_{\mathbb{D}} |F'(z)|^2 dxdy < \infty$. In other words, the Dirichlet space consists of the holomorphic functions in the Sobolev space $W^{1,2}(\mathbb{D})$ (functions with one derivative in $L^2(dxdy)$).

Definition 1. The curve Γ is a quasicircle and $\Gamma = f(\mathbb{T})$, where f is conformal on $\mathbb{D}$ and $\log f'$ is in the Dirichlet class.

This definition of Weil-Petersson curves immediately provides some geometric information about the curve Γ. For a Jordan arc γ, let $\ell(\gamma)$ denote its arclength, and let $\operatorname{crd}(\gamma) = |z-w|$, where z, w are the endpoints of γ. If $\log f'$ is in the Dirichlet class, then $\log f' \in \mathrm{VMOA}$ (vanishing mean oscillation; see Chapter VI of [58]). The John-Nirenberg theorem (e.g., Theorem VI.2.1 of [58]) then implies that f' is in the Hardy space $H^1(\mathbb{D})$, so Γ is rectifiable. Even stronger, a theorem of Christian Pommerenke [104] implies that Γ is asymptotically smooth, i.e., $\ell(\gamma)/\operatorname{crd}(\gamma) \to 1$ as $\ell(\gamma) \to 0$. Thus a Weil-Petersson curve has "no corners", e.g., no polygon is Weil-Petersson. Asymptotic smoothness implies Γ is chord-arc; a fact observed in [56] (see also Theorem 2.8 of [105], but there is a gap due to the non-standard definition of "quasicircle" in a result quoted from [46].)

An estimate of Arne Beurling [13] (simplified and extended by Alice Chang and Don Marshall in [32] and [87]) says that $\log |f'|$ being in the Dirichlet class implies $\int \exp(\alpha \log^2 |f'|^2) ds < \infty$ for all $\alpha \leq 1$. In particular, $|f'| \in L^p(\mathbb{T})$ for every $p < \infty$ (but examples show $|f'|$ need not be bounded). Thus f is almost, but not quite, Lipschitz. We shall describe its precise smoothness later.

It is easy to prove using power series (e.g., Lemma 10.2 of [17]) that for any holomorphic function F on $\mathbb{D}$ the condition

$$\left|F(0)\right|^2 + \int_{\mathbb{D}} \left|F'(z)\right|^2 dxdy < \infty$$

holds if and only if

$$\left|F(0)\right|^2 + \left|F'(0)\right|^2 + \int_{\mathbb{D}} \left|F''(z)\right|^2 \left(1-|z|^2\right)^2 dxdy < \infty.$$

Applying this to $F = \log f'$, we see that

$$\int_{\mathbb{D}} \left|\left(\log f'\right)'\right|^2 dxdy = \int_{\mathbb{D}} \left|\frac{f''}{f'}\right|^2 dxdy < \infty \tag{2.1}$$

could be replaced by the condition

$$\int_{\mathbb{D}} \left|\left(\frac{f'''}{f'}\right) - \left(\frac{f''}{f'}\right)^2\right|^2 \left(1-|z|^2\right)^2 dxdy < \infty. \tag{2.2}$$

This integrand is reminiscent of the Schwarzian derivative of f given by

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.3}$$

The quantities in (2.2) and (2.3) are very similar, except that a factor of 1 has been changed to 3/2. However, this represents a non-linear change, and it is difficult to compare the two quantities directly. Nevertheless, for conformal maps into bounded quasidisks, the integrals of these two quantities are simultaneously finite or infinite.

Definition 2. The curve Γ is quasicircle and $\Gamma = f(\mathbb{T})$, where f is conformal on $\mathbb{D}$ and satisfies

$$\int_{\mathbb{D}} \left|S(f)(z)\right|^2 \left(1-|z|^2\right)^2 dxdy < \infty. \tag{2.4}$$

Proposition 1 of Cui's paper [37] says that Definitions 2 and 1 are equivalent. See also Theorem II.1.12 of [123] and Theorem 1 of [103]. If f is univalent on $\mathbb{D}$, then

$$\sup_{z\in\mathbb{D}} \left|S(f)(z)\right| \left(1-|z|^2\right)^2 \leq 6. \tag{2.5}$$

See Chapter II of [79] for this and other properties of the Schwarzian. If f is holomorphic on the disk and satisfies (2.5) with 6 replaced by 2, then f is injective, i.e., a conformal map. If 2 is replaced by a value $t < 2$, then f also has a K-quasiconformal extension to the plane, where K depends only on t. This is due to Lars Ahlfors and Georges Weill [2], who gave a formula for the extension and its dilatation

$$f(w) = f(z) + \frac{\left(1-|z|^2\right)f'(z)}{\overline{z} - \frac{1}{2}\left(1-|z|^2\right)\left(f''(z)/f'(z)\right)} \tag{2.6}$$

$$\mu(w) = -\frac{1}{2}\left(1-|z|^2\right)^2 S(f)(z) \tag{2.7}$$

where $w \in \mathbb{D}^*$ and $z = 1/\overline{w} \in \mathbb{D}$. See Section 4 of [35] for a lucid discussion of the Ahlfors-Weill extension; it also appears as Formula (3.33) in [100], and Equation (9) of [107]. Equation (2.7) suggests the following definition.

Definition 3. The curve $\Gamma = f(\mathbb{T})$, where f is a quasiconformal map of the plane that is conformal on $\mathbb{D}^*$ and whose dilatation μ on $\mathbb{D}$ satisfies

$$\int_{\mathbb{D}} \frac{\big|\mu(z)\big|^2}{\big(1-|z|^2\big)^2} dxdy < \infty. \tag{2.8}$$

This is equivalent to Definition 2 by Theorem 2 of [37], and

$$\int_{\mathbb{D}} \big|\mu(z)\big|^2 d\mathrm{A}_\rho < \infty, \tag{2.9}$$

where $d\mathrm{A}_\rho$ denotes integration against hyperbolic area; this was one of the definitions of the Weil-Petersson class mentioned in the introduction.

Another variation on this theme is to consider the map $R(z) = f(1/\overline{f^{-1}(z)})$. This is an orientation reversing quasiconformal map of the sphere to itself that fixes Γ pointwise, exchanges the two complementary components of Γ and whose dilatation satisfies

$$\int_{\Omega \cup \Omega^*} \big|\mu(z)\big|^2 d\mathrm{A}_\rho(z) < \infty, \tag{2.10}$$

where $d\mathrm{A}_\rho$ is hyperbolic area on each of the domains Ω, Ω^*. This version is sometimes easier to check, and we will use it interchangeably with Definition 3. The map R is called a quasiconformal reflection across Γ. Definition 13 will give an analogous characterization in terms of biLipschitz involutions in $\mathbb{R}^n$.

A circle homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ is called a conformal welding if $\varphi = f^{-1} \circ g$, where f, g are conformal maps from the two sides of the unit circle to the two sides of a closed Jordan curve Γ. There are many weldings associated to each Γ, but they all differ from each other by compositions with Möbius transformations of $\mathbb{T}$. Not every circle homeomorphism is a conformal welding, but weldings are dense in all circle homeomorphisms in various senses; see [16].

A circle homeomorphism is called M-quasisymmetric if it maps adjacent arcs of equal length to arcs whose length differ by a factor of at most M; we call φ quasisymmetric if it is M-quasisymmetric for some M. The quasisymmetric maps are exactly the circle homeomorphisms that can be continuously extended to quasiconformal self-maps of the disk, and are also exactly the conformal weldings of quasicircles. See [3]. If $I \subset \mathbb{T}$ is an arc, let $m(I)$ denote its midpoint. For a homeomorphism $\varphi : \mathbb{T} \to \mathbb{T}$ and an arc $I \subset \mathbb{T}$, define

$$\mathrm{qs}(\varphi, I) = \frac{\big|\varphi\big(m(I)\big) - m\big(\varphi(I)\big)\big|}{\ell\big(\varphi(I)\big)}.$$

A quasisymmetric homeomorphism φ is called symmetric if $\text{qs}(\varphi, I) \to 0$ as $|I| \to 0$. Pommerenke [104] proved such weldings characterize curves where $\log f'$ is in the little Bloch space ($|(\log(f')'|(1-|z|) = o(1)$); see also [57] by Fred Gardiner and Dennis Sullivan and [120] by Kurt Strebel. It is proven in [17] that φ corresponds to a Weil-Petersson curve if and only if $\text{qs}(\varphi, I) \in \ell^2$.

Definition 4. The curve Γ has a welding map φ that satisfies

$$\sum_I \text{qs}^2(\varphi, I) \leq C < \infty, \tag{2.11}$$

where the sum is over any dyadic decomposition of $\mathbb{T}$ and C is independent of the choice of the decomposition.

Weil-Petersson weldings were first characterized by Yuliang Shen [117] in terms of the Sobolev space $H^{1/2}$. We will describe his result in the next section.

3. **Sobolev conditions**

We start by recalling some standard notation. Given two quantities A, B that both depend on a parameter, we write $A \lesssim B$ if there is a constant C such that $A \leq CB$ holds independent of the parameter. We write $A \gtrsim B$ if $B \lesssim A$, and we write $A \simeq B$ if both $A \lesssim B$ and $A \gtrsim B$ hold. The notation $A \lesssim B$ means the same as the "big-Oh" notation $A = O(B)$.

Definition 1 can be interpreted in terms of Sobolev spaces. The space $H^{1/2}(\mathbb{T}) \subset L^2(\mathbb{T})$ is defined by the finiteness of the seminorm

$$\begin{aligned} D(f) &= \iint_{\mathbb{D}} \left|\nabla u(z)\right|^2 dxdy \\ &= \frac{1}{8\pi} \int_0^{2\pi} \int_0^{2\pi} \left| \frac{f(e^{is}) - f(e^{it})}{\sin \frac{1}{2}(s-t)} \right|^2 dsdt \simeq \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\left|f(z) - f(w)\right|^2}{|z-w|^2} |dz||dw|, \end{aligned}$$

where u is the harmonic extension of f to $\mathbb{D}$. The equality of the first and second integrals is called the Douglas formula, after Jesse Douglas who introduced it in his solution of the Plateau problem [41]. See also Theorem 2.5 of [4] (for a proof of the Douglas formula) and [110] (for more information about the Dirichlet space). For $s \in (0, 1)$ we define the space $H^s(\mathbb{T})$ using

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \frac{\left|f(z) - f(w)\right|^2}{|z-w|^{1+2s}} |dz||dw| < \infty.$$

See [1] and [39] for additional background on fractional Sobolev spaces. See also [93] by Subhashis Nag and Sullivan; in the authors' words, its "purpose is to survey from various different aspects the elegant role of $H^{1/2}$ in universal Teichüller theory" (a role we seek to explore in this paper too).

Shen [117] proved Γ is Weil-Petersson if and only if its welding map φ satisfies $\log \varphi' \in H^{1/2}$. One direction is easy. If Γ is Weil-Petersson, then $\log f'$ and $\log g'$ have boundary values in $H^{1/2}(\mathbb{T})$, where f and g are conformal maps from either side of $\mathbb{T}$ to either side of Γ. See [123]. A simple computation shows $\log \varphi'(x) = -\log f'(\varphi(x)) + \log g'(x)$. Beurling and Ahlfors [12] proved $H^{1/2}(\mathbb{T})$ is invariant under pre-compositions with quasisymmetric circle homeomorphisms, so $\log \varphi' \in H^{1/2}(\mathbb{T})$. The converse is harder; [17] provides a geometric alternative to Shen's operator theoretic proof.

Note that $\log f'(z) = \log |f'(z)| + i \arg f(z) \in W^{1,2}(\mathbb{D})$ if and only if the radial limits $\log |f'|$ and $\arg(f')$ are both in $H^{1/2}(\mathbb{T})$. Since $\arg(f')$ can be unbounded, it is surprising that this is also equivalent to $f'/|f'| \in H^{1/2}$.

Definition 5. The curve $\Gamma = f(\mathbb{T})$ is chord-arc and $\exp(i \arg f') = f'/|f'| \in H^{1/2}(\mathbb{T})$.

It is easy to deduce this from Definition 1. Since $\arg f' \in H^{1/2}(\mathbb{T})$, using $|e^{ix} - e^{iy}| \leq |x - y|$ and the Douglas formula, we get

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{e^{i \arg f'(x)} - e^{i \arg f'(y)}}{x - y} \right|^2 dxdy \leq \int_{\mathbb{T}} \int_{\mathbb{T}} \left| \frac{\arg f'(x) - \arg f'(y)}{x - y} \right|^2 dxdy < \infty.$$

Thus $\exp(i \arg f') \in H^{1/2}(\mathbb{T})$. A direct function theoretic proof of the converse is given in [17]; it also follows from a chain of implications proven later in this paper.

Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parametrization (i.e., a multiplies the arclength of every set by $\ell(\Gamma)/2\pi$). For $z \in \Gamma$, let $\tau(z)$ be the unit tangent direction to Γ with its usual counterclockwise orientation. Then $\tau(a(x)) = a'(x)2\pi/\ell(\Gamma)$, where $a' = \frac{da}{d\theta}$ on $\mathbb{T}$. Thus $a' = \exp(i \arg f') \circ \varphi$, where $\varphi = a^{-1} \circ f$ is a circle homeomorphism. In Section 8, we shall prove that this map φ is quasisymmetric (and hence so is its inverse). As noted above, pre-composing with such maps preserves $H^{1/2}(\mathbb{T})$, so Definition 5 is equivalent to saying $a' \in H^{1/2}(\mathbb{T})$. Every arclength parametrization is Lipschitz, hence absolutely continuous, and therefore the distributional derivative of a equals its pointwise derivative a'. Thus, for arclength parametrizations, $a' \in H^{1/2}(\mathbb{T})$ is the same as $a \in H^{3/2}(\mathbb{T})$. Therefore Definition 5 is equivalent to

Definition 6. The curve Γ is chord-arc, and the arclength parametrization $a : \mathbb{T} \to \Gamma$ is in the Sobolev space $H^{3/2}(\mathbb{T})$.

Proving this is equivalent to Definition 1 gives Theorem 1.1. Previous to Shen's result described earlier, François Gay-Balmaz and Tudor Ratiu [60] had proved that if Γ is Weil-Petersson, then the corresponding welding map φ is in $H^s(\mathbb{T})$ for all $s < 3/2$, but Shen [117] gave examples that are neither in $H^{3/2}(\mathbb{T})$, nor Lipschitz. Thus Theorem 1.1 implies that having an $H^{3/2}$ arclength parametrization is not the same as having an $H^{3/2}$ conformal welding.

These are equivalent conditions for $s > 3/2$, because for such weldings the Sobolev embedding theorem implies that φ' is Hölder continuous, which implies that the conformal mappings f, g have non-vanishing, Hölder continuous derivatives (e.g.,[36]), and therefore φ is biLipschitz. This fact implies that Γ has an H^s arclength parametrization (to see this, just copy the argument following Definition 5, using the fact that biLipschitz circle homeomorphisms preserve $H^s(\mathbb{T})$ for $1/2 < s < 1$, e.g., [22]).

When identified with quasisymmetric circle homeomorphisms, elements of the universal Teichmüller space $T(1)$ form a group under composition. It is not a topological group for the usual topology because left multiplication is not continuous (e.g., Theorem 3.3 in [79] or Remark 6.9 in [73]). However, Takhtajan and Teo [123] proved $T_0(1)$ is a topological group with its Weil-Petersson topology. So even though $H^{3/2}$-diffeomorphisms of the circle are not a group, Theorem 1.1 shows the set of $H^{3/2}$ Jordan curves can be identified with a group via conformal welding, namely $T_0(1)$. Circle diffeomorphisms in $H^s(\mathbb{T})$ with $s > 3/2$ also form a group, e.g., [71] and [117], and by the previous paragraph this means H^s curves are identified with a topological group via conformal welding. Thus our work gives the "endpoint" result of this previously known fact. See [11], [60], [89], and [90] for related discussions of groups, weldings, Sobolev embeddings and immersions.

Next we consider some consequences of Definition 6. Since Γ is chord-arc,

$$\frac{1}{C} \le \frac{\big|a(x) - a(y)\big|}{|x-y|} \le 1, \quad x, y \in \mathbb{T},$$

so setting $z = a(x), w = a(y)$, we have

$$\begin{aligned}\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z-w} \right|^2 |dz||dw| &= \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{a(x) - a(y)} \right|^2 dxdy \\ &= \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{x-y} \cdot \frac{x-y}{a(x) - a(y)} \right|^2 dxdy \\ &\simeq \int_\mathbb{T} \int_\mathbb{T} \left| \frac{a'(x) - a'(y)}{x-y} \right|^2 dxdy.\end{aligned}$$

Thus Definition 6 is equivalent to

Definition 7. The curve Γ is chord-arc and

$$\int_\Gamma \int_\Gamma \left| \frac{\tau(z) - \tau(w)}{z-w} \right|^2 |dz||dw| < \infty.$$

This characterization of the Weil-Petersson class was independently discovered by Shen and Li Wu [118]. They prove that Γ is a Weil-Petersson curve if and only if $\tau(a(x)) = a'(x) = \exp(ib(x))$ for some $b \in H^{1/2}(\mathbb{T})$. Since

$$\big|\tau\big(a(x)\big) - \tau\big(a(y)\big)\big| = O\big(|b(x) - b(y)|\big),$$

it is easy to check that $\tau \circ a \in H^{1/2}$, which gives Definition 7.

In Section 9, we will prove that Definition 7 is equivalent to

Definition 8. The curve Γ has finite Möbius energy, i.e,

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|z-w|^2} - \frac{1}{\ell(z,w)^2} \right) dzdw < \infty.$$

As mentioned in the introduction, Blatt [21] proved directly that Definition 6 is equivalent to Definition 8 (but there is a typo in Theorem 1.1 of [21]: it is stated that $s = (jp-2)/(2p)$, but this should be $s = (jp-1)/(2p)$, as given in his proof).

A Jordan curve with a $H^{3/2}$ arclength parametrization is chord-arc ([21, Lemma 2.1]), because this assumption prevents bending on small scales, but there is no quantitative bound on the chord-arc constant: Jordan curves with $H^{3/2}$ parametrizations can come arbitrarily close to self-intersecting (think of a smooth, Jordan approximation to a figure "8"). However, such a bound is possible in terms of Möb(Γ). This is Lemma 1.2 of [66], but for the reader's convenience, we sketch a proof here.

If $|z-w| \leq \epsilon$, but $\ell(z,w) \geq M\epsilon$, let $\sigma_k, \sigma_k' \subset \gamma(z,w)$ be arcs of length $2^k\epsilon$ that are path distance (on Γ) $2^k\epsilon$ from z and w, respectively, for $k = 1, \dots, K = \lfloor \log_2(M) \rfloor - 4$. Then $\sigma_k \cup \sigma_k'$ has diameter at most $\epsilon(1+2^{k+1})$ in $\mathbb{R}^n$, but these two arcs are at least distance $(M - 2^{k+2})\epsilon \geq M\epsilon/2$ apart on Γ. Thus

$$\begin{aligned}\int_{\sigma_k} \int_{\sigma_k'} \left(\frac{1}{|z-w|^2} - \frac{1}{\ell(v,w)^2} \right) dzdw &\geq \left[\frac{1}{(2^{k+2}\epsilon)^2} - \frac{1}{(M/2)^2} \right] (2^k\epsilon)(2^k\epsilon) \\ &\geq \frac{1}{16} - \frac{2^{2K+2}}{M^2} \\ &\geq \frac{1}{16} - 2^{-6} > \frac{1}{32}.\end{aligned}$$

Summing over k shows Möb$(\Gamma) \geq K/32 \gtrsim \log M$, so we have proven that Möb$(\Gamma) < \infty$ implies Γ is chord-arc.

Using the fact that Γ is chord-arc, we now get

$$\begin{aligned}\text{Möb}(\Gamma) &= \int_\Gamma \int_\Gamma \frac{\ell(z,w)^2 - |z-w|^2}{|z-w|^2 \ell(z,w)^2} dzdw \\ &= \int_\Gamma \int_\Gamma \frac{(\ell(z,w) - |z-w|)(\ell(z,w) + |z-w|)}{|z-w|^2 \ell(z,w)^2} dzdw \\ &\simeq \int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3}.\end{aligned}$$

Thus Definition 8 holds if and only if

Definition 9. The curve Γ is chord-arc and satisfies

$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| < \infty. \tag{3.1}$$

In [56], Eva Gallardo-Gutiérrez, María González, Fernando Pérez-González, Christian Pommerenke and Jouni Rättyä claim that (3.1) follows from Definition 1, but their proof contains a small error. They also state the converse as a conjecture of Jones; our results prove both directions. Definition 9 does not immediately look like a "curvature is square integrable" criterion, but it can easily be put in this form. Set

$$k(z,w) = \sqrt{24} \cdot \sqrt{\frac{\ell(z,w) - |z-w|}{|z-w|^3}}.$$

If Γ is smooth, then it is easy to check that $k(x) = \lim_{y \to x} k(x,y)$ is the usual Euclidean curvature of Γ at x. Thus (3.1) can be rewritten as

$$\int_\Gamma \int_\Gamma k^2(z,w) |dz||dw| < \infty, \tag{3.2}$$

and this has much more of a "L^2-curvature" flavor.

4. β-numbers

A dyadic interval I in $\mathbb{R}$ is one of the form $(2^{-n}j, 2^{-n}(j+1)]$ for $j, n \in \mathbb{Z}$. A dyadic cube in $\mathbb{R}^n$ is the product of n dyadic intervals of equal length. This common length is called the side length of Q and is denoted $\ell(Q)$. Note that $\text{diam}(Q) = \sqrt{n}\ell(Q)$. For $\lambda > 0$, we let λQ denote the cube concentric with Q but with diameter $\lambda \text{diam}(Q)$, e.g., $3Q$ is the "triple" of Q, a union of Q and $3^n - 1$ adjacent copies of itself.

A multi-resolution family in a metric space X is a collection of sets $\{X_j\}$ in X such that there are $N, M < \infty$ such that

(1) For each $r > 0$, the sets in $\{X_j\}$ with diameter between r and Mr cover X;
(2) Each bounded subset of X hits at most N sets X_k in the collection that satisfy $\text{diam}(X)/M \leq \text{diam}(X_k) \leq M\text{diam}(X)$;
(3) Any subset of X with positive, finite diameter is contained in at least one set X_j with $\text{diam}(X_j) \leq M\text{diam}(X)$.

Dyadic intervals are not a multi-resolution family for $\mathbb{R}$, e.g., $[-1, 1]$ is not contained in any dyadic interval, violating (3). However, the family of triples of all dyadic intervals (or cubes) do form a multi-resolution family. Similarly, if we add all translates of dyadic intervals by $\pm 1/3$, we get a multi-resolution family. This is sometimes called the "$\frac{1}{3}$-trick"; see [98]. The analogous construction for dyadic squares in $\mathbb{R}^n$ is to take all translates by elements of $\{-\frac{1}{3}, 0, \frac{1}{3}\}^n$.

During the course of this paper, we will deal with functions α that map a collection of sets into the non-negative reals, and we will wish to decide if the sum $\sum_j \alpha(X_j)$ over some multi-resolution family converges or diverges. We will frequently use the following observation to switch between various multi-resolution families without comment.

LEMMA 4.1. *Suppose $\{X_j\}$, $\{Y_k\}$ are two multi-resolution families on a space X, and suppose that α is a function mapping subsets of X to $[0,\infty)$ that satisfies $\alpha(E) \lesssim \alpha(F)$, whenever $E \subset F$ and $\operatorname{diam}(F) \lesssim \operatorname{diam}(E)$. Then*

$$\sum_j \alpha(X_j) \simeq \sum_k \alpha(Y_k).$$

Proof. By Condition (3) above, each X_j is contained in some set $Y_{k(j)}$ of comparable diameter. Hence $\alpha(X_j) \lesssim \alpha(Y_{k(j)})$ by assumption. Each Y_k is contained in a comparably sized X_m, and X_m can contain at most a bounded number of comparably sized subsets X_j. Thus each Y_k is only chosen boundedly often as a $Y_{k(j)}$. Thus $\sum_j \alpha(X_j) \lesssim \sum_k \alpha(Y_k)$. The opposite direction follows by reversing the roles of the two families. □

For a Jordan arc γ with endpoints z, w, we recall that $\operatorname{crd}(\gamma) = |z-w|$ and define the "excess length" as $\Delta(\gamma) = \ell(\gamma) - \operatorname{crd}(\gamma)$. In Section 10 we will prove that Definition 9 is equivalent to the following:

Definition 10. The curve Γ is chord-arc and

$$\sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)} < \infty \tag{4.1}$$

for some (hence every) multi-resolution family $\{\Gamma_j\}$ of arcs on Γ.

Condition (4.1) is just a reformulation of (1.2); since, if $\ell(\Gamma) = 1$ and $\{\Gamma_j\}$ corresponds to a dyadic decomposition of Γ, we have

$$\sum_n 2^n \big[\ell(\Gamma) - \ell(\Gamma_n)\big] = \sum_j \Delta(\gamma_j)/\ell(\gamma_j). \tag{4.2}$$

Thus, proving that Definition 10 is equivalent to being Weil-Petersson essentially proves Theorem 1.3. There is a slight gap here because Definition 10 uses a sum over a multi-resolution family and Theorem 1.3 is stated in terms of dyadic intervals. However, the theorem assumes a bound that is uniform over all dyadic decompositions, and this includes the $\pm\frac{1}{3}$-translates of a single dyadic family, and the union of these three families form a multi-resolution family (the "$\frac{1}{3}$-trick" from above). Conversely, Corollary 10.3 will show that $\Delta(\gamma) \le \Delta(3\gamma)$, so the dyadic sum can be bounded by the sum over dyadic triples, a multi-resolution family. Thus (4.1) for any multi-resolution family is equivalent to (4.2) with a uniform bound over all dyadic decompositions of Γ.

Given a set $E \subset \mathbb{R}^n$ and a dyadic cube Q, define Jones's β-number as

$$\beta(Q) = \beta_E(Q) = \frac{2\sqrt{2}/3}{\operatorname{diam}(Q)} \inf_L \sup\{\operatorname{dist}(z, L) : z \in 3Q \cap E\},$$

where the infimum is over all lines L that hit $3Q$. See the left side of Figure 5. Jones invented the β-numbers as part of his traveling salesman theorem (TST) [74], which estimates the length of the shortest curve containing a set E. When $E = \Gamma$ is a Jordan curve itself, the TST gives

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \sum_Q \beta_\Gamma(Q)^2 \operatorname{diam}(Q), \tag{4.3}$$

where the sum is over all dyadic cubes Q in $\mathbb{R}^n$. Our main geometric characterization of Weil-Petersson curves is to simply drop the "diam(Q)" term from (4.3).

Definition 11. The closed Jordan curve Γ satisfies

$$\sum_Q \beta_\Gamma(Q)^2 < \infty, \tag{4.4}$$

where the sum is over all dyadic cubes.

This is not very surprising (in retrospect). In the 1990's Jones and I proved (Lemma 3.9 of [19], or Theorem X.6.2 of [59]) that if Γ is an M-quasicircle, then

$$\ell(\Gamma) \simeq \operatorname{diam}(\Gamma) + \iint |f'(z)| \left|S(f)(z)\right|^2 \left(1 - |z|^2\right)^3 dxdy \tag{4.5}$$

with constants depending only on M. By Koebe's distortion theorem, the following holds:

$$|f'(z)|(1 - |z|^2) \simeq \operatorname{dist}(f(z), \partial\Omega),$$

and thus the factor on the left is analogous to the diam(Q) term in Jones's β^2-sum. Dropping this term from (4.5) gives exactly the integral in Definition 2. Thus dropping diam(Q) from (4.3) "should" also characterize Weil-Petersson curves (but proving this requires some work, given later).

Our results characterize $H^{3/2}$ curves in terms of β-numbers. Xavier Tolsa pointed out that related results for graphs of Besov functions (which include H^s as a special case) are given in [40].

It will be convenient to consider several equivalent formulations of condition (4.4). For $x \in \mathbb{R}^n$ and $t > 0$, define

$$\beta_\Gamma(x, t) = \frac{1}{t} \inf_L \max\{\operatorname{dist}(z, L) : z \in \Gamma, |x - z| \le t\},$$

where the infimum is over all lines hitting the ball $B = B(x, t)$, and let $\widetilde{\beta}_\Gamma(x, t)$ be the similar quantity defined using the infimum only over lines L hitting x.

Since this is a smaller collection, clearly $\beta(x,t) \le \widetilde{\beta}(x,t)$, and it is not hard to prove that $\widetilde{\beta}(x,t) \le 2\beta(x,t)$ if $x \in \Gamma$. See the center picture in Figure 5.

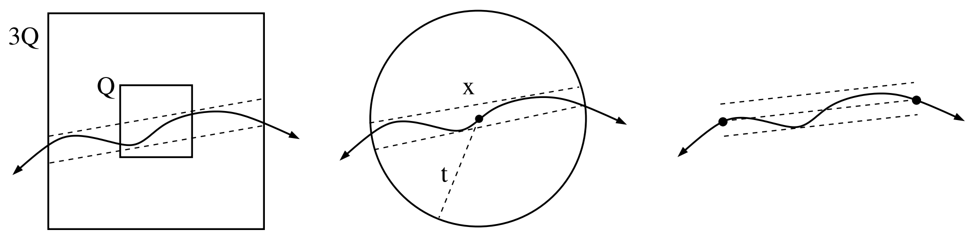

Figure 5. Three equivalent versions of the β-numbers.

Given a Jordan arc γ with endpoints z, w, we define

$$\beta(\gamma) = \frac{\max\{\text{dist}(z,L) : z \in \gamma\}}{|z-w|},$$

where L is the line passing through z and w. See the right side of Figure 5.

Lemma 4.2. *If Γ is a closed Jordan curve or a Jordan arc in $\mathbb{R}^n$ such that* (4.4) *holds, then Γ is a chord-arc curve. For chord-arc curves,* (4.4) *holds if and only if any of the following conditions holds*:

$$\int_0^\infty \iint_{\mathbb{R}^n} \beta^2(x,t) \frac{dxdt}{t^{n+1}} < \infty, \tag{4.6}$$

$$\int_0^\infty \int_\Gamma \widetilde{\beta}^2(x,t) \frac{dsdt}{t^2} < \infty, \tag{4.7}$$

$$\sum_j \beta^2(\Gamma_j) < \infty, \tag{4.8}$$

where dx is volume measure on $\mathbb{R}^n$, ds is arclength measure on Γ, and the sum in (4.8) *is over a multi-resolution family $\{\Gamma_j\}$ for Γ. Convergence or divergence in* (4.6) *and* (4.7) *is not changed if $\int_0^\infty$ is replaced by $\int_0^M$ for any $M > 0$.*

The equivalence of these conditions is fairly standard, and a proof can be found as Lemma B.2 in [18]. Since $\beta(x,t) \simeq \widetilde{\beta}(x,t)$ if $x \in \Gamma$, the integral in (4.7) is finite if and only if it is finite with β replacing $\widetilde{\beta}$. However, putting $\widetilde{\beta}$ into (4.6) gives a divergent integral for every closed Jordan curve Γ.

The Menger curvature of three points $x, y, z \in \mathbb{R}^n$ is $c(x,y,z) = 1/R$, where R is the radius of the circle passing through these points. The perimeter of this triangle with vertices x, y, z is denoted by

$$\ell(x,y,x) = |x-y| + |y-z| + |z-x|.$$

Definition 12. The curve Γ is chord-arc and satisfies

$$\int_\Gamma\int_\Gamma\int_\Gamma \frac{c(x,y,z)^2}{\ell(x,y,z)}|dx||dy||dz| < \infty. \tag{4.9}$$

Again, this is not unexpected in hindsight. It is known that the conditions

$$\int_\Gamma\int_\Gamma\int_\Gamma c(x,y,z)^2|dx||dy||dz| < \infty \tag{4.10}$$

$$\text{and} \quad \sum_Q \beta_\Gamma^2(Q)\ell(Q) < \infty \tag{4.11}$$

are equivalent, and the analog of dropping the length term from (4.11), would be to divide by a term that scales like length in (4.10), which gives (4.9). Indeed, to prove that Definitions 11 and 12 are equivalent, we will simply indicate how to modify the proof of the equivalence of (4.10) and (4.11) in Hervé Pajot's book [102].

Recall that a Whitney decomposition of an open set $W \subset \mathbb{R}^n$ is a collection of dyadic cubes Q with disjoint interiors, whose closures cover W and which satisfy $\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, \partial W)$. The existence of such decompositions is a standard fact (e.g., for each $z \in W$, take the maximal dyadic cube Q so that $z \in Q \subset 3Q \subset W$. See Section I.4 of [59]).

Suppose U is a neighborhood of $\Gamma \subset \mathbb{R}^n$ and $R : U \to U \subset \mathbb{R}^n$ is a homeomorphism with fixed point set Γ. For each Whitney cube Q for $W = \mathbb{R}^n \setminus \Gamma$, with $Q \subset U$, define $\rho(Q)$ to be the infimum of values $\rho > 0$ so that R is $(1+\rho)$-biLipschitz on Q and $\mathrm{dist}(\frac{z+R(z)}{2}, \Gamma) \le \rho \cdot \mathrm{diam}(Q)$ for $z \in Q$ (the latter condition ensures $R(z)$ is on the "opposite" side of Γ from z). The homeomorphism R is called an involution if $R(R(z)) = z$.

Definition 13. There is a homeomorphism R defined on a neighborhood U of Γ that fixes Γ pointwise, and so that

$$\sum_Q \rho^2(Q) < \infty. \tag{4.12}$$

The sum is over all cubes of a Whitney decomposition of $\mathbb{R}^n \setminus \Gamma$ that lie inside U.

We will prove later (Lemma 14.1) that a map R satisfying Definition 13 is biLipschitz in neighborhood $U' \subset U$ of Γ. We can also extend R to be a biLipschitz involution on the sphere $\mathbb{S}^n$, except in the case when Γ is knotted in $\mathbb{R}^3$; the solution of the Smith conjecture implies the fixed set of an orientation preserving diffeomorphic involution of $\mathbb{S}^3$ is an unknotted closed curve. See [92]. So, except for knotted curves in $\mathbb{R}^3$, we can say that Weil-Petersson curves are exactly the fixed point sets of biLipschitz involutions of $\mathbb{S}^n$ that satisfy (4.12). Although the Smith conjecture was stated for diffeomorphisms, John Morgan explains on page 4 of [92] that its proof extends to homeomorphisms when the fixed point set is locally flat (locally ambiently homeomorphic to a segment). This holds in our case by Theorem 4.1 of [66] (finite Möbius energy implies tamely embedded) and the fact that Definition 13 implies Definition 8.

Next, we give a variation of the β-numbers where Γ must avoid a "fat torus" instead of being contained in a "thin cylinder". We start with the definition in the plane. Given a dyadic square Q, we define ε_Γ to be the infimum of the values $\varepsilon \in (0, 1]$ such that there are a line L, a point z and a disk D satisfying the following conditions: (1) L hits $3Q$, (2) D has radius $\ell(Q)/\epsilon$, (3) z is the closest point of D to L, and (4) neither D nor its reflection across L intersects Γ. See Figure 6. If no such line, point and disk exist, we set $\varepsilon_\Gamma(Q) = 1$. It is easy to see that $\beta_\Gamma(Q) = O(\epsilon_\Gamma(Q))$, but the opposite direction can certainly fail for a single square Q. Nevertheless, we will see that the corresponding sums over all dyadic squares are simultaneously convergent or divergent.

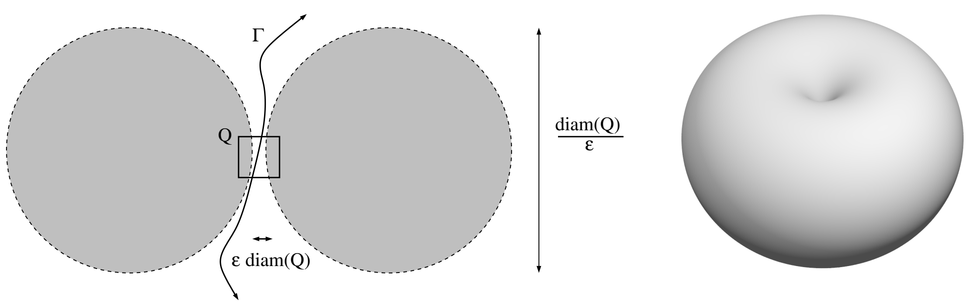

Figure 6. The left side illustrates the definition of $\varepsilon_\Gamma(Q)$ in the plane: Γ passes between two large, almost touching disks. In $\mathbb{R}^3$ the definition says Γ passes through the hole of a "thick torus", as on the right.

Definition 14. The curve Γ is chord-arc and satisfies

$$\sum_Q \varepsilon_\Gamma^2(Q) < \infty \tag{4.13}$$

where the sum is over all dyadic squares Q that hit Γ and satisfy $\operatorname{diam}(Q) \leq \operatorname{diam}(\Gamma)$.

In higher dimensions, the disk D is replaced by a ball B of radius $\operatorname{diam}(Q)/\epsilon$ that attains its distance ϵ from L at $z \in Q$, and so that the full rotation of B around L does not intersect Γ. Thus Γ is surrounded by a "fat torus". The centers of the balls form a $(n-2)$-sphere that lies in a $(n-1)$-hyperplane perpendicular to L.

5. Hyperbolic conditions for planar curves

We start by recalling the basic definitions and then discuss Weil-Petersson curves in the plane. In the next section we describe the changes that must be made for curves in $\mathbb{R}^n$, $n \geq 3$.

The hyperbolic length of a (Euclidean) rectifiable curve in the unit disk $\mathbb{D}$ or in the n-dimensional ball $\mathbb{B}^n$ is given by integrating

$$d\rho = \frac{ds}{1-|z|^2}$$

along the curve. In the upper half-space $\mathbb{H}^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ we integrate $d\rho = ds/t$. Note that this definition differs by a factor of 2 from that given in some sources; we have made our choice so that hyperbolic space has constant Gauss curvature -1. The hyperbolic distance between two points is given by taking the infimum of all hyperbolic lengths of paths connecting the points. In the ball, hyperbolic geodesics are either diameters of the disk or they are circles perpendicular to the boundary. In the half-space model $\mathbb{H}^{n+1}$, hyperbolic geodesics are either vertical rays or semi-circles centered on the boundary.

Given a closed curve $\Gamma \subset \mathbb{R}^n$, the hyperbolic convex hull, denoted $\mathrm{CH}(\Gamma)$, is the convex hull in $\mathbb{H}^{n+1}$ of all infinite geodesics that have both endpoints in Γ. The complement of the convex hull is a union of hyperbolic half-spaces. Each such half-space intersects $\mathbb{R}^n$ in an open Euclidean ball (or half-space or exterior of a ball) that does not hit Γ.

A planar curve Γ divides $\mathbb{R}^2$ into two components, and the boundary of $\mathrm{CH}(\Gamma)$ has two corresponding connected components (unless Γ is a circle) called the domes of the two sides of Γ. Each dome meets $\mathbb{R}^2$ exactly along Γ, and inside $\mathbb{H}^3$ they are disjoint, except when Γ is a circle, in which case they coincide. The dome of the bounded complementary component Ω of Γ is the upper boundary of the union of all hemispheres whose base disk is in Ω. The hemispheres that touch the dome are exactly those whose base disks touch $\partial\Omega$ at two distinct points. Such disks are called medial axis disks, and their centers form the medial axis of Ω, a well studied object in computational geometry. See Figure 7; the figure on the right was drawn by first computing the medial axis shown on the left and then taking the upper envelope of the corresponding hemispheres. The dome of the unbounded complementary component can be defined by inverting around a point in Ω. Figure 8 shows a picture of both domes together.

For a point $z \in \mathrm{CH}(\Gamma)$, we define $\delta(z) = \max(\mathrm{dist}_\rho(z, S_1), \mathrm{dist}_\rho(z, S_2))$, i.e., $\delta(z)$ is the hyperbolic distance to the farther of the two boundary components of $\mathrm{CH}(\Gamma)$. See Figure 9. For z inside the convex hull, $\delta(z)$ measures the

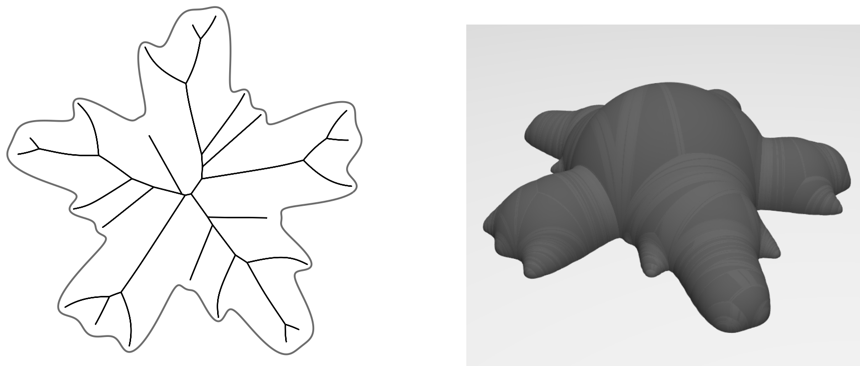

Figure 7. A smooth domain, its medial axis and its dome.

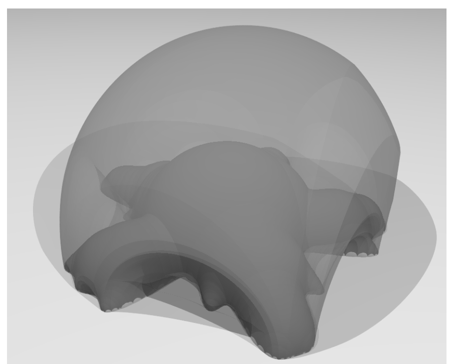

Figure 8. Domes for both sides of the curve in Figure 7; the convex hull of Γ is the region between the two surfaces.

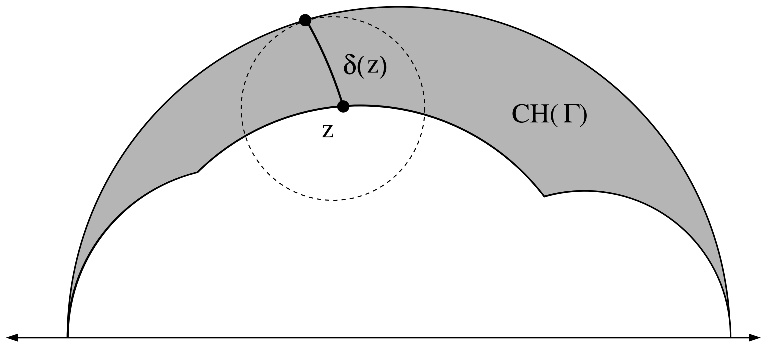

Figure 9. A "side view" of the convex hull. The line at the bottom represents $\mathbb{R}^2$, and region above it represents $\mathbb{H}^3$; $\delta(z)$ is the thickness of the convex hull near z. For quasicircles, δ is uniformly bounded; a curve is a Weil-Petersson curve if and only if $\delta \in L^2$ with respect to hyperbolic area on the boundary of the convex hull.

"thickness" of the convex hull of Γ near z. We will show that Definition 14 implies

Definition 15. The hyperbolic convex hull of the closed curve Γ satisfies

$$\int_{\partial \mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho(z) < \infty, \tag{5.1}$$

where $d\mathrm{A}_\rho$ denotes hyperbolic surface area on $\partial\mathrm{CH}(\Gamma)$.

We have integrated over all of $\partial\mathrm{CH}(\Gamma)$, but the proof will show that if the integral over one component is finite, then so is the integral over the other component.

If Γ is a quasicircle, then each point z of one boundary component is within a uniformly bounded hyperbolic distance $\delta(z)$ of the other boundary component, i.e., if Γ is a quasicircle, then $\delta(z) \in L^\infty(\partial\mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$. This holds because both complementary components of a quasicircle are uniform domains [88], and thus for every $x \in \Gamma$ and $0 < r \leq \mathrm{diam}(\Gamma)$, both complementary components contain disks of diameter $\simeq r$ inside $D(x, r)$. The converse is not true, since non-quasicircles may also have $\delta(z) \in L^\infty$. Definition 15 says that the Weil-Petersson class corresponds to $\delta(z) \in L^2(\partial\mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$. The condition $\delta(z) \in L^1(\partial\mathrm{CH}(\Gamma), d\mathrm{A}_\rho)$ is equivalent to $\mathrm{CH}(\Gamma)$ having finite hyperbolic volume. For a closed curve, this value is either zero (for lines and circles) or infinite (everything else); we leave this fact as an exercise.

For planar closed curves Γ, each boundary surface of $\mathrm{CH}(\Gamma) \subset \mathbb{H}^3$ meets $\mathbb{R}^2$ exactly along Γ, and each is isomorphic to the hyperbolic unit disk when given its hyperbolic path metric. These surfaces are pleated surfaces, i.e., each is a disjoint union of non-intersecting infinite geodesics for $\mathbb{B}^3$ (possibly uncountably many) and at most countably many regions lying on hyperbolic planes, each region bounded by disjoint hyperbolic geodesics. Roughly speaking, each surface is a copy of the hyperbolic disk that has been "bent" along a collection of disjoint geodesics, and there is an associated bending measure that gives the amount of bending on each geodesic. For more about convex hulls and pleated surfaces, see [43] by David Epstein and Al Marden (or the revised version [44]). For an overview of domes and convex hulls see Marden's paper [86]; see also his book [85] for a discussion related to hyperbolic 3-manifolds. Hyperbolic domes and convex hulls have been extensively studied, e.g., [14], [15], [25], [27], [28], [45], and [48].

In general, the bending measure may have both atoms and continuous parts; e.g., the dome of two overlapping disks has a definite angle along an infinite geodesic where two hemi-spheres meet. But for Weil-Petersson curves the bending measure cannot have an atom; this would violate Definition 15 because δ would be bounded away from zero on a fixed neighborhood of an infinite geodesic. For the dome of a planar domain bounded by a Weil-Petersson curve, the amount of bending, $B(z)$, that lies within unit distance of $z \in S_1$ is $O(\delta(z))$. Indeed, the inequality

$$\int_S B^2(z) d\mathrm{A}_\rho < \infty \tag{5.2}$$

gives yet another characterization of Weil-Petersson curves.

If we think of the bending measure as a type of curvature, the following seems reasonable (we will prove it by smoothing the convex hull boundary).

Definition 16. The curve $\Gamma \subset \mathbb{R}^2$ is the boundary of a smooth surface $S \subset \mathbb{H}^3$ such that $\kappa_1(z), \kappa_2(z) \to 0$ as $z \in S$ tends to the boundary of hyperbolic space and

$$\int_S \left(\kappa_1^2(z) + \kappa_2^2(z)\right) d\mathrm{A}_\rho(z) < \infty, \tag{5.3}$$

where κ_1, κ_2 are the principle curvatures of S.

In Section 18 we use a result of Charles Epstein [42], relating curvature, the Gauss map and quasiconformal reflections to show that this implies Definition 3.

We can take the surface in Definition 16 to be minimal. A result of Anderson [7] shows that any closed Jordan curve $\Gamma \subset \mathbb{R}^2$ is the asymptotic boundary of some minimal disk in $\mathbb{H}^3$. An estimate of Andrea Seppi (see Lemma 19.1) says that

$$\max(|\kappa_1(z)|, |\kappa_2(z)|) = O(\delta(z)), \quad x \in S \subset \mathrm{CH}(\Gamma). \tag{5.4}$$

This implies that S has finite total curvature if Definition 15 holds.

Definition 17. The curve Γ is a quasicircle that is the asymptotic boundary of an embedded minimal surface that is topologically a disk and satisfies $\int_S \kappa^2 d\mathrm{A}_\rho < \infty$.

As noted earlier, the Gauss curvature of S satisfies $K(z) = -1 - \kappa^2(z) \leq -1$, so for a compact Jordan sub-domain Ω of S with area A and boundary length L, the isoperimetric equality for such surfaces (e.g., (4.30) of [101]) implies the following:

$$L^2 \geq 4\pi A\chi + A^2,$$

where $\chi = \chi(\Omega)$ is the Euler characteristic of Ω. A short manipulation gives

$$L - A \geq \frac{4\pi A\chi}{L + A}.$$

For $\chi \geq 0$ the right-hand side is non-negative, and if $\chi < 1$ it is greater than or equal to $4\pi\chi$. In either case, the difference is bounded below, depending only on χ. Conversely, we shall prove $L - A$ is bounded above iff Γ is Weil-Petersson.

Definition 18. The closed Jordan curve Γ is the asymptotic boundary of a minimal surface $S \subset \mathbb{H}^3$ that has finite Euler characteristic and that can be written as the nested unions of compact subsets $\Omega_1 \subset \Omega_2 \subset \dots$ such that

$$\limsup_n \left[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n) \right] < \infty.$$

A special case of such compact nested subdomains is given by simply truncating the surface S at Euclidean height t above the boundary of $\mathbb{H}^3$. Define

$$S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s > t\}, \text{ and } \partial S_t = S \cap \{(x, y, s) \in \mathbb{H}^3 : s = t\}.$$

The renormalized area of S is defined as

$$\mathcal{RA}(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t) \right],$$

and we shall prove that the limit always exists (possibly it is $-\infty$):

Definition 19. The closed Jordan curve Γ is the asymptotic boundary of a minimal surface $S \subset \mathbb{H}^3$ with finite Euler characteristic and finite renormalized area.

There is a discrete version of renormalized area that illustrates the connection between our Euclidean and hyperbolic conditions. Define the "dyadic cylinder"

$$X = \bigcup_{n=1}^{\infty} \Gamma_n \times \left(2^{-n}, 2^{-n+1}\right],$$

where $\{\Gamma_n\}$ are the dyadic polygonal approximations to Γ, as in Theorem 1.3. See Figures 10 and 13. The dyadic cylinder X has holes, but in Section 12 we shall describe how to fill them to form a triangulated, simply connected "dyadic dome", that will work too. Theorem 1.3 is equivalent to finite renormalized area for these discrete surfaces:

Definition 20. The dyadic cylinder X corresponding to the closed Jordan curve Γ has finite renormalized area.

6. Hyperbolic conditions in higher dimensions

Next we discuss how the definitions presented in the previous section have to be changed for curves $\Gamma \subset \mathbb{R}^n$. Definition 20 needs no change; the construction of the dyadic cylinder and dome is exactly the same, and the proof that they have finite renormalized area if and only if Γ satisfies Definition 11 is valid in all dimensions. Definition 16 is also unchanged. Rather than smoothing a boundary component of the convex hull, we can smooth the dyadic dome instead. The proof that it implies Definition 3 using the theorem of Charles

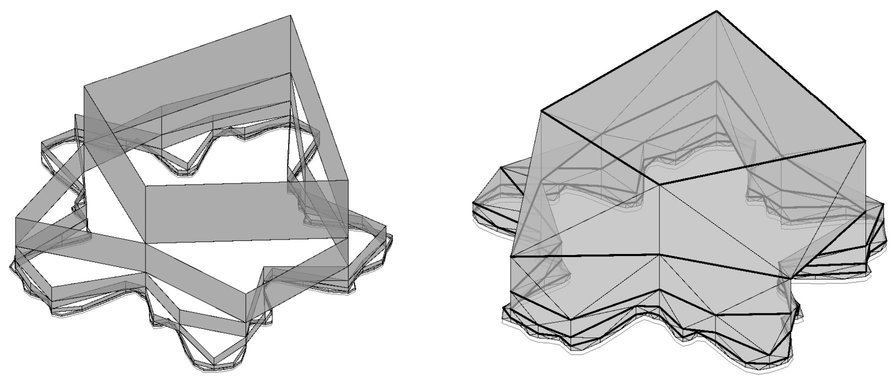

Figure 10. The dyadic cylinder and dyadic dome (same curve as Figure 7).

Epstein on quasiconformality of Gauss maps will be replaced by a construction of a biLipschitz involution fixing Γ and implying Definition 13.

If $\Gamma \subset \mathbb{R}^n$ is the asymptotic boundary of a minimal 2-surface in $\mathbb{H}^{n+1}$, then Definitions 17, 18 and 19 remain the same as before. In general, this need not be the case, and they each require a change of terminology, but not of concept. Anderson's result for $\mathbb{H}^3$ in [7] is replaced by his result from [6] for smooth curves in $\mathbb{R}^n$ giving the existence of a minimal 2-current. This is extended to the existence of minimal 2-chain by Fang-Hua Lin in [83] for C^1 curves, and his proof extends to $H^{3/2}$ curves. For definitions of currents see Herbert Federer's comprehensive text [47] or the more accessible [119] by Leon Simon. Brian White's paper [131] summarizes the basic definitions and results, and [91] by Frank Morgan starts with a very informative example.

In general, one cannot control the global topology of a minimal current or chain, but in [83] Lin proves that if Γ is C^1, then there is a minimal 2-chain with asymptotic boundary Γ that agrees with a smooth surface in a neighborhood of the boundary. His proof of this fact only uses the C^1 assumption to deduce that near the boundary, the 2-chain is close to a vertical 2-plane, and this implication also holds for the curves $\Gamma \subset \mathbb{R}^n$ satisfying Definition 14. He proves that this surface is locally a Lipschitz graph with small norm with respect to this plane, and this also holds under our assumptions. Moreover, this surface is topologically an annulus and is asymptotic to the dyadic dome of Γ. Note that Lin's proof only gives that Γ is the boundary of some such 2-chain, not that every chain with asymptotic boundary Γ has this property.

Definition 17 will thus be replaced by the following: $\Gamma \subset \mathbb{R}^n$ is a closed Jordan curve that is the asymptotic boundary of a minimal 2-chain such that in $\{(x,t) \in \mathbb{H}^{n+1} : 0 < t < t_0\}$ agrees with an annular surface that has finite total curvature.

Definitions 18 and 19 are both changed in the obvious way, replacing the minimal surface by a minimal 2-chain or current which agrees with a surface S near the boundary. In Definition 18, we take Ω_n to be a smooth topological annulus contained in $S_t^* = S \cap \{(x,s) \in \mathbb{H}^{n+1} : s < t\}$ for $t > 0$ small enough. Definition 19 is unchanged, except that it suffices to consider the area of S between heights $0 < s < t$ for t fixed and s tending to zero and show that the corresponding limit exists.

Finally, Definition 15 needs to be changed because in higher dimensions the hyperbolic convex hull of Γ has a single boundary component, and so it does not make sense to measure the thickness of the convex hull by the hyperbolic distance between the two boundary components. However, given a point $z \in \mathrm{CH}(\Gamma)$ and a tangent vector v at z, it does make sense to ask how far it is from z to the boundary of $\mathrm{CH}(\Gamma)$ following a geodesic in direction v. We let $\delta(z,v)$ denote this distance. The thickness of $\mathrm{CH}(\Gamma)$ will be the supremum of these distances over different directions, but we need to avoid moving vertically or parallel to Γ. Therefore we want

$$\delta(z) = \inf_P \sup_{v \perp P} \delta(z,v),$$

where the infimum is over all tangent 2-planes at z generated by the vertical direction and one horizontal direction; the infimum will be attained when the horizontal direction is approximately parallel to Γ. With this definition, we have $\delta(z) = O(\varepsilon_\Gamma(Q))$, where $z = (x,t)$ and Q is a dyadic cube in $\mathbb{R}^n$ with $x \in Q$ and $\mathrm{diam}(Q) \simeq t$.

In Definition 15 we no longer integrate $\delta^2(z)$ over the boundary of the convex hull (the hyperbolic $(n-1)$-measure of a unit ball will be approximately δ^{n-1} instead of $\simeq 1$), but we have to integrate over some appropriate 2-surface, such as the minimal 2-chain or current described above, or the dyadic dome. In the latter case, we do not know that the dyadic dome is contained inside $\mathrm{CH}(\Gamma)$), so $\delta(z)$ as given above is not defined there, but it suffices to integrate $\delta^2(R(z))$ over the dome, where $R : \mathbb{H}^{n+1} \to \mathrm{CH}(\Gamma)$ is the nearest point retraction. We can also state the condition as $\sum_Q \delta^2(Q) < \infty$, where the sum is over dyadic cubes in $\mathbb{R}^n$, and $\delta(Q)$ is defined as the maximum of $\delta(z)$ over $\mathrm{CH}(\Gamma) \cap \mathrm{T(Q)}$, where $T(Q) = Q \times [\frac{1}{2}\ell(Q), \ell(Q)] \subset \mathbb{H}^{n+1}$.

7. **Summary of definitions**

For the reader's convenience, Table 1 gives a summary of the definitions from the preceding sections. For curves in $\mathbb{R}^2$, Definitions (1)–(20) are equivalent. For curves in $\mathbb{R}^n$, $n \geq 3$, Definitions (6)–(20), properly modified, are all equivalent. Definitions (1)–(3) are the previously known function theoretic definition, (4)-(20) are the new definitions proven here and in [17], and (21)–(26) are other known definitions, described in Appendix A. The graph in

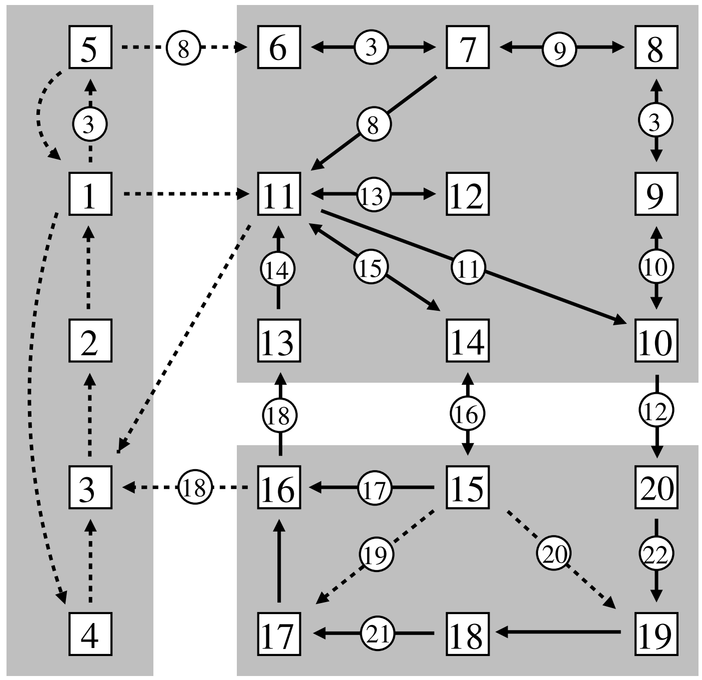

Figure 11. Diagramming the proof that our definitions are all equivalent. Each number in a square refers to a definition and numbers on an arrow give the section of this paper where that implication is proven; unlabeled dashed edges are proven in [17], and unlabeled solid edges are immediate from the definitions. The dashed arrows are only valid for $n = 2$, either because one of the definitions only makes sense there, or we only give the proof in that case. The shaded blocks group definitions based on conformal maps (left), Euclidean geometry (upper right) and hyperbolic geometry (lower right).

Figure 11 has vertices representing definitions and edges representing proofs; the vertex labels correspond to definition numbers, and the edge labels say in which section of this paper the corresponding proof may be found.

8. From function theory to β-numbers: (5) $\Rightarrow$ (6), (7) $\Rightarrow$ (11)

LEMMA 8.1. *Definition* 5 *implies Definition* 6.

Proof. Suppose f is a conformal map from $\mathbb{D}$ to the bounded complementary component of Γ. Let $a : \mathbb{T} \to \Gamma$ be an orientation preserving arclength parametrization, and let $\varphi = a^{-1} \circ f : \mathbb{T} \to \mathbb{T}$. We claim this circle homeomorphism is quasisymmetric. To prove this, consider two adjacent arcs I, J of

Table 1. A summary of 26 definitions of Weil-Petersson curves.

Definition	Description
1	$\log f'$ in Dirichlet class
2	Schwarzian derivative
3	QC dilatation in L^2
4	conformal welding
5	$\exp(i \log f')$ in $H^{1/2}$
6	arclength parametrization in $H^{3/2}$
7	tangents in $H^{1/2}$
8	finite Möbius energy
9	Jones conjecture
10	good polygonal approximations
11	β^2-sum is finite
12	Menger curvature
13	biLipschitz involutions
14	ε^2-sum is finite
15	δ-thickness in L^2
16	finite total curvature surface
17	minimal surface of finite curvature
18	additive isoperimetric bound
19	finite renormalized area
20	dyadic cylinder
21	closure of smooth curves in $T_0(1)$
22	P_φ^- is Hilbert-Schmidt
23	double hits by random lines
24	finite Loewner energy
25	large deviations of SLE(0^+)
26	Brownian loop measure

the same length. Since Definition 5 states that Γ is chord-arc, and chord-arc curves are quasicircles, the conformal map f from $\mathbb{D}$ to the bounded complementary has a quasiconformal extension to the whole plane. Hence f is also a quasisymmetric map on $\mathbb{T}$, and this implies that $f(I)$ and $f(J)$ have comparable diameters; e.g., see [61] or Section 4 of [67]. Since Γ is chord-arc, this implies that $f(I)$ and $f(J)$ have comparable lengths, and hence that $\varphi(I)$ and $\varphi(J)$ also have comparable lengths, since a preserves arclength. This implies that φ is quasisymmetric.

Note that $a' = \exp(i \arg f') \circ \varphi$. In [12], Beurling and Ahlfors proved that $H^{1/2}$ is invariant under composition with a quasisymmetric homeomorphism of $\mathbb{T}$. Thus $a' \in H^{1/2}$ if and only if $\exp(i \arg f) \in H^{1/2}$. Since a is Lipschitz, it is also absolutely continuous, so its weak derivative agrees with its pointwise derivative a'. Hence $a \in H^{3/2}(\mathbb{T})$. □

A direct proof of the converse is given in [17]. A more roundabout proof uses the implications (6) ⇒ (7) ⇒ (11) ⇒ (3) ⇒ (2) ⇒ (1) ⇒ (5). The first two and last implications are proven in this paper, and the other three are proven in [17].

By our remarks in Section 3, we already know that Definition 6 is equivalent to Definition 7, so we need only prove:

Lemma 8.2. *Definition* 7 *implies Definition* 11.

Proof. We let U be the torus $\mathbb{T}^2$ minus the diagonal, and take a Whitney decomposition of U by dyadic squares $\{Q_j\}$, i.e., a covering of U by squares Q with disjoint interiors and the property that $\mathrm{diam}(Q) \simeq \mathrm{dist}(Q, \partial U)$. We will think of $\mathbb{T}$ as $[0,1]$ with its endpoints identified, and use dyadic squares in $[0,1]^2$ as elements of our Whitney decomposition. See Figure 12.

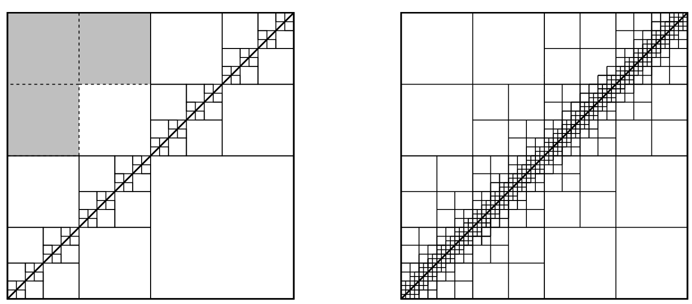

Figure 12. On the left is the obvious packing of $[0,1]^2$ minus the diagonal by maximal dyadic squares, but this is not a Whitney decomposition, since some squares touch the diagonal. However, if we recursively subdivide each of these squares into four sub-squares and keep the three not touching the diagonal (shaded on left), we generate the Whitney decomposition on the right.

Normalize Γ to have length 1 and use the arclength parametrization to identify $\Gamma \times \Gamma$ with the torus $\mathbb{T}^2 = [0,1]^2$. We decompose $\Gamma \times \Gamma$ into pieces denoted $\{W_j\}$, where each piece is a product of two dyadic arcs $W_j = \gamma_j \times \gamma_j'$ of equal length (these correspond to dyadic intervals in $[0,1]$). For each Whitney piece $W_j = \gamma_j \times \gamma_j' \subset \Gamma \times \Gamma$, choose a $w_0 \in \gamma_j'$ such that

$$\ell(\gamma_j') \int_{\gamma_j} \bigl|\tau(z) - \tau(w_0)\bigr|^2 |dz| \le 2 \int_{\gamma_j'} \int_{\gamma_j} \bigl|\tau(z) - \tau(w)\bigr|^2 |dz||dw|.$$

We can do this because a positive measurable function must take a value that is less than or equal to twice its average. Let L be the line through one endpoint of γ_j' in direction $\tau(w_0)$. Then the maximum distance d that γ_j can attain from L satisfies

$$d \lesssim \int_{\gamma_j} \bigl|\tau(z) - \tau(w_0)\bigr||dz| \le \left(\int_{\gamma_j} \bigl|\tau(z) - \tau(w_0)\bigr|^2 |dz| \right)^{1/2} \ell(\gamma_j)^{1/2}.$$

Therefore (using the fact that γ is chord-arc),

$$\begin{aligned} \beta^2(\gamma_j) \simeq d^2/\mathrm{diam}(\gamma_j) &\lesssim \frac{1}{\ell(\gamma_j)} \int_{\gamma_j} \bigl|\tau(z) - \tau(w_0)\bigr|^2 |dz| \\ &\le \frac{2}{\ell(\gamma_j)^2} \int_{\gamma_j} \int_{\gamma_j'} \bigl|\tau(z) - \tau(w)\bigr|^2 |dz||dw| \\ &\lesssim \int_{\gamma_j} \int_{\gamma_j'} \left| \frac{\tau(z) - \tau(w)}{z - w} \right|^2 |dz||dw|. \end{aligned}$$

Summing over all Whitney pieces proves that the β^2-sum is finite when taken over all arcs of the form $\{\gamma_j\}$. By construction (see Figure 12), every dyadic interval in $[0, 1]$ (except for $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ and $[0, 1]$) occurs as a γ_j at least once, and at most three times, so this bounds the sum of $\beta^2(\gamma)$ over all dyadic subintervals of Γ, for a fixed base point, with an estimate independent of the basepoint. Thus it holds for some multi-resolution family of arcs (recall the $\frac{1}{3}$-trick for making such a family from three translates of the dyadic family). Because of Lemma 4.2, this proves the lemma. □

9. Tangents control Möbius energy: (7) ⇔ (8)

The following proof is similar to an argument in [21].

Lemma 9.1. *Definition* 7 *is equivalent to Definition* 8.

Proof. We want to show that

$$\text{Möb}(\Gamma) = \int_\Gamma \int_\Gamma \frac{1}{|z - w|^2} - \frac{1}{\ell(z, w)^2} |dz||dw| < \infty \tag{9.1}$$

if and only if

$$\int_\Gamma \int_\Gamma \frac{\bigl|\tau(x) - \tau(y)\bigr|^2}{|x - y|^2} |dx||dy| < \infty. \tag{9.2}$$

First we collect a few relevant formulas about rectifiable arcs.

Suppose γ is rectifiable with endpoints z, w, and let $\tau(x)$ denote the unit tangent vector at $x \in \gamma$ (well defined almost everywhere on γ; we assume γ is oriented from w to z). Then

$$u = \frac{z-w}{|z-w|} = \frac{1}{|z-w|} \int_\gamma \tau(y)|dy|$$

is the unit vector in direction $z-w$, and hence

$$|z-w|^2 = |z-w| \int_\gamma \langle \tau(x), u \rangle |dx| = \int_\gamma \int_\gamma \langle \tau(x),\ \tau(y) \rangle |dy||dx|. \tag{9.3}$$

Next, using $|\tau| = 1$, we get

$$\begin{aligned}
\int_\gamma \int_\gamma \left|\tau(x) - \tau(y)\right|^2 |dx||dy| &= \int_\gamma \int_\gamma \langle \tau(x) - \tau(y), \tau(x) - \tau(y) \rangle |dx||dy| \\
&= \int_\gamma \int_\gamma \left(\left|\tau(x)\right|^2 - 2\langle \tau(y), \tau(x) \rangle + \left|\tau(y)\right|^2 \right) |dx||dy| \\
&= 2\ell(\gamma)^2 - 2 \int_\gamma \int_\gamma \langle \tau(x), \tau(y) \rangle |dx||dy|.
\end{aligned}$$

Let $\gamma = \gamma(z, w) \subset \Gamma$ be the shorter sub-arc with endpoints z, w. Combining the equality above with (9.3) and the assumption that Γ is chord-arc, we get

$$\begin{aligned}
\text{Möb}(\Gamma) &= \int_\Gamma \int_\Gamma \frac{\ell(z,w)^2 - |z-w|^2}{\ell(z,w)^2 |z-w|^2} |dz||dw| \\
&\simeq \int_\Gamma \int_\Gamma \frac{\ell(z,w)^2 - \int_\gamma \int_\gamma \langle \tau(x), \tau(y) \rangle |dx||dy|}{|z-w|^4} |dz||dw| \\
&= \frac{1}{2} \int_\Gamma \int_\Gamma \frac{\int_\gamma \int_\gamma \left|\tau(x) - \tau(y)\right|^2 |dx||dy|}{|z-w|^4} |dz||dw|
\end{aligned}$$

Given $x, y \in \Gamma$ with $\ell(x,y) \leq \frac{1}{8}\ell(\Gamma)$, set $\sigma(x,y) = \{(z,w) \in \Gamma \times \Gamma : x, y \in \gamma(z,w)\}$. If, in addition, $0 < t < \ell(\Gamma)/2$, let $\sigma(x,y,t) \subset \Gamma$ be the arc of length t with one endpoint x that is disjoint from the arc $\gamma(x,y)$. Using the fact that Γ is chord-arc, it is not hard to show that if $m \geq 2$, $t \in [\ell(x,y), \text{diam}(\Gamma/8)]$, and $w \in \sigma(y,x,t)$, then

$$\int\limits_{\sigma(x,y,t)} \frac{|dz|}{|z-w|^m} \simeq \frac{1}{|x-w|^{m-1}}. \tag{9.4}$$

(Hint: divide the integral using the annuli $\{z : 2^n|w-x| < |z-w| \le 2^{n+1} \cdot |w-x|\}$). Set $s = \ell(\Gamma)/2$ and set $t = \ell(\Gamma)/8$. Note that

$$\begin{aligned} \ell(x,y) \le t, w \in \sigma(y,x,t), z \in \sigma(x,y,t) \quad &\Rightarrow \quad (z,w) \in \sigma(x,y) \\ \ell(x,y) \le t, (z,w) \in \sigma(x,y) \quad &\Rightarrow \quad z \in \sigma(x,y,s), \\ & \quad\quad w \in \sigma(y,x,s). \end{aligned}$$

Let $\Sigma(t) = \{(x,y) \in \Gamma \times \Gamma : \ell(x,y) \le t\}$. By the first implication and Fubini's theorem,

$$\begin{aligned} &\iint_{\Gamma\times\Gamma} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y)} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\ &\qquad \ge \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y)} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\ &\qquad \gtrsim \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{z\in\sigma(x,y,t)} \int_{w\in\sigma(y,x,t)} \frac{|dz||dw|}{|z-w|^4}|dx||dy|, \end{aligned}$$

and using (9.4),

$$\begin{aligned} &\simeq \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{w\in\sigma(y,x,t)} \frac{|dw|}{|x-w|^3}|dx||dy| \\ &\simeq \iint_{\Sigma(t)} \frac{|\tau(x)-\tau(y)|^2}{|x-y|^2}|dx||dy|. \end{aligned}$$

This proves that Definition 8 implies Definition 7, since the integral over $(\Gamma \times \Gamma) \setminus \Sigma(t)$ is obviously bounded (depending on t) since pairs of points (x,y) in this set are separated by distance $\gtrsim t$.

To prove the opposite implication, we want to show that Möb(Γ) is finite if the τ-integral is. As above, it suffices to evaluate the energy integral (9.1) over $\Sigma(t)$. A calculation similar to the one above gives

$$\begin{aligned} &\iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \iint_{(z,w)\in\sigma(x,y))} \frac{|dz||dw|}{|z-w|^4}|dx||dy| \\ &\qquad \lesssim \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{z\in\sigma(x,y,s)} \int_{w\in\sigma(y,x,s)} \frac{|dz||dw|}{|x-w|^4}|dx||dy| \\ &\qquad \simeq \iint_{\Sigma(t)} |\tau(x)-\tau(y)|^2 \int_{w\in\sigma(y,x,s)} \frac{|dw|}{|x-w|^3}|dx||dy| \\ &\qquad \simeq \iint_{\Sigma(t)} \frac{|\tau(x)-\tau(y)|^2}{|x-y|^2}|dx||dy|. \end{aligned}$$

This proves that Definitions 8 and 7 are equivalent. $\square$

10. **Continuous and discrete asymptotic smoothness: (9) $\Leftrightarrow$ (10)**

Lemma 10.1. *Definition* 9 *is equivalent to Definition* 10.

Proof. We use notation similar to Section 8. Without loss of generality, we may rescale Γ so that it has length 1. As before, we identify $\Gamma\times\Gamma$ with the torus $\mathbb{T}^2 = [0,1]^2$, we let U be this torus minus the diagonal, and we take a Whitney decomposition of U by dyadic squares $\{Q_j\}$ as in the proof of Lemma 8.2. Also as before, $\Gamma\times\Gamma$, minus the diagonal, has a Whitney decomposition with elements denoted $\{W_j\}$, and each of these pieces is a product of dyadic arcs $W_j = \gamma_j \times \gamma_j'$. For each W_j, we can write $\gamma_j \cup \gamma_j' = \Gamma_j \setminus \Gamma_j'$ so that $\gamma_j, \gamma_j', \Gamma_j, \Gamma_j'$ all have comparable lengths.

Recall that $\mathrm{crd}(\gamma) = |z-w|$, where z, w are the endpoints of γ, and that $\Delta(\gamma) \equiv \ell(\gamma) - \mathrm{crd}(\gamma)$. We sometimes write $\Delta(z,w)$ for $\Delta(\gamma)$ when γ has endpoints z, w, and when it is clear from context which arc connecting these points we mean. We say two subarcs of Γ are adjacent if they have disjoint interiors, but share a common endpoint.

Lemma 10.2. *If* $\gamma, \gamma' \subset \Gamma$ *are adjacent, then* $\Delta(\gamma) + \Delta(\gamma') \le \Delta(\gamma\cup\gamma')$.

Proof. Note that $\ell(\gamma\cup\gamma') = \ell(\gamma)+\ell(\gamma')$ and $\mathrm{crd}(\gamma\cup\gamma') \le \mathrm{crd}(\gamma)+\mathrm{crd}(\gamma')$, so the following holds:

$$\begin{aligned}\Delta(\gamma\cup\gamma') &= \ell(\gamma\cup\gamma') - \mathrm{crd}(\gamma\cup\gamma')\\ &\ge \ell(\gamma)+\ell(\gamma') - \mathrm{crd}(\gamma) - \mathrm{crd}(\gamma') = \Delta(\gamma)+\Delta(\gamma'). \qquad \square\end{aligned}$$

Corollary 10.3. *If* $\gamma \subset \gamma'$ *then* $\Delta(\gamma) \le \Delta(\gamma')$.

Now, fix j and consider the Whitney box $W_j = \gamma_j \times \gamma_j'$. If $\gamma \subset \Gamma_j$ is any arc with one endpoint in γ_j and the other in γ_j', then $\Gamma_j' \subset \gamma \subset \Gamma_j$, and hence $\Delta(\Gamma_j') \le \Delta(\gamma) \le \Delta(\Gamma_j)$. Because Γ is chord-arc, if $z \in \gamma_j'$ and $w \in \gamma_j$, then $|z-w| \gtrsim \ell(\Gamma_j') \simeq \ell(\Gamma_j)$. We can therefore write the integral from Definition 9 as

$$\begin{aligned}\int_\Gamma\int_\Gamma \frac{\ell(z,w)-|z-w|}{|z-w|^3}|dz||dw| &= \sum_j \int_{W_j} \frac{\Delta(z,w)}{|z-w|^3}|dz||dw|\\ &\lesssim \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)^3}\ell(\Gamma_j)^2 = \sum_j \frac{\Delta(\Gamma_j)}{\ell(\Gamma_j)}.\end{aligned}$$

Thus Definition 10 implies Definition 9.

Reversing the argument, now assume Γ_j' is some dyadic subinterval of Γ and let γ_j, γ_j' be the equal length dyadic arcs adjacent to Γ_j'. We have

$$\int_{\gamma_j}\int_{\gamma_j'} \frac{\ell(z,w)-|z-w|}{|z-w|^3}|dz||dw| \gtrsim \frac{\Delta(\Gamma_j')}{\ell(\Gamma_j')}.$$

The squares $W_j = \gamma_j \times \gamma_j'$ arising in this way have bounded overlap, so

$$\int_\Gamma \int_\Gamma \frac{\ell(z,w) - |z-w|}{|z-w|^3} |dz||dw| \gtrsim \sum_j \frac{\Delta(\Gamma_j')}{\ell(\Gamma_j')},$$

where the sum is over all dyadic subintervals of Γ. This works for any dyadic decomposition $\{\Gamma_j\}$ of Γ, and hence for a multi-resolution family. This gives the equivalence of Definitions 9 and 10. □

11. The β^2-sum implies asymptotic smoothness: (11) ⇒ (10)

The following is where we use Theorem 1.5, the strengthening of Jones's traveling salesman theorem (TST) mentioned in the introduction. The proof given in [18] is somewhat involved so, although this section is short, the following implication may be the most difficult one represented by an arrow in Figure 11.

Lemma 11.1. *Definition* 11 *implies Definition* 10.

Proof. We let $\{\gamma_j\}$ be a dyadic decomposition of Γ. For each j, choose a dyadic cube Q_j that hits γ_j and has diameter between $\text{diam}(\gamma_j)$ and $2{\cdot}\text{diam}(\gamma_j)$. Note that any such dyadic cube can only be associated to a uniformly bounded number of arcs γ_j in this way, because there are only a bounded number of arcs γ_j that have the correct size and are close enough to Q_j; this uses the fact that Γ is chord-arc. Also, because Γ is chord-arc, $\text{diam}(\gamma_j) \simeq \ell(\gamma_j) \simeq \text{diam}(Q_j)$. Therefore, by the strengthened TST (1.5),

$$\Delta(\gamma_j) = \ell(\gamma_j) - \text{crd}(\gamma_j) \simeq \sum_{Q \subset 3Q_j} \beta^2_{\gamma_j}(Q)\ell(Q).$$

Since $\beta_{\gamma_j}(Q) \le \beta_\Gamma(Q)$, we get

$$\begin{aligned}\sum_j \frac{\Delta_j}{\ell(\gamma_j)} &\simeq \sum_j \sum_{Q \subset 3Q_j} \beta^2_{\gamma_j}(Q) \frac{\ell(Q)}{\ell(Q_j)} \\ &\lesssim \sum_j \sum_{Q \subset 3Q_j} \beta^2_\Gamma(Q) \frac{\ell(Q)}{\ell(Q_j)} \simeq \sum_Q \beta^2_\Gamma(Q) \cdot \sum_{j: Q \subset 3Q_j} \frac{\ell(Q)}{\ell(Q_j)}.\end{aligned}$$

Note that for each Q with $\text{diam}(Q) \le \text{diam}(\Gamma)$ and $Q \cap \Gamma \ne \emptyset$, there is a cube of the form Q_j from above that has diameter comparable to $\text{diam}(Q)$ and such that $Q \subset 3Q_j$. Moreover, there are only a uniformly bounded number of distinct dyadic cubes Q_j of a given size so that $3Q_j$ contains Q, so each Q_j can only be chosen a bounded number of times. Thus the sum over the j's in the last line above is bounded by a multiple of a geometric series, and so it is uniformly bounded. Therefore $\sum_j \frac{\Delta(\gamma_j)}{\ell(Q_j)} \lesssim \sum_Q \beta^2_\Gamma(Q)$. □

12. Dyadic cylinders and domes: (10) ⇔ (20)

If $\Gamma \subset \mathbb{R}^n$ is rectifiable, set $Y_t = \Gamma \times (t, 1] \subset \mathbb{H}^{n+1}$ and $Y = Y_0$. Then

$$\mathrm{A}_\rho(Y_t) = \int_t^1 \int_\Gamma \frac{dsdt}{t^2} = \ell(\Gamma)\left(\frac{1}{t} - 1\right) = L_\rho(\Gamma_t) - \ell(\Gamma).$$

Thus the vertical cylinder Y has finite renormalized area for any rectifiable curve. Roughly speaking, we expect renormalized area to measure how orthogonal the surface is to the boundary. The cylinder is perfectly vertical; a minimal surface with the same boundary curve necessarily deviates from vertical over regions where Γ has some curvature. We can make this vague idea precise using a discrete analog of a minimal surface.

Define a "dyadic cylinder" associated to Γ by $X = \bigcup_{n=0}^\infty \Gamma_n \times (2^{-n-1}, 2^{-n}]$, where Γ_n is the 2^n-gon inscribed in Γ corresponding to a dyadic decomposition of Γ into subarcs of length $2^{-n}\ell(\Gamma)$. Note that is depends on a choice of base point for the dyadic decomposition.

Each "layer" of X between heights 2^{-n} and 2^{-n+1} consists of 2^n Euclidean rectangles (or "panels") in vertical planes that meet along vertical edges (called "hinges"). See Figure 13. Alternate vertices of the top edge of one layer agree with the bottom vertices of the next layer up, but there are triangular horizontal "holes" between the layers.

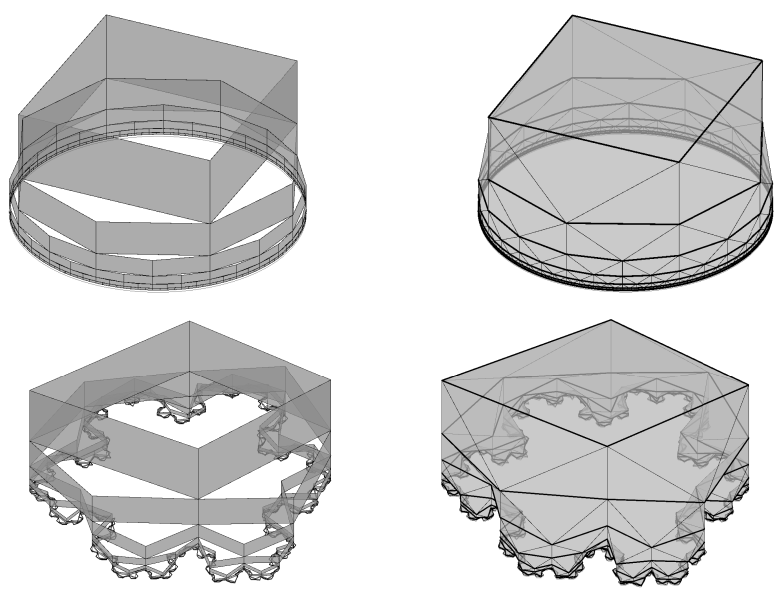

Figure 13. The dyadic cylinder and dome of a circle and a snowflake; the first has finite renormalized area and the second does not.

If desired, these holes can be eliminated as follows. Suppose $[z,w]$ is an edge segment of Γ_n and let $a_1, a_2 \in \mathbb{H}^3$ be the points of height 2^{-n} and 2^{-n-1} above z. Similarly b_1, b_2 above w. Let v be the vertex of Γ_{n+1} between z and w and let c_2 be the point at height 2^{-n-1} above v. The rectangular face of the dyadic cylinder X with corners a_1, a_2, b_2, b_1 is replaced by the three Euclidean triangles with vertices (a_1, b_1, c_2), (c_2, b_2, b_1) and (a_1, c_2, b_1). Doing this for every edge of Γ_n and adding the interior of the polygon Γ_2 raised to height $1/4$ defines a closed surface that we will call a dyadic dome of Γ. See Figure 13 for two examples of dyadic cylinders and the corresponding domes.

Lemma 12.1. *If Γ is a closed rectifiable Jordan curve, then Γ is Weil-Petersson if and only if every corresponding dyadic cylinder X has finite renormalized area, with a bound independent of the choice of base point.*

Proof. First we show that the Weil-Petersson condition implies finite renormalized area. A simple calculation, as above, shows that the part of X between heights 2^{-n} and 2^{-n+1} has hyperbolic area $2^{n-1}\ell(\Gamma_n)$. Similarly, if $2^{-n-1} < t \leq 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) = \sum_{k=0}^{n} 2^{k-1}\ell(\Gamma_k) + \left(\frac{1}{t} - 2^n\right)\ell(\Gamma_{n+1}),$$

and hence

$$\begin{aligned}
\mathrm{A}_\rho(X_t) - \frac{1}{t}\ell(\Gamma) &= \mathrm{A}_\rho(X_t) - \left(\frac{1}{t} - 2^n + 1 + \sum_{k=1}^{n} 2^{k-1}\right)\ell(\Gamma) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k\left[\ell(\Gamma) - \ell(\Gamma_k)\right] + \left(\frac{1}{t} - 2^n\right)\left(\ell(\Gamma) - \ell(\Gamma_{n+1})\right) \\
&= -\ell(\Gamma) - \sum_{k=1}^{n} 2^k\left[\ell(\Gamma) - \ell(\Gamma_k)\right] + O\big(2^n[\ell(\Gamma) - \ell(\Gamma_{n+1})]\big) \\
&\to -\ell(\Gamma) - \sum_{k=1}^{\infty} 2^k\left[\ell(\Gamma) - \ell(\Gamma_k)\right]
\end{aligned}$$

since the infinite series is convergent when Γ is Weil-Petersson by (1.2). Finally, for $2^{-n-1} < t \leq 2^{-n}$, note that $\ell(\partial X_t) = \ell(\Gamma_{n+1})/t$, so

$$\frac{1}{t}\left[\ell(\partial X_t) - \ell(\Gamma)\right] \leq 2^{n+1}\left[\ell(\Gamma_{n+1}) - \ell(\Gamma)\right] \to 0,$$

since these are terms of a summable series. Thus $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ has a finite limit and X has finite renormalized area.

Next we consider the converse: finite renormalized area implies Γ is Weil-Petersson. Suppose that $\mathcal{RA}(X) < \infty$. First we deduce that Γ is rectifiable.

If $t = 2^{-n}$, then

$$\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) = \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n \ell(\Gamma_n) = O(1),$$

or, equivalently,

$$\ell(\Gamma_n) = \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-2}) + \cdots + 2^{-n}\ell(\Gamma_1) + O(2^{-n}),$$

and hence (since $\{\ell(\Gamma_n)\}$ is non-decreasing in n),

$$\begin{aligned} \ell(\Gamma_n) &\le \frac{1}{2}\ell(\Gamma_{n-1}) + \frac{1}{4}\ell(\Gamma_{n-1}) + \cdots + O(2^{-n}) \\ &\le \ell(\Gamma_{n-1}) + O(2^{-n}), \end{aligned}$$

which implies $\ell(\Gamma) < \infty$. To show Γ is Weil-Petersson, take $t = 2^{-n}$ and note that

$$\begin{aligned} \mathrm{A}_\rho(X_t) - L_\rho(\partial X_t) &= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - 2^n \ell(\Gamma_n) \\ &= \left(\sum_{k=1}^{n} 2^{k-1}\ell(\Gamma_k)\right) - \left(1 + 1 + 2 + \ldots 2^{n-1}\right)\ell(\Gamma_n) \\ &= -\frac{1}{2}\sum_{k=1}^{n} 2^k \left[\ell(\Gamma_n) - \ell(\Gamma_k)\right] - \ell(\Gamma_n). \end{aligned}$$

By the monotone convergence theorem (for counting measure on $\mathbb{N}$), as $n \nearrow \infty$ this tends to

$$-\frac{1}{2}\sum_{k=1}^{\infty} 2^k \left[\ell(\Gamma) - \ell(\Gamma_k)\right] - \ell(\Gamma).$$

Thus if $\mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ is bounded below, then

$$\sum_{k=1}^{\infty} 2^k \left[\ell(\Gamma) - \ell(\Gamma_k)\right] < \infty$$

with a bound independent of the choice of the dyadic decomposition. Hence finite renormalized area of X implies Γ is Weil-Petersson by Theorem 1.3. □

It is not hard to show that the dyadic dome has finite renormalized area if and only if the dyadic cylinder does, by considering a horizontal projection between the surfaces that changes hyperbolic area and lengths by at most a bounded additive factor. A very similar argument will be used in Section 22 to show that Weil-Petersson curves bound minimal surfaces with finite renormalized area, so we leave the details until then.

13. The β's and Menger curvature are equivalent: (11) $\Leftrightarrow$ (12)

In this section, we prove that Definitions 11 and 12 are equivalent. The estimates follow arguments contained in Pajot's book [102]; we will just indicate where to find them and how to modify the proofs given there.

We start with bounding Menger curvature using the β-numbers. This is similar to the proof of Theorem 31 of [102]. In our proof, we will take μ to be arclength measure on Γ; this satisfies the linear growth condition of Theorem 31 in [102] because Γ is chord-arc. Pajot defines

$$c^2(\mu) = \int_\Gamma \int_\Gamma \int_\Gamma c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z),$$

and on the bottom of page 37 he notes that

$$c^2(\mu) \leq 3\overline{c}^2(\mu), \tag{13.1}$$

where

$$\overline{c}^2(\mu) = \int_A c^2(x, y, z) d\mu(x) d\mu(y) d\mu(z),$$

and

$$A = \{(x, y, z) \in \Gamma \times \Gamma \times \Gamma : |x - z| \leq |x - y|, |y - z| \leq |x - y|\}.$$

He states that

$$\overline{c}^2(\mu) \leq \sum_Q \int_{(x,z)\in 3Q} \left(\sum_{R\subset Q} \int_{x,y\in \widetilde{R}} c^2(x, y, z) d\mu(y) \right) d\mu(x) d\mu(z),$$

where the inner sum is over dyadic sub-cubes $R \subset Q$ and where

$$\widetilde{R} = \{(x, y) \in 3R : |x - y| \geq \operatorname{diam}(R)/3\}.$$

Recall that $\ell(x, y, z) = |x-y|+|y-z|+|z-y|$ is defined as the perimeter of the triangle with vertices (x, y, z), and it is comparable to the longest of the three sides. Note that, for $(x, y, z) \in A$ and $(x, y) \in \widetilde{R}$, we have $\ell(x, y, z) \simeq |x - y| \simeq \operatorname{diam}(R)$. Thus we can replace (13.1) by the analogous estimate

$$\begin{aligned}
&\int_\Gamma \int_\Gamma \int_\Gamma \frac{c^2(x, y, z)}{\ell(x, y, z)} d\mu(x) d\mu(y) d\mu(z) \\
&\qquad \lesssim \sum_Q \int_{x,z\in 3Q} \left(\sum_{R\subset Q} \int_{x,y\in \widetilde{R}} \frac{c^2(x, y, z)}{\ell(x, y, z)} d\mu(y) \right) d\mu(x) d\mu(z) \\
&\qquad \simeq \sum_Q \int_{x,z\in 3Q} \left(\sum_{R\subset Q} \int_{x,y\in \widetilde{R}} \frac{c^2(x, y, z)}{\operatorname{diam}(R)} d\mu(y) \right) d\mu(x) d\mu(z).
\end{aligned}$$

We now follow the rest of the proof on page 38 of [102], replacing the factor $\mathrm{diam}(R)^{-2}$ that occurs throughout by $\mathrm{diam}(R)^{-3}$. At the end we obtain

$$\int_\Gamma \int_\Gamma \int_\Gamma \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x)d\mu(y)d\mu(z) \lesssim \sum_Q \beta^2(Q).$$

Thus Definition 11 implies Definition 12, as desired.

Next we deal with the opposite inequality: bounding $\sum \beta^2$ in terms of the Menger curvature. The relevant estimates are given in the proof of Theorem 38 of [102]. On the bottom of page 43, Pajot gives the inequality

$$\beta_\Gamma^2(Q)\mathrm{diam}(Q) \lesssim \sum_{P \subset Q} \int_{P^*}\int c^2(x,y,z)d\mu(x)d\mu(y)d\mu(z)\left(\frac{\mathrm{diam}(P)}{\mathrm{diam}(Q)}\right)^{1/2}, \tag{13.2}$$

where

$$P^* = \{(x,y,z) \in (3P)^3 : |x-y| \simeq |x-z| \simeq |y-z| \simeq \mathrm{diam}(P)\}.$$

Divide both sides of (13.2) by $\mathrm{diam}(Q)$, and note that for $(x,y,z) \in P^*$ we have $\ell(x,y,z) \simeq \mathrm{diam}(P)$. This gives

$$\begin{aligned}\beta_\Gamma^2(Q) &\lesssim \sum_{P \subset Q} \int_{3P}\int \frac{c^2(x,y,z)}{\mathrm{diam}(Q)} d\mu(x)d\mu(y)d\mu(z)\left(\frac{\mathrm{diam}(P)}{\mathrm{diam}(Q)}\right)^{1/2} \\ &\lesssim \sum_{P \subset Q} \int_{3P}\int \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x)d\mu(y)d\mu(z)\left(\frac{\mathrm{diam}(P)}{\mathrm{diam}(Q)}\right)^{1/2}\end{aligned}$$

On the top of page 44, this modified expression leads to

$$\sum_{S \subset Q} \beta_\Gamma^2(S) \lesssim \int_Q \int_Q \int_Q \frac{c^2(x,y,z)}{\ell(x,y,z)} d\mu(x)d\mu(y)d\mu(z).$$

Since $d\mu$ is arclength measure, this shows that Definition 12 implies Definition 11.

14. Reflections control β's: (13) $\Rightarrow$ (11)

Next we show that Definition 13 implies Definition 11; i.e., if Γ is the fixed point set of a involution R defined on a neighborhood U of Γ, and whose distortion satisfies certain L^2 estimates, then the β^2-sum for Γ is finite. We start by showing that such an involution is a biLipschitz map.

LEMMA 14.1. *A map $R : U \to U$ satisfying Definition* 13 *is biLipschitz on U.*

Proof. Choose a neighborhood $U' \subset U$ of Γ such that for any pair of points $z, w \in U'$ with $|z-w| \le 3\max(\mathrm{dist}(z,\Gamma), \mathrm{dist}(w,\Gamma))$ the segment between z and w is inside U. Fix two such points $z, w \in U'$. Without loss of generality, we may assume $\mathrm{dist}(z,\Gamma) \geq \mathrm{dist}(w,\Gamma)$. Let S be the segment between z and w. Then $R(S)$ is an arc connecting $R(z)$ and $R(w)$, so we must have $|R(z) - R(w)| \leq \ell(R(S))$. The segment S may hit Γ, but R is the identity at such points, and $S \setminus \Gamma$ consists of at most countably many open sub-segments, each covered by its intersection with Whitney cubes Q for $\mathbb{R}^n \setminus \Gamma$. The length of each such intersection is increased by at most a factor of $\rho(Q)$. Therefore,

$$|R(z) - R(w)| - |z - w| \lesssim \sum_{Q\cap S\neq\emptyset} \rho(Q)\mathrm{diam}(Q),$$

where the sum is over all Whitney cubes that hit S. By the Cauchy-Schwarz inequality, the right side of the equation above is less than

$$\begin{aligned} &\lesssim \left(\sum_{Q\cap S\neq\emptyset} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2} \left(\sum_{Q\cap S\neq\emptyset} \mathrm{diam}(Q)\right)^{1/2} \\ &\lesssim \left(\sum_{Q\cap S\neq\emptyset} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2} (\ell(S))^{1/2}. \end{aligned}$$

Let Q' be the minimal dyadic cube containing w with $\ell(Q') \geq 6\mathrm{dist}(z,\Gamma)$. By our assumptions on z and w, any dyadic cube Q in the above sum is a subcube of $3Q'$, and hence this sum is bounded by

$$\lesssim \left(\ell(S) \sum_{Q\subset 3Q'} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2},$$

where we sum over Whitney cubes inside $3Q'$. If we define

$$P(Q') = \left(\frac{1}{\mathrm{diam}(Q')} \sum_{Q\subset 3Q'} \rho^2(Q)\mathrm{diam}(Q)\right)^{1/2}, \tag{14.1}$$

then this estimate can be written more compactly as

$$|R(z) - R(w)| - |z - w| \lesssim P(Q')\mathrm{diam}(Q').$$

Since $P(Q') \leq (\sum_Q \rho^2(Q))^{1/2} < \infty$, we get $|R(z) - R(w)| = O(|z - w|)$ for all $z, w \in U$ with $|z - w| \leq 3\mathrm{dist}(z,\Gamma)$. Reversing the roles of z and w gives the same estimate when $|z - w| \leq 3\mathrm{dist}(w,\Gamma)$. When $|z - w| \geq 3\max(\mathrm{dist}(z,\Gamma), \mathrm{dist}(w,\Gamma))$, we can choose $z', w' \in \Gamma$, with $|z - z'| = \mathrm{dist}(z,\Gamma)$ and $|w - w'| = \mathrm{dist}(w,\Gamma)$. Since z' and w' are both fixed by R, we have

$$|R(z) - R(w)| \leq |R(z) - z'| + |z' - w'| + |w' - R(w)| \lesssim |z - w|.$$

Thus R is Lipschitz. Since $R = R^{-1}$ is an involution, it is automatically biLipschitz. $\square$

LEMMA 14.2. *Definition* 13 *implies Definition* 11.

Proof. For $n = 2$, it is clear that Definition 13 implies Definition 3, which in turn implies Definition 11 by using results from [17] and implications proven earlier in this paper. Thus we may assume $n \geq 3$.

First note that that if P is as defined in (14.1), then

$$\begin{aligned}\sum_{Q'} P^2(Q') &= \sum_{Q'} \sum_{Q \subset 3Q'} \rho^2(Q) \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \\ &= \sum_{Q} \rho^2(Q) \sum_{Q' : Q \subset 3Q'} \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \lesssim \sum_{Q} \rho^2(Q),\end{aligned}$$

since the sum over Q' only involves $O(1)$ cubes of each size. Thus it suffices to show that $\beta(Q') = O(P(Q'))$. Normalize so $\ell(Q') = 1$. Choose two points $p, q \in \Gamma \cap 3Q'$ with $|p - q| \simeq 1$, and let L be the line through p and q. Choose some $w \in \Gamma \cap 3Q'$ that maximizes the distance from L. Let $\beta = \operatorname{dist}(w, L)$. It suffices to show that $\beta = O(P(Q'))$. We may fix a large $M < \infty$ and assume that $P(Q') \leq 1/M^2$ and $MP(Q') \leq \beta \leq 1/M$, for otherwise there is nothing to do. We will show that this gives a contradiction if M is large enough, and hence that $\beta \leq M \cdot P(Q')$.

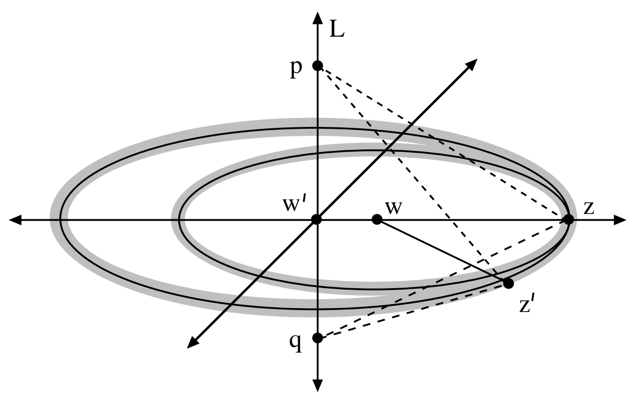

Figure 14. Proof that $\beta = O(P)$. The two points z, z' cannot be almost equidistant from p, q and w without their average being far from from L, contradicting how all these points were chosen.

Let w' be the closest point on L to w, and let z be the point on the ray from w' through w so that $\operatorname{dist}(z, L) = \frac{1}{2}\ell(Q')$. See Figure 14. Let Q be the Whitney square for $\mathbb{R}^n \setminus \Gamma$ containing z, and let $z' = R(z)$. Note that the p, q, w, w', z, z' all lie in a three dimensional subspace, so, without loss of generality, we may assume that L is the z-axis in $\mathbb{R}^3$, $w' = 0$, $w = (\beta, 0, 0)$, and $z = (1, 0, 0)$. The points p, q satisfy $|p| \simeq |q| \simeq |p - q| \simeq 1$. Since z and z'

are approximately the same distance from these two points, up to a factor of $O(P(Q'))$, we deduce that z' lies inside a $O(P(Q'))$-neighborhood of the circle $x^2 + y^2 = 1$ in the xy-plane. See Figure 14 again.

Similarly, since z and z' are equidistant from w, up to a factor of $O(P(Q'))$, the points z' lies within a $O(P(Q'))$ neighborhood of the sphere of radius $1 - \beta$ around z. However, since $P(Q') \ll \beta \ll 1$, these two regions only intersect in the half-space $\{x > 0\}$, and thus z' also lies in this half-space. Thus $q = (z + z')/2$ has x-coordinate $\geq 1/2$, and, by the definition of ρ, it is within $\rho(Q)$ of a point $q' \in \Gamma$. But $\rho(Q) \lesssim P(Q') \ll 1$, since it is one of the cubes in the sum defining $P(Q')$. This implies there is a point q' of Γ that is about unit distance from L, which contradicts the assumption that the maximum distance was $\beta \leq 1/M \ll 1$.

This contradiction implies $\beta(Q') \leq M \cdot P(Q')$, as desired, so we have proven that Definition 13 implies Definition 11. □

15. The β^2-sum is equivalent to the ε_Γ^2-sum: (11) ⇔ (14)

Recall from Section 4 that for a dyadic cube Q hitting Γ, $\varepsilon_\Gamma(Q)$ is the infimum of $\epsilon \in (0, 1]$ so that there is a line L hitting $3Q$ and a ball B of radius $\mathrm{diam}(Q)/\epsilon$, so that B attains its minimum distance $\leq \epsilon$ from L at a point $z \in Q$, and so that every rotation of B around L is disjoint from Γ. Thus Γ is "surrounded" by the solid torus obtained by rotating B around L.

Lemma 15.1. *Definition* 11 *is equivalent to Definition* 14.

Proof. It is easy to see that $\beta_\Gamma(Q) \lesssim \varepsilon_\Gamma(Q)$, so one direction is clear. It is also easy to find examples where $\beta_\Gamma(Q) = 0$, but $\varepsilon_\Gamma(Q) > 0$, so the opposite inequality does not hold for individual cubes. However, we will prove that $\varepsilon_\Gamma(Q)$ can be bounded by a weighted sum of $\beta_\Gamma(Q_k)$ over a sequence of cubes containing Q, and this estimate will imply that the sum of $\varepsilon_\Gamma^2(Q)$ over all dyadic cubes is bounded, if the corresponding sum of $\beta_\Gamma^2(Q)$ is bounded.

Fix $x \in \Gamma$ and a dyadic cube Q_0 containing x with $\mathrm{diam}(Q_0) \leq \mathrm{diam}(\Gamma)$. Renormalize so that $\mathrm{diam}(Q_0) = 1$. For $k \geq 1$, let Q_k be the unique dyadic cube containing Q_0 and satisfying $\mathrm{diam}(Q_k) = 2^k \mathrm{diam}(Q_0)$. Set

$$(15.1) \quad \epsilon = \epsilon(Q) = 2A \sum_{k=1}^{\infty} 2^{-k} \beta_\Gamma(Q_k) = 2A \sum_{Q' : Q \subset Q'} \beta_\Gamma(Q') \frac{\mathrm{diam}(Q)}{\mathrm{diam}(Q')},$$

where the constant $1 \leq A < \infty$ will be chosen later, independent of Q and Γ. We will prove that $\varepsilon_\Gamma(Q) \lesssim \epsilon$ (with a constant depending only on A).

By definition, $\varepsilon_\Gamma \in [0, 1]$, so if $\epsilon > 1/(20A) \geq 1/20$, there is nothing to do. Therefore, we may assume that $\epsilon \leq 1/(20A)$ is fairly small. Let L be a line through x that minimizes in the definition of $\beta_\Gamma(Q_0)$. Let $L^\perp$ be the perpendicular hyperplane through x, and let $z \in L^\perp$ be distance $1/\epsilon \geq 20$

from x. Let $B = B(z,r)$, where $r = (1/\epsilon) - \epsilon \geq 19$. In particular, $B \setminus 3Q_0$ is nonempty. We claim that B is disjoint from Γ (and so are all its rotations around L).

Note that $\mathrm{dist}(B,L) = \epsilon$. For $0 \leq m \leq N = \lfloor \log_2 \frac{1}{\epsilon} \rfloor$, simple trigonometry shows that $\mathrm{dist}(B \setminus 3Q_m, L) \geq C_1 \epsilon 2^{2m}$ (we can do the calculation in the plane generated by L and z; see Figure 15 and recall that $\mathrm{diam}(Q_0) = 1$). On the other hand, the distance between $\Gamma \cap 3Q_m$ and L is $\leq C_2 \sum_{k=0}^m \beta_\Gamma(Q_k) 2^k$, because the angle between the best approximating lines for Q_k and Q_{k+1} is $O(\beta_\Gamma(Q_{k+1}))$. Therefore, if for every $0 \leq m \leq N$ we have

$$\sum_{k=0}^{m} \beta_\Gamma(Q_k) 2^k < (C_1/C_2)\epsilon 2^{2m},$$

then B and $\Gamma \cap 2Q_N$ will be disjoint. Note that if we take $A = \frac{1}{2} C_2/C_1$, then

$$\begin{aligned} \max_{0 \leq m \leq N} 2^{-2m} \sum_{k=0}^{m} \beta_\Gamma(Q_k) 2^k &\leq \sum_{m=0}^{N} 2^{-2m} \sum_{k=0}^{m} \beta_\Gamma(Q_k) 2^k \\ \leq \sum_{k=0}^{N} \beta_\Gamma(Q_k) 2^k \sum_{m=k}^{N} 2^{-2m} &\leq \sum_{k=0}^{N} \beta_\Gamma(Q_k) 2^{-k} = \epsilon/(2A) = (C_1/C_2)\epsilon. \end{aligned}$$

Since this holds for every choice of z in $L^\perp$ such that $\mathrm{dist}(z,L) = 1/\epsilon$, we have proven that $\varepsilon_\Gamma(Q) \lesssim \epsilon$, as claimed.

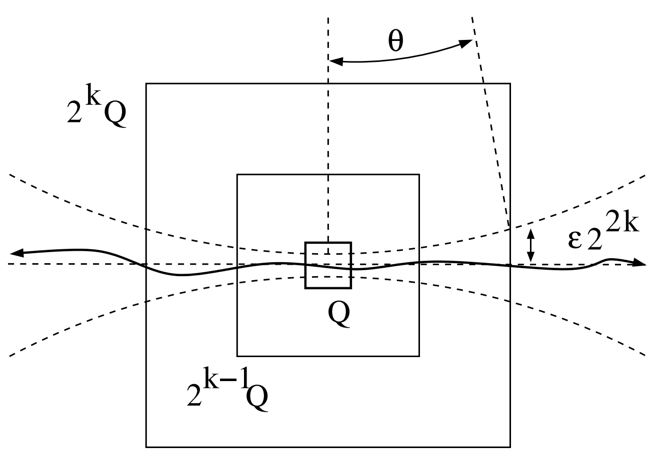

Figure 15. The part of the ball of radius $\mathrm{diam}(Q)/\varepsilon(Q)$ that lies in $2^k Q \setminus 2^{k-1} Q$ makes angle $\theta \simeq \varepsilon 2^k$ with the perpendicular ray from L to z and hence (since we are assuming $\mathrm{diam}(Q) = 1$), it is distance approximately $\varepsilon^{-1}(1 - \cos(\theta)) \simeq \varepsilon\theta^2 = \varepsilon 2^{2k}$ from the line L.

Summing the rightmost term of (15.1) over all dyadic cubes gives

$$\sum_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q') \frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')} \right]^2$$

$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma(Q') \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/4} \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/4} \right]^2,$$

and by Cauchy-Schwarz we get

$$\lesssim \sum_Q \left[\sum_{Q':Q\subset Q'} \beta_\Gamma^2(Q') \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{3/2} \right] \cdot \left[\sum_{Q':Q\subset Q'} \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(Q')}\right)^{1/2} \right].$$

The second term is dominated by a geometric series; hence it is bounded. Thus

$$\sum_Q \varepsilon_\Gamma^2(Q) \lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{Q:Q\subset Q'} \frac{\operatorname{diam}(Q)^{3/2}}{\operatorname{diam}(Q')^{3/2}}.$$

Since Definition 11 implies Γ is chord-arc, the number of dyadic cubes inside Q' of size $\operatorname{diam}(Q')2^{-k}$ and hitting Γ is at most $O(2^k)$. Thus the right side is bounded by

$$\lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{k=0}^\infty O(2^k) 2^{-3k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q') \sum_{k=0}^\infty 2^{-k/2} \lesssim \sum_{Q'} \beta_\Gamma^2(Q'),$$

and so the ε_Γ^2-sum is finite if the β_Γ^2-sum is finite, as desired. $\square$

It is sometimes convenient to assume that the balls in the definition of ε_Γ are small compared to $\operatorname{diam}(\Gamma)$. This is easy to obtain if we replace $\varepsilon_\Gamma(Q)$ by

$$\widetilde{\varepsilon}_\Gamma(Q) = \max\Big(\varepsilon_\Gamma, \big(\operatorname{diam}(Q)/\operatorname{diam}(\Gamma)\big)^\alpha\Big)$$

for some $1/2 < \alpha < 1$. Then only balls of diameter $d \lesssim \operatorname{diam}(\Gamma) \cdot \operatorname{diam}(Q)^{1-\alpha}$ are needed to bound $\widetilde{\varepsilon}_\Gamma(Q)$. Clearly $\epsilon_\Gamma(Q) \le \widetilde{\epsilon}_\Gamma(Q)$, and

$$\sum_{Q:Q\cap\Gamma\neq\emptyset} \widetilde{\varepsilon}_\Gamma^2(Q) \lesssim \sum_Q \varepsilon_\Gamma^2(Q) + \sum_Q \left(\frac{\operatorname{diam}(Q)}{\operatorname{diam}(\Gamma)}\right)^{2\alpha}$$

where the second sum is finite for chord-arc curves because $\alpha > 1/2$, and the number of dyadic squares of size $\simeq 2^{-n}$ hitting Γ is $O(2^n)$. Thus bounding $\sum \widetilde{\epsilon}_\Gamma^2$ is equivalent to bounding $\sum \epsilon_\Gamma^2$ for chord-arc curves. This is helpful when one wants to control $\epsilon_\Gamma(Q)$ in terms of more local behavior of Γ.

16. **The ε_Γ-sum is equivalent to the δ-integral: (14) $\Leftrightarrow$ (15)**

Recall that each ball $B \subset \mathbb{R}^n$ is the boundary of a hyperbolic half-space H in $\mathbb{H}^{n+1}$, and two balls are disjoint if and only if the corresponding half-spaces are disjoint.

LEMMA 16.1. *Suppose $B_1, B_2 \subset \mathbb{R}^n$ are disjoint balls of radius r that are distance ϵ apart. Then ρ, the hyperbolic distance between the corresponding half-spaces, satisfies $\rho \simeq \sqrt{\epsilon/r}$.*

Proof. The nearest points on the half-spaces will occur inside the copy of the hyperbolic upper half-plane lying above the line connecting the centers of B_1 and B_2. Thus it suffices to do this calculation in this plane; this is a simple calculus exercise. Using Möbius transformations, we can normalize so the balls both have radius 1, and the distance between them is $\eta = \epsilon/r$. The intersection of the hemispheres with this plane are two half-circles. At height t above $\mathbb{R}^2$, these circles are Euclidean distance $\eta + O(t^2)$ apart, hence hyperbolic distance $\rho \simeq t + \eta/t$ apart. This is minimized when $t = \sqrt{\eta} = \sqrt{\epsilon/r}$. $\square$

LEMMA 16.2. *If $z \in \mathrm{CH}(\Gamma)$, then $\delta(w) = O(\delta(z))$ for all $w \in \mathrm{CH}(\Gamma) \cap$ $\mathrm{B}_\rho(\mathrm{z}, 1)$.*

Proof. The point is that if H_1, H_2 are two disjoint hyperbolic half-planes that are both within distance δ of a point z, then their boundaries remain within distance $O(\delta)$ of each other inside $B_\rho(z, 1)$ (imagine $z = 0$ in the ball model). $\square$

LEMMA 16.3. *Definition* 14 *implies Definition* 15.

Proof. Lemmas 16.1 and 16.2 imply that if $\varepsilon_\Gamma(Q)$ is small (say less than $1/100$), then $\delta(z) \lesssim \varepsilon_\Gamma(Q)$ for every point $z \in T(Q) = Q \times [\ell(Q)/2, \ell(Q)]$ $\subset \mathbb{H}^{n+1}$. This gives Definition 15. $\square$

In the higher dimensional version of Definition 15, we can either sum $\delta^2(Q)$, or we can integrate δ^2 over any surface with $\mathrm{A}_\rho(S \cap T(Q)) = O(1)$, e.g., the dyadic dome of Γ, or a smoothed version of the dyadic dome, or a minimal surface with asymptotic boundary Γ, or (in the case $n = 2$) a boundary component of the hyperbolic convex hull of Γ.

LEMMA 16.4. *Definition* 15 *implies Definition* 14.

Proof. If $\Gamma \subset \mathbb{R}^2$ is a circle, then $\delta(z)$ vanishes everywhere, but ε_Γ does not. Thus $\varepsilon_\Gamma(Q)$ cannot be bounded using δ alone; there must also be some dependence on the size of Q. Without loss of generality, we assume $\mathrm{diam}(\Gamma) = 1$, $\mathrm{diam}(Q) \le 1/100$, and $\delta(z) < 1/100$ for z in $\mathrm{CH}(\Gamma) \cap \mathrm{T(Q)}$, where we define $T(Q) \subset \mathbb{H}^{n+1}$ as the points that project vertically into Q, and have height between $\ell(Q)/2$ and $\ell(Q)$. For $z \in T(Q)$ and each normal direction at z that is perpendicular to an optimal plane in the definition of $\delta(z)$, there are a pair of disjoint hyperbolic half-spaces H and H^* connected by a geodesic segment of hyperbolic length at most $O(\delta)$ running through z and perpendicular to each half-space. These half-spaces intersect $\mathbb{R}^n$ in disjoint regions B, B^* that do not hit Γ and are bounded by spheres.

If both regions are bounded balls, they are separated by a $(n-1)$-plane, which extends to a vertical n-plane in $\mathbb{H}^{n+1}$ which separates the hyperbolic half-spaces and thus comes within $O(\delta)$ of the point z. This implies B, B^* each have radius $\gtrsim \delta(z) \cdot \text{diam}(Q)$. Otherwise one region, say B, is a bounded ball and the other region B^* is the exterior of a ball. Since B^* does not hit Γ, its boundary sphere must have diameter ≥ 1, and therefore it makes angle of at most $O(\text{diam}(Q))$ with the vertical near z. Since the other half-space H is also within $O(\delta)$ of z, it makes an angle of at most $\theta = O(\delta(z)) + O(\text{diam}(Q))$ with the vertical, and hence B has radius $r \gtrsim \text{diam}(Q)/(\delta(z) + \text{diam}(Q))$.

In either case, we have $\varepsilon_\Gamma^2(Q) = O(\delta^2(z)) + O(\text{diam}^2(Q))$. The δ^2-sum is bounded by assumption. This assumption also implies that given $\delta_0 = 2^{-m} > 0$, all but finitely many terms of the δ sum are less than δ_0. Assume we are at a scale below which all cubes satisfy this. Given such a cube Q, $\Gamma \cap 3Q$ can hit only $O(1/\delta_0)$ sub-dyadic-cubes of $3Q$ of size $\delta_0 \text{diam}(Q)$. Iterating, we see that Γ hits at most $O(C^k \delta^k) = O(2^{(m+\log_2 C)k})$ dyadic cubes of size $s \gtrsim 2^{-mk}$. Thus

$$\sum_{k=0}^{\infty} \sum_{\substack{Q: 2^{-m(k+1)} \\ < \ell(Q) \leq 2^{-mk}, \\ Q \cap \Gamma \neq \emptyset}} \text{diam}^2(Q) \leq \sum_k 2^{(m+\log_2 C - 2m)k} < \infty$$

if $m > \log_2 C$, which occurs if δ_0 is small enough. This proves the lemma. □

17. The δ-integral controls surface curvature: (15) ⇒ (16)

LEMMA 17.1. *Definition* 15 *implies Definition* 16.

Proof. For dimensions $n = 2$ and higher we create a triangulated surface where adjacent triangles are very close to parallel, and we smooth this surface to obtain a surface with small principle curvatures. In dimensions higher than 2, the discrete surface can be the dyadic dome, introduced in Section 12, and the principle curvatures of a smoothed version are controlled by the β-numbers. In the special case $n = 2$, we can also use a discretization of the usual hyperbolic dome of one side of Γ. Since this case is of particular interest, we describe it first.

Suppose S is one component of $\partial\text{CH}(\Gamma)$. It is known that S, with its hyperbolic path metric, is isomorphic to the hyperbolic disk (see, for example,

[44], [86], and [85]). The hyperbolic unit disk can be triangulated by geodesic triangles with hyperbolic diameters $d \simeq 1$ and angles bounded strictly between 0 and π; e.g., take the tesselation corresponding to a Fuchsian triangle group.

Fix such a triangulation of $\mathbb{D}$ and map the vertices to S via the isometry. Each triple of image vertices corresponding to a triangle on $\mathbb{D}$ lies on a hyperbolic plane and determines a triangle on this plane. Create a new surface S_1 by gluing these triangles together along their edges. Because the vertices lie in $\mathrm{CH}(\Gamma)$, convexity implies that each triangle, and hence all of S_1, also lies in $\mathrm{CH}(\Gamma)$.

Consider two triangles T_1, T_2 in S_1 that meet along a common edge e. Normalize so that one endpoint of e is the origin in the ball model of hyperbolic 3-space, e lies along the x axis, and T_1 lies in the xy-plane. Then T_2 lies in Euclidean plane that makes some angle θ with the xy-plane, and by our assumptions, it contains a point p (e.g., the vertex of T_2 not on e) that is hyperbolic distance $\rho \simeq 1$ from 0 and Euclidean distance $d \simeq 1$ from the x-axis. Then p is Euclidean distance $r \simeq \theta$ from the xy-plane. Because both triangles lie inside $\mathrm{CH}(\Gamma)$ and, since $\mathrm{CH}(\Gamma)$ is trapped between two hyperbolic half-planes that each come within hyperbolic distance $\delta(0)$ of the origin, we must have $\theta \lesssim \delta(0)$ (we are using Lemma 16.2).

If T is component triangle of S_1, let $\theta(T)$ be the maximum angle T makes with any of its neighboring triangles, and think of $\theta(z)$ as a function on S_1 that is constant on triangles. Since $\theta(z)$ can be bounded by a uniform multiple of $\delta(w)$ for a point w that is a uniform hyperbolic distance away, we get

$$\int_{S_1} \theta^2(z) dA_\rho(z) \lesssim \int_{S_1} \delta^2(z) dA_\rho(z) < \infty.$$

The principle curvatures of S_1 are zero inside each triangle and are a multiple of arclength measure along the edges. However, by smoothing S_1 we can obtain a surface S_2 such that the principle curvatures of S_2 tend to zero as we approach infinity and are bounded by $O(\max_{T^*} \theta(z))$, where T^* denotes the union of all component triangles that touch T (including those that only touch at a vertex). Then

$$\int_{S_2} |K+1|^2(z) dA_\rho(z) \lesssim \int_{S_1} \delta^2(z) dA_\rho(z) < \infty.$$

For $n \geq 2$, essentially the same proof works if we take the dyadic dome for our triangulated surface with asymptotic boundary Γ. The angles between adjacent faces are easily bounded by the β-numbers of the corresponding arcs of Γ, which, after smoothing, proves that Definition 11 implies Definition 16. □

18. Surface curvature bounds QC reflections: (16) $\Rightarrow$ (3)

LEMMA 18.1. *For* $n = 2$, *Definition* 16 *implies Definition* 3.

Proof. For $n = 2$, this implication is due to Charles Epstein [42]. He proves that for a surface $S \subset \mathbb{H}^3$ whose principle curvatures $|\kappa_1(p)|, |\kappa_2(p)|$ are bounded strictly below 1, the Gauss map from the surface to the plane at infinity is quasiconformal. Recall that the Gauss map sends a point p on S to the endpoint on $\mathbb{R}^2$ of the hyperbolic geodesic ray starting at p that is normal to S. There are actually two Gauss maps from S to $\mathbb{R}^2$ depending on which "side" of S the geodesic ray is on. In the case when the surface has asymptotic limit $\Gamma \subset \mathbb{R}^2$, the composition of one of these maps with the inverse of the other defines a quasiconformal reflection across Γ. By Proposition 5.1 of [42], the dilatation of the composed Gauss maps is

$$\begin{aligned} D(z) &= \max\left(\left| \frac{1+\kappa_1(p)}{1-\kappa_1(p)} \cdot \frac{1-\kappa_2(p)}{1+\kappa_2(p)} \right|^{1/2}, \left| \frac{1-\kappa_1(p)}{1+\kappa_1(p)} \cdot \frac{1+\kappa_2(p)}{1-\kappa_2(p)} \right|^{1/2} \right) \\ &= 1 + O\big(|\kappa_1(p)| + |\kappa_2(p)|\big), \end{aligned}$$

where $p \in S$ is the point corresponding to $z \in \mathbb{R}^2$. Therefore the dilatation satisfies

$$|\mu(z)| = O(|\kappa_1(p)| + |\kappa_2(p)|).$$

Moreover, on page 121 of [42], Epstein shows that the Jacobian J of this map satisfies

$$C_1\big|(1 \mp \kappa_1)(1 \mp \kappa_2)\big| \leq J \leq C_2\big|(1 \pm \kappa_1)(1 \pm \kappa_2)\big|.$$

In particular, $J \simeq 1$ if $|\kappa_1|, |\kappa_2|$ are both uniformly bounded below 1.

Definition 16 implies that κ_1, κ_2 are both small outside some compact ball B around the origin. Thus the Gauss map for S defines a quasiconformal reflection in some neighborhood U of Γ and inside this neighborhood

$$\int_U \big|\mu(z)\big|^2 d\mathrm{A}_\rho(z) \lesssim \int_{S \setminus B} \big|\mathcal{K}_0(z)\big|^2 d\mathrm{A}_\rho(z),$$

where $d\mathrm{A}_\rho$ in the left-hand and right-hand sides is the hyperbolic area measure on $\mathbb{R}^2 \setminus \Gamma$ and S, respectively, and $\mathcal{K}_0$ is the trace-free second fundamental form of S. Extend this reflection to the rest of $\mathbb{R}^2$ by some diffeomorphism of one component of $\mathbb{R}^2 \setminus U$ to the other that agrees with the reflection given by the Gauss map on ∂U. This gives a global quasiconformal reflection across Γ that satisfies (2.10), as desired. $\square$

Next we consider higher dimensions.

LEMMA 18.2. *For* $n \geq 2$, *Definition* 16 *implies Definition* 13.

Proof. We consider only points z on S that are at height not exceeding t_0 above $\mathbb{R}^n$ where t_0 is chosen so small that if $z = (x, t) \in \mathbb{R} \times (0, t_0)$, then the principle curvatures of S at z are all very small, say $\leq 1/100$. There is an $(n-1)$-sphere of directions in the tangent space of $\mathbb{H}^{n+1}$ at z that are perpendicular to S. These directions define a tangent $(n-1)$-dimensional hyperbolic hyperplane H_z that passes through z, and the boundary of H_z on $\mathbb{R}^n$ is a Euclidean $(n-2)$-sphere S_z whose center is within $O(t \cdot \sup_w \max_j |\kappa_j(w)|)$ of the point x (the vertical projection of z onto $\mathbb{R}^n$). We define R on this sphere by taking the antipodal map.

We claim that such spheres foliate a neighborhood U of Γ and that R is Lipschitz. If so, then R is a biLipschitz involution that fixes Γ. Let $K_r = K_r(z)$ be an upper bound for $\max |\kappa_j|$ in a hyperbolic r-ball around z. Given $z, w \in S$ that are $t < r$ apart in the hyperbolic metric, let γ be the geodesic segment in $\mathbb{H}^{n+1}$ connecting them. The perpendicular hyperplanes H_z, H_w are both within $O(K_r)$ of being orthogonal to γ. Hence the corresponding spheres S_z, S_w are within $O(K_r \cdot t)$ of each other, but they are also at least distance $\gtrsim K \cdot t$ apart (this is easiest to see in the ball model of hyperbolic space, setting $z = 0 \in \mathbb{B}^{n+1}$). Thus using the antipodal map on each boundary sphere preserves the distance between points on the same sphere, and it increases the distance between points on different spheres by at most $O(K \cdot t)$. Thus R is Lipschitz, as desired. Moreover, if two such spheres intersect the same Whitney cube Q of $\mathbb{R}^n \setminus \Gamma$, then both have radii $r \simeq \ell(Q)$, and their centers are within $O(\ell(Q))$ of each other. Thus the corresponding points on S are within hyperbolic distance $O(1)$ of each other. The argument above implies that $\rho(Q) = O(K_r(z))$ for some point $z \in S$, and thus $\sum_Q \rho^2(Q)$ is finite if $\int_S |K_r(z)|^2$ is. Hence Definition 16 implies Definition 13. $\square$

19. Minimal surfaces with finite total curvature ($n = 2$): (15) $\Rightarrow$ (17)

We already know that Γ is Weil-Petersson if and only if it is the boundary of some surface in $\mathbb{H}^{n+1}$ that is asymptotically flat and has finite total curvature. Next we prove this surface can be taken to be minimal if $n = 2$. First, we need to know that a minimal surface that is trapped between parallel planes has small principle curvatures. This is obvious for minimal surfaces in $\mathbb{R}^n$ because they can be parameterzed by harmonic functions, and in hyperbolic 3-space, the corresponding estimate is due to Andrea Seppi [115]:

LEMMA 19.1. *Suppose S is an embedded minimal disk in $\mathbb{B}^3$ that has an asymptotic bounding quasicircle $\Gamma \subset \mathbb{S}^2$. Suppose that $0 \in S$ and that S lies between two disjoint hyperbolic planes that both lie at most distance ϵ from 0, on opposite sides of the xy-plane. Then the tangent plane of S at 0 makes angle at most $O(\epsilon)$ with the xy-plane, and the absolute values of the principle curvatures of S at 0 are both bounded by $O(\epsilon)$.*

This is essentially Propositions 4.14 and 4.15 of [115]; see Equation (32) in particular. Given a minimal surface S that is trapped between two hyperbolic planes P_-, P_+, Seppi considers the function $u(z) = \sinh(\text{dist}(z, P_-))$ for $z \in S$ and uses the fact that this function satisfies the equation $\Delta_S u - 2u = 0$, where Δ_S is the Laplace-Beltrami operator for the surface S. The Schauder estimates for this equation imply that

$$\|u\|_{C^2(B(x,r/2))} \leq C\|u\|_{C^0(B(x,r))}.$$

In order to get a uniform bound for C, we must bound the curvature of S, and Seppi gives an argument for this assuming the boundary of S is a quasicircle (this covers our application, since Weil-Petersson curves are quasicircles). Finally, the sup norm of u is bounded in terms of the distance between P_- and P_+ near z. In our case, this distance is $O(\delta(z))$ by Lemma 16.2. One small technical point is that Seppi requires the point z to be on a geodesic segment that meets both P_- and P_+ orthogonally. However, it is very simple to see that if z is between two disjoint hyperbolic planes that each come within ϵ of z, then there are also two disjoint planes that come within $O(\epsilon)$ and satisfy the orthogonality condition for z.

The lemma implies that near the boundary of hyperbolic space we have

$$\int_S \left(\kappa_1^2(z) + \kappa_2^2(z)\right) d\mathrm{A}_\rho \lesssim \int_{\partial\mathrm{CH}(\Gamma)} \delta^2(z) d\mathrm{A}_\rho < \infty$$

when Γ is Weil-Petersson. Thus for $n = 2$, Definition 15 implies Definition 17.

20. Renormalized area ($n = 2$): (15) ⇒ (19)

As we discussed in Section 1, a 2-surface $S \subset \mathbb{H}^{n+1}$ with boundary curve $\Gamma \subset \mathbb{R}^n$ is said to have finite renormalized area if

$$\mathcal{RA}(S) = \lim_{t \searrow 0} \left[\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t)\right]$$

exists and is finite, where

$$S_t = \{(x, y, s) \in S : s \geq t\}, \quad \partial S_t = \{(x, y, s) \in S : s = t\}.$$

Lemma 20.1. *For $n = 2$, Definition 15 implies Definition 19.*

Proof. Using the Gauss-Bonnet theorem (in the third equality), we have

$$\begin{aligned}
\mathrm{A}_\rho(S_t) - L_\rho(\partial S_t) &= \int_{S_t} 1 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= \int_{S_t} (1 + \kappa^2) d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho \\
&= -\int_{S_t} K d\mathrm{A}_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho
\end{aligned}$$

$$= -2\pi\chi(S_t) + \int_{\partial S_t} \kappa_g dL_\rho - \int_{S_t} \kappa^2 d\mathrm{A}_\rho - \int_{\partial S_t} 1 dL_\rho$$
$$= -2\pi\chi(S_t) - \int_{S_t} \kappa^2 d\mathrm{A}_\rho + \int_{\partial S_t} (\kappa_g - 1) dL_\rho,$$

where κ_g is the geodesic curvature of ∂S_t in S_t. Since we are assuming that Definition 15 holds, we know from earlier results that the β's tend to zero, and this implies that near the boundary any minimal surface is nearly vertical (trapped between nearly touching hyperbolic planes), and therefore the surface has finite Euler characteristic.

The estimate of Seppi discussed in Section 19 shows that

$$\int_{S_t} \kappa^2 d\mathrm{A}_\rho = O(\int_{S_t} \delta^2 d\mathrm{A}_\rho).$$

Since Γ is Weil-Petersson, our earlier results imply that this area integral converges to a finite limit as $t \searrow 0$.

Therefore it suffices to show that the boundary integral of $\kappa_g - 1$ tends to zero as $t \searrow 0$. The geodesic curvature κ_g of the boundary curve comes from two components. There is a vertical component of size 1 due to the curve lying on the horizontal plane. There is a horizontal component due to the curvature of ∂S_t in this plane. This component has size bounded by the principle curvatures of the surface, that by Seppi's estimate are bounded by $O(\delta(z))$. The geodesic curvature κ_g is given by projecting this vector onto the tangent space of S_t, and our previous estimates show this tangent space makes an angle at most $O(\delta)$ with the vertical. Thus $|\kappa_g| = 1 + O(\delta^2)$. Hence

$$\int_{\partial S_t} (\kappa_g - 1) ds = O\left(\int_{\partial S_t} \delta^2(z) ds\right). \tag{20.1}$$

Note that, since δ^2 has finite integral over the whole surface, its integral over the annulus $A_t = S_t \setminus S_{t+1}$ tends to zero with t. Moreover, Lemma 16.2 implies that the integral of $\delta^2(z)$ over ∂S_t is dominated by a multiple of the area integral over A_t and hence the boundary integral in (20.1) must tend to zero. This proves the lemma (and also shows that the formula (1.6) holds.) □

The estimate $|\kappa_g| = 1 + O(\delta^2)$ also follows from equation (2.4) of [34]:

$$\kappa_g = \frac{1}{\nabla r}\big(\coth r + \langle \mathcal{K}(e, e), \nabla^\perp r\rangle\big),$$

where r is the hyperbolic distance to some fixed point (say the origin in the ball model), Dr is the gradient of r in $\mathbb{H}^{n+1}$, ∇r is the projection of Dr onto the tangent space of S, $\nabla^\perp r$ is the projection of Dr onto the normal space of S, and $\mathcal{K}$ is the second fundamental form of S.

Seppi's paper [115] is written for minimal 2-surfaces in $\mathbb{H}^3$; extending his bound on principle curvatures to surfaces in $\mathbb{H}^{n+1}$ would extend the lemma to this case since the rest of this argument is valid in higher dimensions. My reading of his paper indicates such an extension is true, but the details have not been written down, so far as I know.

21. Isoperimetric inequalities: (18) ⇒ (17)

Suppose that $S \subset \mathbb{H}^{n+1}$ is a minimal surface with asymptotic boundary curve Γ in $\mathbb{R}^n$. As before, for $t > 0$ let $S_t = S \cap \{(x,s) \in \mathbb{R}^n \times (t,\infty)\}$ be the part of S above height t and let $S_t^* = S \setminus S_t$ be the part below height t. We assume that for t small enough, S_t^* is real analytic and a topological annulus. Suppose $\Omega \subset S_t^*$ is a compact subannulus with one boundary component equal to $\Gamma_t = S(\cap\mathbb{R}^n \times \{t\})$, and the other boundary component is a smooth curve $\Gamma(0)$. Let $T = T(\Omega)$ be the distance in S between $\Gamma(0)$ and Γ_t. For $0 \leq s \leq T$, let

$$\Omega(s) = \{z \in \Omega : d_S(z, \Gamma(0)) > s\}, \quad \Gamma(s) = \{z \in \Omega : d_S(z,\Gamma(0)) = s\}.$$

Here d_S refers to distance on the surface S. Note that $\Omega(0) = \Omega$. Also note that $\chi(\Omega) = 0$ (it is an annulus) and $\chi(\Omega(s)) \geq 0$ since $\Omega(s)$ is the union of a topological annulus and possibly some disks. Let $A(s)$ be the hyperbolic area of $\Omega(s)$, and let $L(s)$ be the hyperbolic length of $\Gamma(s) = \partial\Omega(s) \setminus \Gamma_t$. In particular, $A(0) = \mathrm{A}_\rho(\Omega)$ and $L(0) = L_\rho(\Gamma)$. The Gauss-Bonnet theorem says that

$$\int_{\Omega(s)} K d\mathrm{A}_\rho + \int_{\partial\Omega(s)} \kappa_g dL_\rho = 2\pi\chi\big(\Omega(s)\big),$$

where κ_g is the geodesic curvature of $\partial\Omega$ in Ω. For points in $\Gamma_t \subset \partial\Omega$, this is the negative of κ_g^S, the geodesic curvature of Γ_t in S_t. Since $\partial\Omega(s) = \Gamma_t \cup \Gamma(s)$ and $\chi(\Omega(s)) \geq 0$, we get

$$-\int_{\Gamma(s)} \kappa_g dL_\rho \leq \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho. \tag{21.1}$$

Lemma 21.1. *Suppose* $\frac{2}{T} < \epsilon \leq 1$. *With notation as above,*

$$L_\rho(\partial\Omega) - \mathrm{A}_\rho(\Omega) \geq -C(S,t) + (1-\epsilon)\int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho,$$

where

$$C(S,t) = \max\left(\int_{\Gamma_t} \kappa_g^S dL_\rho, L_\rho(\Gamma_t)\right).$$

Proof. This follows from known facts about the isoperimetric inequality on negatively curved surfaces. Our presentation follows that of Issac Chavel and Edgar Feldman [33], although they attribute the basic facts to Félix Faila [52].

As shown in [52], the function $A(s)$ is continuously differentiable and decreasing on $[0,T]$, and $A'(s) = -L(s)$ (Theorem 5 of [52]). Similarly, by Theorem 3 of [52], $L(s)$ is continuous on $[0,T]$, analytic except for finitely many points, and (except for these points)

$$L'(s) \le -\int_{\Gamma(s)} \kappa_g dL_\rho.$$

Using (21.1), we get

$$L'(s) \le \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho. \tag{21.2}$$

Thus

$$\begin{aligned} L'(s) - A'(s) &\le \int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega(s)} K d\mathrm{A}_\rho + L(s) \\ &= \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(s)} (1+\kappa^2) d\mathrm{A}_\rho + L(s), \end{aligned}$$

which implies

$$L'(s) - A'(s) \le L(s) - A(s) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega_s} \kappa^2 d\mathrm{A}_\rho. \tag{21.3}$$

By the isoperimetric inequality for surfaces with $K \le -1$ (e.g., equation (4.30) of [101]), we have

$$L_\rho\big(\partial\Omega(s)\big)^2 = \big(L(s) + L_\rho(\Gamma_t)\big)^2 \ge 4\pi\chi(\Omega_s)A(s) + A(s)^2,$$

and this implies

$$L(s) - A(s) \ge \frac{4\pi\chi(\Omega_s)A(s)}{L(s) + L_\rho(\Gamma_t) + A(s)} - L_\rho(\Gamma_t) \ge -L_\rho(\Gamma_t) \tag{21.4}$$

since $\chi(\Omega_s) \ge 0$. Assume for the moment that

$$L(0) - A(0) \le -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{21.5}$$

Then we claim there must be a $s \in [0, 1/\epsilon]$ such that

$$L'(s) - A'(s) \ge -\epsilon \int_{\Omega_{1/\epsilon}} \kappa^2 d\mathrm{A}_\rho. \tag{21.6}$$

If not, then, by integrating and using (21.5), we get

$$\begin{aligned} L(\frac{1}{\epsilon}) - A(\frac{1}{\epsilon}) &= L(0) - A(0) + \int_0^{1/\epsilon} L'(x) - A'(x) dx \\ &< -L_\rho(\Gamma_t) + \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho + \frac{1}{\epsilon}\left[-\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho\right] = -L_\rho(\Gamma_t), \end{aligned}$$

which contradicts (21.4) for $s = 1/\epsilon$, proving that there is at least one such point s.

Let a be the infimum of values s where (21.6) holds. Since we have assumed that κ is not constant zero, the right side of (21.6) is negative if ϵ is small enough. Thus on $[0,a]$ the function $L(s) - A(s)$ has a negative derivative except for finitely many points. Therefore $L(a) - A(a) \le L(0) - A(0)$. Using (21.6) and (21.3) with $s = a$, we get

$$\begin{aligned} -\epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho &\le L'(a) - A'(a) \\ &\le L(a) - A(a) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \\ &\le L(0) - A(0) + \int_{\Gamma_t} \kappa_g dL_\rho - \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \end{aligned}$$

This implies the following:

$$L(0) - A(0) \ge -\int_{\Gamma_t} \kappa_g dL_\rho + \int_{\Omega_t} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho.$$

Now, since $0 \le a \le 1/\epsilon$, we have $\Omega(1/\epsilon) \subset \Omega(a)$, so

$$\int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho - \epsilon \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho \ge (1-\epsilon) \int_{\Omega(a)} \kappa^2 d\mathrm{A}_\rho \ge (1-\epsilon) \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho,$$

and hence

$$L(0) - A(0) \ge -\int_{\Gamma_t} \kappa_g dL_\rho + (1-\epsilon) \int_{\Omega(1/\epsilon)} \kappa^2 d\mathrm{A}_\rho. \tag{21.7}$$

Thus either (21.5) fails or (21.7) holds. In either case, we have proven the lemma. □

LEMMA 21.2. *Definition* 18 *implies* 17.

Proof. Fix a point $z \in S$ and a large disk $D = D(z, R)$ around z. For n large enough, Ω_n contains $D(z, 2R)$, and so $\Omega_n(R)$ contains $D(z, R)$. So if R is large enough, κ is as small as we wish in $\Omega_n^*(R) = \Omega_n \setminus \Omega_n(R)$. Lemma 21.1 with $\epsilon = 1/2$ then implies

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho \le 2C(S,t) + 2\big[L_\rho(\partial\Omega_n) - \mathrm{A}_\rho(\Omega_n)\big].$$

The first term on the right is independent of n, and Definition 18 says the second term is bounded independently of n. Therefore

$$\int_{D(z,R)} \kappa^2 d\mathrm{A}_\rho = O(1),$$

with a bound independent of R. Taking $R \nearrow \infty$ and applying the monotone convergence theorem shows $\int_{S_t^*} \kappa^2 d\mathrm{A}_\rho < \infty$, as desired. □

Proof of Corollary 1.8. The inequality

$$\mathcal{RA}(S) \le \sup\{\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)\}$$

is obvious since the truncated surfaces in the definition of $\mathcal{RA}(S)$ are among the domains used in the supremum on the right.

To prove the other direction, note that if $D(z,R) \subset \Omega$, then $\chi(S) = \chi(\Omega) = \chi(\Omega_t)$ for all $0 \le t \le T/2$ if R is large enough. Then by Lemma 21.1

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \le -(2\pi \mp \epsilon)\chi\big(D(z,R/2)\big) - (1-\epsilon)\int_{D(z,R/2)} \kappa^2 d\mathrm{A}_\rho.$$

Taking $R \nearrow \infty$, and applying the monotone convergence theorem, we get

$$\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega) \le -(2\pi \mp \epsilon)\chi(\Omega) - (1-\epsilon)\int_S \kappa^2 d\mathrm{A}_\rho.$$

Then taking $\epsilon \searrow 0$ gives

$$\limsup_{R\nearrow\infty} \sup_{\Omega:\Omega\supset D(z,R)} [\mathrm{A}_\rho(\Omega) - L_\rho(\partial\Omega)] \le -2\pi\chi(\Omega) - \int_S \kappa^2 d\mathrm{A}_\rho. \qquad \square$$

22. From dyadic domes to renormalized area: (20) ⇒ (19)

In Section 20 we showed that Definition 15 ($\delta \in L^2$) implies Definition 19 ($\mathcal{RA} < \infty$) for planar curves by using a result of Seppi [115] that bounds the principle curvatures at a point z of a minimal surface in terms of $\delta(z)$, the local thickness of the hyperbolic convex hull of Γ. Seppi's proof is written for curves in $\mathbb{R}^2$ and surfaces in $\mathbb{H}^3$, but it seems very likely that his estimate remains valid for curves in $\mathbb{R}^n$ and minimal currents or chains in $\mathbb{H}^{n+1}$. However, since we lack an explicit reference for this extension, we provide an alternative approach for the higher dimensional case. We will show that Definition 19 follows from Definition 20 using a result from Seppi's paper [115], which does easily extend to higher dimensions.

We recall from the discussion of minimal currents and 2-chains in Section 6 that, if Γ satisfies Definition 11, then it is the asymptotic boundary of a minimal 2-chain whose restriction to $\mathbb{H}^{n+1}_t = \{(x,s) \in \mathbb{H}^{n+1} : s < t\}$ agrees with a minimal surface S that is a topological annulus, that has one boundary component on $\mathbb{R}^n \times \{t\}$, and that has asymptotic boundary Γ. Lemma 1.4 in Lin's paper [83] shows that on a unit hyperbolic neighborhood of any point $z \in S$, S is a Lipschitz graph with respect to a vertical 2-plane with Lipschitz constant $o(1)$; i.e., it tends to 0 as $t \searrow 0$. In particular, the path metric on S is comparable to the ambient metric with constant tending to 1 as $t \searrow 0$. We want to show that this Lipschitz constant, near $z = (x,t) \in S$, is bounded by $O(\varepsilon_\Gamma(Q))$, where ε_Γ is as in Definition 14 and $Q \subset \mathbb{R}^n$ is the dyadic cube containing x and with $t < \ell(Q) \le 2t$.

Suppose P is a n-dimensional geodesic plane in $\mathbb{H}^{n+1}$, and suppose $u(z) = \sinh d_{\mathbb{H}}(z, P)$, where $d_{\mathbb{H}}$ denotes the signed distance in $\mathbb{H}^{n+1}$ from z to P. Then Proposition 2.4 in [115] proves for $n = 2$ that

$$\Delta_S u = 2u, \tag{22.1}$$

where $\Delta_S = \text{trace}(\nabla_v^S u)$ is the Laplace-Beltrami operator on S. The same proof works in higher codimension, except that certain terms that give projections onto the normal vector to S are replaced by the projection into the (multi-dimensional) space of normal vectors.

Definition 14 says that if $z \in S$ and Q are as above, then S is trapped between disjoint half-spaces that are at most $O(\varepsilon_\Gamma(Q))$ apart (in the hyperbolic metric) and are separated by a vertical n-plane. Thus, as explained in [115], the Schauder estimates for elliptic PDE imply that $|\nabla u(z)| = O(\varepsilon_\Gamma(Q))$. The same estimate holds for $(n-1)$ mutually orthogonal choices of hyperplanes P passing through z that are also orthogonal to the vertical direction and to the direction locally parallel to Γ. Since the distance function to each of these on S is Lipschitz with constant $O(\varepsilon_\Gamma(Q))$, we see that S can be parameterzed by a Lipschitz function on the $(n-1)$-plane normal to the vertical 2-plane. The Schauder estimates also require that we have uniform bounds on the curvature of S, but this is standard and explained in [115].

An alternative approach that avoids using the Schauder estimates is to consider a conformal map φ from the unit disk into a neighborhood of the point z on S. By standard potential theory on the disk, Equation (22.1) implies that $u \circ \varphi$ can be written as the sum of a harmonic function U bounded by $O(\varepsilon_\Gamma(Q))$ and the convolution V of $\log 1/|z|$ against a function bounded by $O(\delta)$. On a strictly smaller disk, $|\nabla U|$ is bounded by $O(\varepsilon_\Gamma(Q))$ by Harnack's inequality, and $|\nabla V|$ satisfies the same estimate because the gradient is given by convolution of $1/z$, which is in $L^1(dxdy)$, with a function bounded by $O(\varepsilon_\Gamma(Q))$. Combined with Lin's estimate showing the intrinsic path metric and ambient metrics are comparable, this gives an alternate proof that u restricted to S is Lipschitz with constant $O(\varepsilon_\Gamma(Q))$.

LEMMA 22.1. *Let X denote the dyadic cylinder associated to Γ. If X has finite renormalized area, then $\mathrm{A}_\rho(S_t) - \mathrm{A}_\rho(X_t)$ has a finite limit as $t \searrow 0$*

Proof. If Definition 20 holds, so does Definition 14 (we have already proved a path from Figure 11 between them). For each vertical rectangle R making up a side (or a "panel") of X, we have a Lipschitz map from this panel to a portion of S that changes area by at most an additive factor of $O(\varepsilon_\Gamma^2(Q))$, where Q is the dyadic cube associated to the center of R. Because of the vertical "hinges" between adjacent panels, some points of S might be hit twice or not at all by the Lipschitz maps associated to those panels. However, the angles between these panels are bounded by $O(\varepsilon_\Gamma(Q))$, and the hyperbolic distance between S and X is also bounded by $O(\varepsilon_\Gamma(Q))$. Thus the total error is at most $O(\varepsilon_\Gamma^2(Q))$, which is summable over all the panels of X. Thus the difference between the hyperbolic areas of S and X above height t has a finite limit $t \searrow 0$. □

LEMMA 22.2. *With X as above, $L_\rho(S_t) - L_\rho(X_t)$ had a finite limit as $t \searrow 0$*

Proof. The same argument as in the previous lemma works again: the Lipschitz map from each panel of X to S preserves length up to an additive factor of $O(\varepsilon_\Gamma^2(Q))$, and the errors caused by the corners are bounded by the same magnitude. □

If Definition 20 holds, then $\lim_{t\searrow 0} \mathrm{A}_\rho(X_t) - L_\rho(\partial X_t)$ exists and is finite. The preceding lemmas imply the same for S, so it also has finite renormalized area.

We have now proven all the implications in Figure 11.

23. **Remarks and questions**

Comparing different quantities: The work of Takhtajan and Teo [123], Rohde and Wang [109] and Viklund and Wang [124] includes many explicit formulas relating the Dirichlet norm of $\log f'$ to the Kahler potential of the Weil-Petersson metric on universal Teichmuller space and to the Loewner energy of the curve Γ. Are there similar formulas that relate these quantities to quantities discussed in this paper, e.g., Möbius energy, β^2-sums, Menger integrals, the curvature integral of a minimal surface associated to Γ or the renormalized area of this curve? If there is more than one such minimal surface, which surface? A closely related result has recently been announced in [26], equating the Loewner energy of a planar curve to the renormalized volume between two associated Epstein-Poincaré surfaces of constant mean curvature in hyperbolic space.

Other knot energies: There are a variety of other knot energies besides Möbius energies. For example,

$$E^{j,p}(\Gamma) = \int_\Gamma \int_\Gamma \left(\frac{1}{|x-y|^j} - \frac{1}{\ell(x,y)^j} \right)^p dxdy$$

blows up for self-intersections if $jp \geq 2$ and is finite for smooth curves if $jp \leq 2p+1$. Sobolev smoothness properties for curves with finite $E^{j,p}$ energy are studied by Blatt in [21] (but there is a typo in Theorem 1.1; s should be $s = (jp-1)/(2p)$).

Another class of knot energies considered in [121] are the Menger energies

$$\mathcal{M}_p(\Gamma) = \int_\Gamma \int_\Gamma \int_\Gamma c^p(x,y,z)|dx||dy||dz|$$

with $\mathcal{M}_2(\Gamma)$ being the usual condition that is equivalent to rectifiability (see Section 4). They show that, for $p \geq 3$, finite energy curves are Jordan curves, and that for $p > 3$ these curves are $C^{1,\alpha}$. The endpoint case $p = 3$ seems the most interesting as this is the only scale-invariant Menger energy. Since $c(x,y,z) \lesssim 1/\ell(x,y,z)$, $\mathcal{M}_3(\Gamma)$ is less restrictive than the Weil-Petersson condition. What are the corresponding geometric characterizations of these curves?

Length convergence on minimal surfaces: The estimates in this paper prove that if Γ is Weil-Petersson, S is a minimal surface with asymptotic boundary Γ and $\Gamma(t)$ is the curve on S at height t above the boundary, then

$$\int_0^1 \left|\ell(\Gamma(t)) - \ell(\Gamma)\right| \frac{dt}{t^2} < \infty.$$

Does the converse hold? The direction stated above follows by writing

$$\left|\ell\big(\Gamma(t)\big) - \ell(\Gamma)\right| \leq \left|\ell\big(\Gamma(t)\big) - \ell(\Gamma_n)\right| + \left|\ell(\Gamma_n) - \ell(\Gamma)\right|,$$

where Γ_n is the usual dyadically inscribed polygon with $2^{-n-1} < t \leq 2^{-n}$. The second term on the right is integrable by Theorem 1.3, and it is controlled by using the β-numbers at scales smaller than t. The first term is controlled by Seppi's estimate and the ε-numbers at scale t; these, in turn, are controlled by sums of β-numbers over scales larger than t. Thus the question is whether there are non-Weil-Petersson curves for which $\ell(\Gamma(2^{-n}))$ is a much better approximation to $\ell(\Gamma)$ than $\ell(\Gamma_n)$ is?

Möbius energy and SLE: As we will discuss briefly in Appendix A, Weil-Petersson curves are related to the large deviations theory of Schramm-Loewner evolutions (SLE) as the parameter κ tends to zero. It is intriguing that they are also characterized, via Möbius energy, in terms of the rate of blow-up of a self-repulsive energy that prevents self-intersections. Is there some more direct connection between these two ideas? A SLE(κ) curve has Hausdorff dimension $1+\kappa/8$ for $0 < \kappa \leq 8$, and we expect the ϵ-truncation of the energy integral for an α-dimensional measure and kernel $|x|^{2-d}$ to grow like $\epsilon^{2-d+\alpha}$. Do SLE paths have energy that grows like $\epsilon^{-1+\kappa/8}$, and are they, in some sense, optimal among such curves?

Is there something interesting to say regarding hyperbolic convex hulls and minimal surfaces of an SLE path when $\kappa > 0$; e.g., can we compute an "expected curvature" for the corresponding minimal surface? When $\kappa \geq 8$, the paths become plane filling, but do the corresponding minimal surfaces still make sense, and if so, can we characterize their properties (e.g., growth rate of renormalized area) in terms of κ? In [125], Viklund and Wang consider connections between WP curves and SLE(κ) as $\kappa \nearrow \infty$.

Renormalized volume of hyperbolic 3*-manifolds*: Let G be a quasi-Fuchsian group, M its hyperbolic quotient 3-manifold, R_1, R_2 the two Riemann surfaces comprising the boundary at ∞ of M, and Γ its limit set. There are a variety of papers that relate the volume $\mathrm{CH}(\Gamma)$, the renormalized volume of M, and the Weil-Petersson distance between R_1 and R_2. For example, see [29], [30], [76], and [113]. The ideas in these papers seem very similar to our results characterizing Weil-Petersson curves Γ in terms of the "thickness" of the hyperbolic convex hull of Γ and the renormalized area of a surface with boundary Γ. Is there a precise connection between the results of this paper and the papers mentioned above? In [123], Takhtajan and Teo show that the usual Weil-Petersson metric for compact surfaces can be recovered from their Weil-Petersson metric on the universal Teichmüller space. Is this helpful in making the connection suggested above?

Detecting Weil-Petersson components of $T(1)$: The Hilbert manifold topology of Takhtajan and Teo divides the universal Teichmüller space into uncountable many connected components. Can we geometrically characterize when two curves belong to the same component? The current paper has done this for the component containing the unit circle. Perhaps some condition can be given saying that the convex hulls are quasi-isometric with constants that tend to 1 in a square integrable sense near the boundary of hyperbolic space. Are Γ_1, Γ_2 in the same component if and only if $\Gamma_2 = f(\Gamma_1)$ for some planar QC map f whose dilatation is in L^2 for hyperbolic area on the complement of Γ_1? A related problem suggested by Takhtajan is to construct a natural section for universal Teichmüller space, i.e., a natural choice of one quasicircle from each connected component. A good starting point might be Rohde's paper [108], which gives such a choice for quasicircles modulo biLipschitz images.

Angles of inscribed dyadic polygons: Suppose $\{z_j^n\}$ are a choice of dyadic points in Γ, as in Theorem 1.3, and define

$$\theta(n,k) = \arg\left(\frac{z_{j+1}^n - z_j^n}{z_j^n - z_{j-1}^n}\right)$$

to be the angles between adjacent nth generation segments. Using Theorems 1.3 and 1.5, it is not hard to show that if Γ is Weil-Petersson, then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{2^n} \theta^2(n,k) < \infty$$

with a uniform bound independent of the choice of dyadic base point. Is the converse true? What if we also assume that Γ is chord-arc? In general, θ can be zero at a point, even if β is large, e.g., at the center of a spiral. This problem is reminiscent of the longstanding ϵ^2-conjecture recently proved by Benjamin Jaye, Xavier Tolsa and Michele Villa [72]. Do their methods apply here?

The medial axis: The medial axis $\mathrm{MA}(\Omega)$ of a domain Ω is the set of centers of disks $D(x,r) \subset \Omega$ such that $\mathrm{dist}(x,y) = r$ for at least two points $y \in \partial\Omega$. See [54] for its basic properties (it is called the skeleton of Ω there). Mumford has asked if Weil-Petersson curves can be characterized in terms of the medial axis of their complementary domains. This means we know both the set and the distance function to the boundary (a line segment, with different distance functions, can be the medial axis of both WP and non-WP curves). The cleanest statement we are aware of is the following: The region $\Omega \setminus \mathrm{MA}(\Omega)$ is foliated by directed line segments that connect each point to its unique nearest point on $\partial\Omega$. For each hyperbolic unit ball $B_\rho(w,1)$ in Ω we assign the supremum of the difference between directions for the segments hitting B. Then Γ is Weil-Petersson if and only if Γ is chord-arc and this function is in $L^2(\Omega, d\mathrm{A}_\rho)$. See Figure 16. This says Γ is Weil-Petersson if and only if the nearest point foliation is "flat" up to an L^2 error. Is there a more elegant characterization in terms of the medial axis itself?

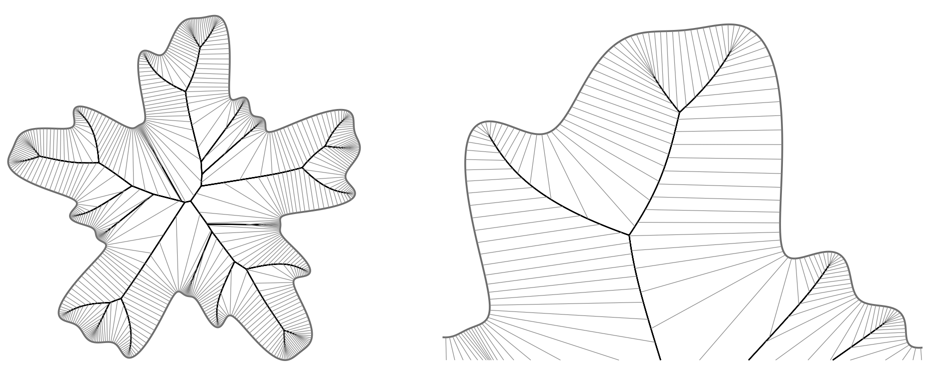

Figure 16. A medial axis, the nearest point foliation and an enlargement.

New characterizations of old curve families: The function theoretic characterizations of the Weil-Petersson class are exactly analogous to known characterizations of other classes, e.g., when $\log f'$ is in VMO [104] or BMO [8] and [19]. See [17] for a table comparing these results precisely. Do VMO and BMO curves have other characterizations analogous to the ones discussed in this paper? Do they extend to higher dimensions? For example, Michel Zinsmeister has asked if anything interesting can be said about the domes and minimal surfaces associated to boundaries of BMO domains.

Appendix A. Other characterizations of Weil-Petersson curves

This appendix lists some further equivalent definitions of the Weil-Petersson class in the plane; these definitions were never used in our proofs, but we include them to illustrate the variety of problems in which the Weil-Petersson class naturally occurs. We start with Takhtajan and Teo's definition that gives the class its name.

Teichmüller theory: Recall that $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}^* = \{|z| > 1\}$. Let $\mathbb{B}_1^\infty(\mathbb{D}^*)$ denote the unit ball of $L^\infty(\mathbb{D}^*)$. By the measurable Riemann mapping theorem, each $\mu \in \mathbb{B}_1^\infty(\mathbb{D}^*)$ determines a quasiconformal map w^μ of the plane that is conformal inside $\mathbb{D}$, and satisfies $f(0) = f''(0) = 0$, $f'(0) = 1$. We say that μ and ν are equivalent if $w^\mu = w^\nu$ on $\mathbb{T}$, and we define $T(1)$ be $L^\infty(\mathbb{D}^*)_1$ quotiented by this equivalence relation. This is the universal Teichmüller space. In [123], Takhtajan and Teo define a Weil-Petersson metric on $T(1)$ as follows. Let U be the set of holomorphic ϕ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}^*} \big|\phi(z)\big|^2 \big(1 - |z|^2\big)^2 dxdy < \sqrt{\pi/3},$$

and, for each $\phi \in U$, define a dilatation μ on $\mathbb{D}^*$ by

$$\mu(z) = -\frac{1}{2}\left(1 - |z|^2\right)^2 \phi(1/\overline{z}) z^{-4}.$$

Given a fixed dilatation ν on $\mathbb{D}^*$, consider the set of all dilations of the form

$$\lambda = \nu * \mu^{-1}\left(\frac{\nu - \mu}{1 - \overline{\mu}\nu}\right) \cdot \frac{(w_\mu)_z}{(w^\mu)_{\overline{z}}} \circ w^\mu.$$

(This follows from the usual formula for computing the complex dilatation of a composition of two quasiconformal mappings.) This defines a set $V_\nu \subset \mathbb{B}_1^\infty(\mathbb{D}^*)$ that contains ν. Projecting these sets into $T(1)$ defines a neighborhood of each point $[\nu] \in T(1)$, and $T_0(1)$ is the connected component of the identity in this topology.

Definition 21. The curve $\Gamma = f(\mathbb{T})$, where f is a quasiconformal map of the plane, conformal inside $\mathbb{D}$ and whose dilatation on $\mathbb{D}^*$ represents a point of $T_0(1)$.

Operator theory: Given a circle homeomorphism φ, we can define an operator on harmonic functions on the unit disk by precomposing the boundary values of u with φ, taking the harmonic extension back to the disk, and subtracting the value at the origin (so the resulting harmonic function, denoted $P_\varphi u$, is zero at the origin). Given a holomorphic function in the Dirichlet class, we can apply this operator and follow it by orthogonal projection onto the anti-holomorphic Dirichlet functions and finally apply $f(z) \to f(\bar{z})$ to make it holomorphic. Nag and Sullivan [93] proved that this operator, denoted P_φ^-, is bounded from the Dirichlet class to itself if and only if φ is quasisymmetric, and Yun Hu and Yuliang Shen [70] proved it is Hilbert-Schmidt if and only if φ is Weil-Petersson (an operator T on a Hilbert space is Hilbert-Schmidt if $\sum_j \|Te_j\|^2 < \infty$ for any orthonormal basis $\{e_j\}$; equivalently TT^* is trace class).

Definition 22. The operator P_φ^- is Hilbert-Schmidt on the Dirichlet space.

Another operator theoretic characterization of the Weil-Petersson class is given by Takhtajan and Teo ([123, Corollary II.2.9]) in terms of Grunsky operators on ℓ^2.

Integral geometry: Another measure of how much Γ deviates from a straight line can be given in terms of how random lines hit Γ. Suppose we parameterze lines L in $\mathbb{R}^2 \setminus \{0\}$ by $(r,\theta) \in (0,\infty) \times (0, 2\pi]$, where $z = r\exp(i\theta)$ is the point of L closest to 0. It is a well known fact from integral geometry (e.g., [112]) that the measure $d\mu = drd\theta$ on lines is invariant under Euclidean isometries of the plane, and the measure of the set of lines hitting a non-degenerate convex set X equals the length of the boundary of X (for a line segment, it is twice the length of the segment). For a dyadic cube Q, let $S(Q,\Gamma)$ be the set of lines hitting $3Q$ that also hit both $\Gamma \cap \frac{5}{3}Q$ and $\Gamma \cap (3Q \setminus 2Q)$.

Definition 23. Any translate of $\Gamma \subset \mathbb{R}^2$ satisfies

$$\sum_Q \frac{\mu\big(S(Q,\Gamma)\big)}{\operatorname{diam}(Q)} < \infty. \tag{A.1}$$

The equivalence of Definitions 11 and 23 follows immediately from Theorem 10.2.1 of [20]. Since $12\operatorname{diam}(Q)/\sqrt{2}$ is the perimeter of $3Q$, it equals the μ measure of the random lines hitting $3Q$. Normalizing, each term becomes the probability that if a line L hits $3Q$, then $L \in S(Q,\Gamma)$. Thus (A.1) becomes

$$\sum_Q \mathbb{P}\big(S(Q,\Gamma)|3Q\big) < \infty. \tag{A.2}$$

These terms represent the probability that a random line hitting $3Q$ will hit $\Gamma \cap 3Q$ at two points approximately $\ell(Q)$ apart. The particular values $\frac{5}{3}$, 2 and 3 in the definition of $S(L,\Gamma)$ are probably not important, just convenient for the proof in [20].

Loewner energy: Suppose $\Omega = \mathbb{C} \setminus [0,\infty)$ and suppose that $\gamma \subset \Omega$ is a curve that connects 0 to ∞. Suppose also that this curve corresponds to the driving function W via Loewner's equation. Then the chordal Loewner energy of γ is defined by Peter Friz and Atul Shekhar [55] and Yilin Wang [127] to be

$$I(\gamma) = \frac{1}{2}\int_0^\infty \dot{W}(t)^2 dt.$$

This was generalized to simple closed curves Γ on the Riemann sphere by Rohde and Wang [109] as follows. Given two points $z, w \in \Gamma$, define $\Gamma_{z,w} \subset \Gamma$ to be the subarc connecting z and w, and let $\Omega_{z,w}$ be the complement of $\Gamma_{z,w}$ on the Riemann sphere. If we conformally map $\Omega_{z,w}$ to Ω with z, w mapping to $0, \infty$, respectively, then $\Gamma \setminus \Gamma_{z,w}$ maps to an arc from 0 to ∞ in Ω, so its energy can be defined as above. The image of $\Gamma \setminus \Gamma_{z,w}$ is now an arc from 0 to ∞ in Ω, so its energy is defined as above. The energy of the loop Γ rooted at z is defined as the limit of these energies as $w \to z$; Rohde and Wang showed this is independent of the choice of z.

Definition 24. The curve Γ has finite Loewner energy.

The equivalence with the earlier definitions was proven by Wang [127]. She showed that the Loewner energy equals $\mathbf{S_1}(\varphi)/\pi$ where $\mathbf{S_1}(\varphi)$ is the universal Liouville action (and Kähler potential for the Weil-Petersson metric) defined by Takhtajan and Teo:

$$\mathbf{S_1}(\varphi) = \iint_{\mathbb{D}} \left| \frac{f''(z)}{f'(z)} \right|^2 dxdy + \iint_{\mathbb{D}^*} \left| \frac{g''(z)}{g'(z)} \right|^2 dxdy + 4\pi \log \frac{|f'(0)|}{|g'(\infty)|},$$

where $f : \mathbb{D} \to \Omega$, $g : \mathbb{D}^* \to \Omega^*$ are the conformal maps from the two sides of the unit circle to the two sides of Γ.

Large deviations of Schramm-Loewner evolutions: In [126], Wang interprets finite energy curves γ from 0 to ∞ in $\mathbb{H}^2$ in terms of large deviations of SLE(κ) as $\kappa \searrow 0$. Roughly speaking, the Loewner energy of γ is equal to

$$\lim_{\epsilon \to 0} \left[\lim_{\kappa \searrow 0} \log \mathbb{P}\big[\text{SLE}(\kappa) \in B(\gamma, \epsilon)\big] \right].$$

Thus the Loewner energy is the exponential rate of decay of the probability that SLE stays in an ϵ-neighborhood of γ. In fact, Wang's result is not stated using Hausdorff neighborhoods, but in terms of sets of curves that pass to the left or right of a specified finite set of points. A little more precisely, suppose we are given a finite set Z of points $\{z_n\}$ in the upper half-plane, and each point is labeled with ± 1. A curve γ from 0 to ∞ cuts the upper half-plane into simply connected regions, that we call the "left side" and "right side". A curve γ is called admissible for Z (written $\gamma \in \mathcal{A}(Z)$) if every point labeled $+1$ is on the right side of γ and every point labeled -1 is on the left side. If γ is admissible for Z, then we say that Z is consistent with γ, and we write $Z \in \mathcal{Z}(\gamma)$. Wang shows that given a set Z,

$$-\lim_{\kappa \to 0} \kappa \cdot \log \mathbb{P}\big[\text{SLE}(\kappa) \in \mathcal{A}(Z)\big] = \inf\{I(\gamma) : \gamma \in \mathcal{A}(Z)\}.$$

Thus the Weil-Petersson class can be defined using the following condition:

Definition 25. The curve Γ is Weil-Petersson if and only if

$$\sup_{Z\in\mathcal{Z}(\gamma)}\lim_{\kappa\to 0}(-\kappa)\log\mathbb{P}\left[\mathrm{SLE}(\kappa)\in\mathcal{A}(Z)\right]<\infty.$$

Roughly speaking, a curve in $\mathbb{H}^2$ from 0 to ∞ is a (subarc of a spherical) Weil-Petersson curve iff for any finite set of labeled points Z consistent with γ, the probability that $\mathrm{SLE}(\kappa)$ is also consistent with Z decays at most exponentially quickly as $\kappa\to 0$. See [126] for precise statements and further details.

Brownian loop soup: The Brownian loop measure, introduced by Greg Lawler and Wendelin Werner in [77], is a measure on closed loops in a domain Ω. It is conformally invariant and if $\Omega'\subset\Omega$, then the loop measure on Ω' is just the restriction of the loop measure for Ω to loops that are contained in Ω'. Given disjoint compact subsets of Ω, we define $\mathcal{W}(A,B;\Omega)$ to be the loop measure of closed curves γ in Ω so that the outer boundary of γ hits both A and B. Suppose Γ^r is the image of the circle $\{|z|=r\}$ under a conformal map from $\mathbb{D}$ to the interior of Γ. In [128], Wang proves that the Loewner energy of Γ is 12 times

$$\lim_{r\to 1}\left[\mathcal{W}(S^1,r\cdot S^1,\mathbb{C})-\mathcal{W}(\Gamma,\Gamma^r,\mathbb{C})\right].$$

Thus being a Weil-Petersson curve is equivalent to:

Definition 26. The curve Γ satisfies

$$\lim_{r\to 1}\left[\mathcal{W}(S^1,r\cdot S^1,\mathbb{C})-\mathcal{W}(\Gamma,\Gamma^r,\mathbb{C})\right]<\infty.$$

It is interesting to note that this is a type of renormalization of a divergent quantity, just as the renormalized area is divergent. Similarly, Möbius energy can be written as the Hadamard renormalization of a divergent energy integral involving an inverse cube law, e.g., the repulsive force exerted by distributing charge according to arclength on a curve in $\mathbb{R}^4$. Is there some underlying connection between these different renormalizations?

Acknowledgments. The author is partially supported by NSF Grant DMS-2303987.

References

[1] R. A. Adams and J. J. F. Fournier (eds.), *Sobolev Spaces*, second ed., *Pure and Applied Mathematics* (*Amsterdam*) **140**, Elsevier/Academic Press, Amsterdam, 2003. MR 2424078. Zbl 1098.46001.

[2] L. Ahlfors and G. Weill, A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962), 975–978. MR 0148896. Zbl 0106.28504. https://doi.org/10.2307/2034099.

[3] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, second ed., *University Lecture Series* **38**, American Mathematical Society, Providence, RI, 2006, With supplemental chapters by C. J. Earle, I. Kra, M. Shishikura and J. H. Hubbard. MR 2241787. Zbl 1103.30001. https://doi.org/10.1090/ulect/038.

[4] L. V. Ahlfors, *Conformal Invariants*, AMS Chelsea Publishing, Providence, RI, 2010, Topics in geometric function theory, Reprint of the 1973 original, With a foreword by Peter Duren, F. W. Gehring and Brad Osgood. MR 2730573. Zbl 1211.30002. https://doi.org/10.1090/chel/371.

[5] S. Alexakis and R. Mazzeo, Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds, *Comm. Math. Phys.* **297** no. 3 (2010), 621–651. MR 2653898. Zbl 1193.53131. https://doi.org/10.1007/s00220-010-1054-3.

[6] M. T. Anderson, Complete minimal varieties in hyperbolic space, *Invent. Math.* **69** no. 3 (1982), 477–494. MR 0679768. Zbl 0515.53042. https://doi.org/10.1007/BF01389365.

[7] M. T. Anderson, Complete minimal hypersurfaces in hyperbolic n-manifolds, *Comment. Math. Helv.* **58** no. 2 (1983), 264–290. MR 0705537. Zbl 0549.53058. https://doi.org/10.1007/BF02564636.

[8] K. Astala and M. Zinsmeister, Teichmüller spaces and BMOA, *Math. Ann.* **289** no. 4 (1991), 613–625. MR 1103039. Zbl 0896.30028. https://doi.org/10.1007/BF01446592.

[9] M. Badger and S. McCurdy, Subsets of rectifiable curves in Banach spaces I: Sharp exponents in traveling salesman theorems, *Illinois J. Math.* **67** no. 2 (2023), 203–274. MR 4593892. Zbl 07724273. https://doi.org/10.1215/00192082-10592363.

[10] M. Badger and S. McCurdy, Subsets of rectifiable curves in Banach spaces II: Universal estimates for almost flat arcs, *Illinois J. Math.* **67** no. 2 (2023), 275–331. MR 4593893. Zbl 07724274. https://doi.org/10.1215/00192082-10592390.

[11] M. Bauer, P. Harms, and P. W. Michor, Sobolev metrics on shape space of surfaces, *J. Geom. Mech.* **3** no. 4 (2011), 389–438. MR 2888014. Zbl 1262.58004. https://doi.org/10.3934/jgm.2011.3.389.

[12] A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* **96** (1956), 125–142. MR 0086869. Zbl 0072.29602. https://doi.org/10.1007/BF02392360.

[13] A. Beurling, *Études sur un problème de majoration*, Ph.D. thesis, Uppsala University, 1933. Zbl 0008.31802.

[14] C. J. Bishop, Quasiconformal Lipschitz maps, Sullivan's convex hull theorem and Brennan's conjecture, *Ark. Mat.* **40** no. 1 (2002), 1–26. MR 1948883. Zbl 1034.30013. https://doi.org/10.1007/BF02384499.

[15] C. J. Bishop, An explicit constant for Sullivan's convex hull theorem, in *In the Tradition of Ahlfors and Bers, III*, *Contemp. Math.* **355**, Amer. Math. Soc., Providence, RI, 2004, pp. 41–69. MR 2145055. Zbl 1069.30030. https://doi.org/10.1090/conm/355/06444.

[16] C. J. Bishop, Conformal welding and Koebe's theorem, *Ann. of Math.* (2) **166** no. 3 (2007), 613–656. MR 2373370. Zbl 1144.30007. https://doi.org/10.4007/annals.2007.166.613.

[17] C. J. Bishop, Function theoretic characterizations of Weil-Petersson curves, *Rev. Mat. Iberoam.* **38** no. 7 (2022), 2355–2384. MR 4526317. Zbl 1516.30028. https://doi.org/10.4171/rmi/1398.

[18] C. J. Bishop, The traveling salesman theorem for Jordan curves, *Adv. Math.* **404** no. part A (2022), Paper No. 108443, 27. MR 4420442. Zbl 1492.28002. https://doi.org/10.1016/j.aim.2022.108443.

[19] C. J. Bishop and P. W. Jones, Harmonic measure, L^2 estimates and the Schwarzian derivative, *J. Anal. Math.* **62** (1994), 77–113. MR 1269200. Zbl 0801.30024. https://doi.org/10.1007/BF02835949.

[20] C. J. Bishop and Y. Peres, *Fractals in Probability and Analysis*, *Cambridge Studies in Advanced Mathematics* **162**, Cambridge University Press, Cambridge, 2017. MR 3616046. Zbl 1390.28012. https://doi.org/10.1017/9781316460238.

[21] S. Blatt, Boundedness and regularizing effects of O'Hara's knot energies, *J. Knot Theory Ramifications* **21** no. 1 (2012), 1250010, 9. MR 2887901. Zbl 1238.57007. https://doi.org/10.1142/S0218216511009704.

[22] G. Bourdaud, Changes of variable in Besov spaces. II, *Forum Math.* **12** no. 5 (2000), 545–563. MR 1779495. Zbl 0971.46022. https://doi.org/10.1515/form.2000.018.

[23] M. J. Bowick and S. G. Rajeev, The holomorphic geometry of closed bosonic string theory and Diff S^1/S^1, *Nuclear Phys. B* **293** no. 2 (1987), 348–384. MR 0908048. https://doi.org/10.1016/0550-3213(87)90076-9.

[24] M. J. Bowick and S. G. Rajeev, String theory as the Kähler geometry of loop space, *Phys. Rev. Lett.* **58** no. 6 (1987), 535–538. MR 0873068. https://doi.org/10.1103/PhysRevLett.58.535.

[25] M. Bridgeman, R. Canary, and A. Yarmola, An improved bound for Sullivan's convex hull theorem, *Proc. Lond. Math. Soc.* (3) **112** no. 1 (2016), 146–168. MR 3458148. Zbl 1344.30018. https://doi.org/10.1112/plms/pdv064.

[26] M. Bridgeman, K. Bromberg, F. V. Pallete, and Y. Wang, Universal Liouville action as a renormalized volume and its gradient flow, 2023, *Duke Math. J.*, to appear. arXiv 2311.18767.

[27] M. Bridgeman and R. D. Canary, The Thurston metric on hyperbolic domains and boundaries of convex hulls, *Geom. Funct. Anal.* **20** no. 6 (2010), 1317–1353. MR 2738995. Zbl 1218.30123. https://doi.org/10.1007/s00039-010-0102-7.

[28] M. Bridgeman and R. D. Canary, Uniformly perfect domains and convex hulls: improved bounds in a generalization of a theorem of Sullivan, *Pure Appl. Math. Q.* **9** no. 1 (2013), 49–71. MR 3126500. Zbl 1291.30124. https://doi.org/10.4310/PAMQ.2013.v9.n1.a2.

[29] M. Bridgeman and R. D. Canary, Renormalized volume and the volume of the convex core, *Ann. Inst. Fourier* (*Grenoble*) **67** no. 5 (2017), 2083–2098. MR 3732685. Zbl 1404.57031. https://doi.org/10.5802/aif.3130.

[30] J. F. Brock, The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores, *J. Amer. Math. Soc.* **16** no. 3 (2003), 495–535. MR 1969203. Zbl 1059.30036. https://doi.org/10.1090/S0894-0347-03-00424-7.

[31] M. Bruveris and F.-X. Vialard, On completeness of groups of diffeomorphisms, *J. Eur. Math. Soc.* (*JEMS*) **19** no. 5 (2017), 1507–1544. MR 3635359. Zbl 1370.58003. https://doi.org/10.4171/JEMS/698.

[32] S.-Y. A. Chang and D. E. Marshall, On a sharp inequality concerning the Dirichlet integral, *Amer. J. Math.* **107** no. 5 (1985), 1015–1033. MR 0805803. Zbl 0578.30010. https://doi.org/10.2307/2374345.

[33] I. Chavel and E. A. Feldman, Isoperimetric inequalities on curved surfaces, *Adv. in Math.* **37** no. 2 (1980), 83–98. MR 0591721. Zbl 0461.53012. https://doi.org/10.1016/0001-8708(80)90028-6.

[34] Q. Chen and Y. Cheng, Chern-Osserman inequality for minimal surfaces in H^n, *Proc. Amer. Math. Soc.* **128** no. 8 (2000), 2445–2450. MR 1664325. Zbl 0955.53036. https://doi.org/10.1090/S0002-9939-00-05635-5.

[35] M. Chuaqui and B. Osgood, Ahlfors-Weill extensions of conformal mappings and critical points of the Poincaré metric, *Comment. Math. Helv.* **69** no. 4 (1994), 659–668. MR 1303231. Zbl 0826.30013. https://doi.org/10.1007/BF02564508.

[36] M. Lanza de Cristoforis and L. Preciso, Differentiability properties of some nonlinear operators associated to the conformal welding of Jordan curves in Schauder spaces, *Hiroshima Math. J.* **33** no. 1 (2003), 59–86. MR 1966652. Zbl 1031.30006. https://doi.org/10.32917/hmj/1150997868.

[37] G. Cui, Integrably asymptotic affine homeomorphisms of the circle and Teichmüller spaces, *Sci. China Ser. A* **43** no. 3 (2000), 267–279. MR 1766456. Zbl 0965.30018. https://doi.org/10.1007/BF02897849.

[38] G. C. David and R. Schul, The analyst's traveling salesman theorem in graph inverse limits, *Ann. Acad. Sci. Fenn. Math.* **42** no. 2 (2017), 649–692. MR 3701642. Zbl 1376.28003. https://doi.org/10.5186/aasfm.2017.4260.

[39] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** no. 5 (2012), 521–573. MR 2944369. Zbl 1252.46023. https://doi.org/10.1016/j.bulsci.2011.12.004.

[40] J. R. Dorronsoro, Mean oscillation and Besov spaces, *Canad. Math. Bull.* **28** no. 4 (1985), 474–480. MR 0812124. Zbl 0648.46032. https://doi.org/10.4153/CMB-1985-058-3.

[41] J. Douglas, Solution of the problem of Plateau, *Trans. Amer. Math. Soc.* **33** no. 1 (1931), 263–321. MR 1501590. Zbl 0001.14102. https://doi.org/10.2307/1989472.

[42] C. L. Epstein, The hyperbolic Gauss map and quasiconformal reflections, *J. Reine Angew. Math.* **372** (1986), 96–135. MR 0863521. Zbl 0591.30018. https://doi.org/10.1515/crll.1986.372.96.

[43] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Analytical and Geometric Aspects of Hyperbolic Space* (*Coventry/Durham*, 1984), *London Math. Soc. Lecture Note*

Ser. **111**, Cambridge Univ. Press, Cambridge, 1987, pp. 113–253. MR 0903852. Zbl 0612.57010.

[44] D. B. A. Epstein and A. Marden, Convex hulls in hyperbolic space, a theorem of Sullivan, and measured pleated surfaces, in *Fundamentals of Hyperbolic Geometry: Selected Expositions*, *London Math. Soc. Lecture Note Ser.* **328**, Cambridge Univ. Press, Cambridge, 2006, pp. 117–266. MR 2235711.

[45] D. B. A. Epstein and V. Markovic, The logarithmic spiral: a counterexample to the $K = 2$ conjecture, *Ann. of Math.* (2) **161** no. 2 (2005), 925–957. MR 2153403. Zbl 1076.30048. https://doi.org/10.4007/annals.2005.161.925.

[46] K. J. Falconer and D. T. Marsh, Classification of quasi-circles by Hausdorff dimension, *Nonlinearity* **2** no. 3 (1989), 489–493. MR 1005062. Zbl 0684.58023. Available at http://stacks.iop.org/0951-7715/2/489.

[47] H. Federer, *Geometric Measure Theory*, *Die Grundlehren der mathematischen Wissenschaften, Band* 153, Springer-Verlag New York, Inc., New York, 1969. MR 0257325. Zbl 0176.00801.

[48] M. Feiszli, Extremal distance, hyperbolic distance, and convex hulls over domains with smooth boundary, *Ann. Acad. Sci. Fenn. Math.* **36** no. 1 (2011), 195–214. MR 2797691. Zbl 1222.30017. https://doi.org/10.5186/aasfm.2011.3612.

[49] M. Feiszli, S. Kushnarev, and K. Leonard, Metric spaces of shapes and applications: compression, curve matching and low-dimensional representation, *Geom. Imaging Comput.* **1** no. 2 (2014), 173–221. MR 3396622. Zbl 1314.30122. https://doi.org/10.4310/GIC.2014.v1.n2.a1.

[50] M. Feiszli and A. Narayan, Numerical computation of Weil-Peterson geodesics in the universal Teichmüller space, *SIAM J. Imaging Sci.* **10** no. 3 (2017), 1322–1345. MR 3686793. Zbl 1383.30012. https://doi.org/10.1137/15M1043947.

[51] F. Ferrari, B. Franchi, and H. Pajot, The geometric traveling salesman problem in the Heisenberg group, *Rev. Mat. Iberoam.* **23** no. 2 (2007), 437–480. MR 2371434. Zbl 1142.28004. https://doi.org/10.4171/RMI/502.

[52] F. Fiala, Le problème des isopérimètres sur les surfaces ouvertes à courbure positive, *Comment. Math. Helv.* **13** (1941), 293–346. MR 0006422. Zbl 0025.23003. https://doi.org/10.1007/BF01378068.

[53] M. H. Freedman, Z.-X. He, and Z. Wang, Möbius energy of knots and unknots, *Ann. of Math.* (2) **139** no. 1 (1994), 1–50. MR 1259363. Zbl 0817.57011. https://doi.org/10.2307/2946626.

[54] D. H. Fremlin, Skeletons and central sets, *Proc. London Math. Soc.* (3) **74** no. 3 (1997), 701–720. MR 1434446. Zbl 0949.54050. https://doi.org/10.1112/S0024611597000233.

[55] P. K. Friz and A. Shekhar, On the existence of SLE trace: finite energy drivers and non-constant κ, *Probab. Theory Related Fields* **169** no. 1-2 (2017), 353–376. MR 3704771. Zbl 1407.60113. https://doi.org/10.1007/s00440-016-0731-3.

[56] E. A. Gallardo-Gutiérrez, M. J. González, F. Pérez-González, C. Pommerenke, and J. Rättyä, Locally univalent functions, VMOA and

the Dirichlet space, *Proc. Lond. Math. Soc.* (3) **106** no. 3 (2013), 565–588. MR 3048550. Zbl 1276.30026. https://doi.org/10.1112/plms/pds040.

[57] F. P. Gardiner and D. P. Sullivan, Symmetric structures on a closed curve, *Amer. J. Math.* **114** no. 4 (1992), 683–736. MR 1175689. Zbl 0778.30045. https://doi.org/10.2307/2374795.

[58] J. B. Garnett, *Bounded Analytic Functions*, *Pure and Applied Mathematics* **96**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981. MR 0628971. Zbl 0469.30024.

[59] J. B. Garnett and D. E. Marshall, *Harmonic Measure*, *New Mathematical Monographs* **2**, Cambridge University Press, Cambridge, 2008, Reprint of the 2005 original. MR 2450237. Zbl 1139.31001.

[60] F. Gay-Balmaz and T. S. Ratiu, The geometry of the universal Teichmüller space and the Euler-Weil-Petersson equation, *Adv. Math.* **279** (2015), 717–778. MR 3345193. Zbl 1320.32018. https://doi.org/10.1016/j.aim.2015.04.005.

[61] F. W. Gehring, The definitions and exceptional sets for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I* **281** (1960), 28. MR 0124488. Zbl 0090.05303.

[62] F. W. Gehring and K. Hag, *The Ubiquitous Quasidisk*, *Mathematical Surveys and Monographs* **184**, American Mathematical Society, Providence, RI, 2012, With contributions by Ole Jacob Broch. MR 2933660. Zbl 1267.30003. https://doi.org/10.1090/surv/184.

[63] Z. Geng and F. Lin, The two-dimensional liquid crystal droplet problem with a tangential boundary condition, *Arch. Ration. Mech. Anal.* **243** no. 3 (2022), 1181–1221. MR 4381139. Zbl 1487.76007. https://doi.org/10.1007/s00205-021-01733-5.

[64] C. R. Graham and E. Witten, Conformal anomaly of submanifold observables in AdS/CFT correspondence, *Nuclear Phys. B* **546** no. 1-2 (1999), 52–64. MR 1682674. Zbl 0944.81046. https://doi.org/10.1016/S0550-3213(99)00055-3.

[65] H. Guo, Integrable Teichmüller spaces, *Sci. China Ser. A* **43** no. 1 (2000), 47–58. MR 1766239. Zbl 0948.30052. https://doi.org/10.1007/BF02903847.

[66] Z.-X. He, The Euler-Lagrange equation and heat flow for the Möbius energy, *Comm. Pure Appl. Math.* **53** no. 4 (2000), 399–431. MR 1733697.

[67] J. Heinonen and P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, *Acta Math.* **181** no. 1 (1998), 1–61. MR 1654771. Zbl 0915.30018. https://doi.org/10.1007/BF02392747.

[68] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* no. 7 (1998), Paper 23, 12. MR 1644988. Zbl 0958.81083. https://doi.org/10.1088/1126-6708/1998/07/023.

[69] N. J. Hicks, *Notes on Differential Geometry*, *Van Nostrand Mathematical Studies, No.* 3, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London, 1965. MR 0179691. Zbl 0132.15104.

[70] Y. Hu and Y. Shen, On quasisymmetric homeomorphisms, *Israel J. Math.* **191** no. 1 (2012), 209–226. MR 2970868. Zbl 1258.30007. https://doi.org/10.1007/s11856-011-0204-4.

[71] H. Inci, T. Kappeler, and P. Topalov, On the regularity of the composition of diffeomorphisms, *Mem. Amer. Math. Soc.* **226** no. 1062 (2013), vi+60. MR 3135704. Zbl 1293.58004. https://doi.org/10.1090/S0065-9266-2013-00676-4.

[72] B. Jaye, X. Tolsa, and M. Villa, A proof of Carleson's ε^2-conjecture, *Ann. of Math.* (2) **194** no. 1 (2021), 97–161. MR 4276285. Zbl 1472.28005. https://doi.org/10.4007/annals.2021.194.1.2.

[73] Y. Jiang, *Renormalization and Geometry in One-Dimensional and Complex Dynamics*, *Advanced Series in Nonlinear Dynamics* **10**, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. MR 1442953. Zbl 0864.58018. https://doi.org/10.1142/9789814350105.

[74] P. W. Jones, Rectifiable sets and the traveling salesman problem, *Invent. Math.* **102** no. 1 (1990), 1–15. MR 1069238. Zbl 0731.30018. https://doi.org/10.1007/BF01233418.

[75] J. Krandel, The traveling salesman theorem for Jordan curves in Hilbert space, 2023, Advance Publication, pp. 1–73. arXiv 2107.07107. https://doi.org/10.1307/mmj/20226254.

[76] K. Krasnov and J.-M. Schlenker, The Weil-Petersson metric and the renormalized volume of hyperbolic 3-manifolds, in *Handbook of Teichmüller Theory. Volume III*, *IRMA Lect. Math. Theor. Phys.* **17**, Eur. Math. Soc., Zürich, 2012, pp. 779–819. MR 2952776. Zbl 1256.30001. https://doi.org/10.4171/103-1/15.

[77] G. F. Lawler and W. Werner, The Brownian loop soup, *Probab. Theory Related Fields* **128** no. 4 (2004), 565–588. MR 2045953. Zbl 1049.60072. https://doi.org/10.1007/s00440-003-0319-6.

[78] O. Lehto and K. I. Virtanen, *Quasiconformal Mappings in the Plane*, second ed., *Die Grundlehren der mathematischen Wissenschaften*, *Band* 126, Springer-Verlag, New York-Heidelberg, 1973, Translated from the German by K. W. Lucas. MR 0344463. Zbl 0267.30016.

[79] O. Lehto, *Univalent Functions and Teichmüller Spaces*, *Graduate Texts in Mathematics* **109**, Springer-Verlag, New York, 1987. MR 0867407. Zbl 0606.30001. https://doi.org/10.1007/978-1-4613-8652-0.

[80] S. Li, Stratified β-numbers and traveling salesman in Carnot groups, *J. Lond. Math. Soc.* (2) **106** no. 2 (2022), 662–703. MR 4477201. Zbl 1538.28012. https://doi.org/10.1112/jlms.12582.

[81] S. Li and R. Schul, The traveling salesman problem in the Heisenberg group: upper bounding curvature, *Trans. Amer. Math. Soc.* **368** no. 7 (2016), 4585–4620. MR 3456155. Zbl 1350.53044. https://doi.org/10.1090/tran/6501.

[82] S. Li and R. Schul, An upper bound for the length of a traveling salesman path in the Heisenberg group, *Rev. Mat. Iberoam.* **32** no. 2 (2016), 391–417. MR 3512421. Zbl 1355.28005. https://doi.org/10.4171/RMI/889.

[83] F.-H. Lin, Asymptotic behavior of area-minimizing currents in hyperbolic space, *Comm. Pure Appl. Math.* **42** no. 3 (1989), 229–242. MR 0982349. Zbl 0688.49042. https://doi.org/10.1002/cpa.3160420302.

[84] J. Maldacena, Wilson loops in large N field theories, *Phys. Rev. Lett.* **80** no. 22 (1998), 4859–4862. MR 1732582. Zbl 0947.81128. https://doi.org/10.1103/PhysRevLett.80.4859.

[85] A. Marden, *Outer Circles*, Cambridge University Press, Cambridge, 2007, An Introduction to Hyperbolic 3-Manifolds. MR 2355387. Zbl 1149.57030. https://doi.org/10.1017/CBO9780511618918.

[86] A. Marden, The view from above, *Pure Appl. Math. Q.* **7** no. 2, Special Issue: In honor of Frederick W. Gehring, Part 2 (2011), 383–394. MR 2815381. Zbl 1291.30244. https://doi.org/10.4310/PAMQ.2011.v7.n2.a6.

[87] D. E. Marshall, A new proof of a sharp inequality concerning the Dirichlet integral, *Ark. Mat.* **27** no. 1 (1989), 131–137. MR 1004727. Zbl 0692.30028. https://doi.org/10.1007/BF02386365.

[88] O. Martio and J. Sarvas, Injectivity theorems in plane and space, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** no. 2 (1979), 383–401. MR 0565886. Zbl 0406.30013. https://doi.org/10.5186/aasfm.1978-79.0413.

[89] P. W. Michor and D. Mumford, Riemannian geometries on spaces of plane curves, *J. Eur. Math. Soc.* (*JEMS*) **8** no. 1 (2006), 1–48. MR 2201275. Zbl 1101.58005. https://doi.org/10.4171/JEMS/37.

[90] P. W. Michor and D. Mumford, An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach, *Appl. Comput. Harmon. Anal.* **23** no. 1 (2007), 74–113. MR 2333829. Zbl 1116.58007. https://doi.org/10.1016/j.acha.2006.07.004.

[91] F. Morgan, A regularity theorem for minimizing hypersurfaces modulo ν, *Trans. Amer. Math. Soc.* **297** no. 1 (1986), 243–253. MR 0849477. Zbl 0641.49027. https://doi.org/10.2307/2000466.

[92] J. W. Morgan, The Smith conjecture, in *The Smith Conjecture* (*New York*, 1979), *Pure Appl. Math.* **112**, Academic Press, Orlando, FL, 1984, pp. 3–6. MR 0758460. https://doi.org/10.1016/S0079-8169(08)61632-3.

[93] S. Nag and D. Sullivan, Teichmüller theory and the universal period mapping via quantum calculus and the $H^{1/2}$ space on the circle, *Osaka J. Math.* **32** no. 1 (1995), 1–34. MR 1323099. Zbl 0820.30027. Available at http://projecteuclid.org/euclid.ojm/1200785862.

[94] T. Nishioka, S. Ryu, and T. Takayanagi, Holographic entanglement entropy: an overview, *J. Phys. A* **42** no. 50 (2009), 504008, 35. MR 2566335. Zbl 1179.81138. https://doi.org/10.1088/1751-8113/42/50/504008.

[95] J. O'Hara, Energy of a knot, *Topology* **30** no. 2 (1991), 241–247. MR 1098918. Zbl 0733.57005. https://doi.org/10.1016/0040-9383(91)90010-2.

[96] J. O'Hara, Energy functionals of knots, in *Topology Hawaii* (*Honolulu*, *HI*, 1990), World Sci. Publ., River Edge, NJ, 1992, pp. 201–214. MR 1181493. Zbl 1039.58500.

[97] J. O'Hara, Family of energy functionals of knots, *Topology Appl.* **48** no. 2 (1992), 147–161. MR 1195506. Zbl 0769.57006. https://doi.org/10.1016/0166-8641(92)90023-S.

[98] K. Okikiolu, Characterization of subsets of rectifiable curves in $\mathbf{R}^n$, *J. London Math. Soc.* (2) **46** no. 2 (1992), 336–348. MR 1182488. Zbl 0758.57020. https://doi.org/10.1112/jlms/s2-46.2.336.

[99] G. de Oliveira Filho, Compactification of minimal submanifolds of hyperbolic space, *Comm. Anal. Geom.* **1** no. 1 (1993), 1–29. MR 1230271. Zbl 0794.53038. https://doi.org/10.4310/CAG.1993.v1.n1.a1.

[100] B. Osgood, Old and new on the Schwarzian derivative, in *Quasiconformal Mappings and Analysis* (*Ann Arbor*, *MI*, 1995), Springer, New York, 1998, pp. 275–308. MR 1488455. Zbl 0894.30001. https://doi.org/10.1007/978-1-4612-0605-7_16.

[101] R. Osserman, The isoperimetric inequality, *Bull. Amer. Math. Soc.* **84** no. 6 (1978), 1182–1238. MR 500557. Zbl 0411.52006. https://doi.org/10.1090/S0002-9904-1978-14553-4.

[102] H. Pajot, *Analytic Capacity*, *Rectifiability*, *Menger Curvature and the Cauchy Integral*, *Lecture Notes in Mathematics* **1799**, Springer-Verlag, Berlin, 2002. MR 1952175. Zbl 1043.28002. https://doi.org/10.1007/b84244.

[103] F. Pérez-González and J. Rättyä, Dirichlet and VMOA domains via Schwarzian derivative, *J. Math. Anal. Appl.* **359** no. 2 (2009), 543–546. MR 2546769. Zbl 1178.30005. https://doi.org/10.1016/j.jmaa.2009.06.023.

[104] C. Pommerenke, On univalent functions, Bloch functions and VMOA, *Math. Ann.* **236** no. 3 (1978), 199–208. MR 0492206. Zbl 0385.30013. https://doi.org/10.1007/BF01351365.

[105] D. Radnell, E. Schippers, and W. Staubach, Dirichlet problem and Sokhotski-Plemelj jump formula on Weil-Petersson class quasidisks, *Ann. Acad. Sci. Fenn. Math.* **41** no. 1 (2016), 119–127. MR 3467700. Zbl 1338.31006. https://doi.org/10.5186/aasfm.2016.4108.

[106] D. Radnell, E. Schippers, and W. Staubach, Quasiconformal Teichmüller theory as an analytical foundation for two-dimensional conformal field theory, in *Lie Algebras*, *Vertex Operator Algebras*, *and Related Topics*, *Contemp. Math.* **695**, Amer. Math. Soc., Providence, RI, 2017, pp. 205–238. MR 3709713. Zbl 1393.30034. https://doi.org/10.1090/conm/695/14003.

[107] E. Reich, Quasiconformal mappings of the disk with given boundary values, in *Advances in Complex Function Theory* (*Proc. Sem.*, *Univ. Maryland*, *College Park*, *Md.*, 1973–1974), 1976, pp. 101–137. Lecture Notes in Math., Vol. 505. MR 0419758. Zbl 0328.30018. https://doi.org/10.1007/BFb0081102.

[108] S. Rohde, Quasicircles modulo bilipschitz maps, *Rev. Mat. Iberoamericana* **17** no. 3 (2001), 643–659. MR 1900898. Zbl 1003.30013. https://doi.org/10.4171/RMI/307.

[109] S. Rohde and Y. Wang, The Loewner energy of loops and regularity of driving functions, *Int. Math. Res. Not. IMRN* no. 10 (2021), 7715–7763. MR 4259153. Zbl 1482.30068. https://doi.org/10.1093/imrn/rnz071.

[110] W. T. Ross, The classical Dirichlet space, in *Recent Advances in Operator-related Function Theory*, *Contemp. Math.* **393**, Amer. Math. Soc., Providence, RI, 2006, pp. 171–197. MR 2198379. Zbl 1135.31007. https://doi.org/10.1090/conm/393/07378.

[111] S. Ryu and T. Takayanagi, Aspects of holographic entanglement entropy, *J. High Energy Phys.* no. 8 (2006), 045, 48. MR 2249925. https://doi.org/10.1088/1126-6708/2006/08/045.

[112] L. A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976, With a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1. MR 0433364. Zbl 0342.53049.

[113] J.-M. Schlenker, The renormalized volume and the volume of the convex core of quasifuchsian manifolds, *Math. Res. Lett.* **20** no. 4 (2013), 773–786. MR 3188032. Zbl 1295.30101. https://doi.org/10.4310/MRL.2013.v20.n4.a12.

[114] R. Schul, Subsets of rectifiable curves in Hilbert space—the analyst's TSP, *J. Anal. Math.* **103** (2007), 331–375. MR 2373273. Zbl 1152.28006. https://doi.org/10.1007/s11854-008-0011-y.

[115] A. Seppi, Minimal discs in hyperbolic space bounded by a quasicircle at infinity, *Comment. Math. Helv.* **91** no. 4 (2016), 807–839. MR 3566524. Zbl 1356.53063. https://doi.org/10.4171/CMH/403.

[116] E. Sharon and D. Mumford, 2D-Shape analysis using conformal mapping, *Int. J. Comput. Vis.* **70** (2006), 55–75. Zbl 1477.68492. https://doi.org/10.1007/s1263-006-6121-z.

[117] Y. Shen, Weil-Petersson Teichmüller space, *Amer. J. Math.* **140** no. 4 (2018), 1041–1074. MR 3828040. Zbl 1421.30059. https://doi.org/10.1353/ajm.2018.0023.

[118] Y. Shen and L. Wu, Weil-Petersson Teichmüller space III: dependence of Riemann mappings for Weil-Petersson curves, *Math. Ann.* **381** no. 1-2 (2021), 875–904. MR 4322630. Zbl 1554.30022. https://doi.org/10.1007/s00208-020-02067-5.

[119] L. Simon, *Lectures on Geometric Measure Theory*, *Proceedings of the Centre for Mathematical Analysis*, *Australian National University* **3**, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 0756417. Zbl 0546.49019.

[120] K. Strebel, On the existence of extremal Teichmueller mappings, *J. Analyse Math.* **30** (1976), 464–480. MR 0440031. Zbl 0334.30013. https://doi.org/10.1007/BF02786734.

[121] P. Strzelecki and H. von der Mosel, Menger curvature as a knot energy, *Phys. Rep.* **530** no. 3 (2013), 257–290. MR 3105400. Zbl 1358.57019. https://doi.org/10.1016/j.physrep.2013.05.003.

[122] T. Takayanagi, Entanglement entropy from a holographic viewpoint, *Classical Quantum Gravity* **29** no. 15 (2012), 153001, 25. MR 2960962. Zbl 1247.83005. https://doi.org/10.1088/0264-9381/29/15/153001.

[123] L. A. Takhtajan and L.-P. Teo, Weil-Petersson metric on the universal Teichmüller space, *Mem. Amer. Math. Soc.* **183** no. 861 (2006), viii+119. MR 2251887. Zbl 1243.32010. https://doi.org/10.1090/memo/0861.

[124] F. Viklund and Y. Wang, Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines, *Geom. Funct. Anal.* **30** no. 1 (2020), 289–321. MR 4080509. Zbl 1436.30009. https://doi.org/10.1007/s00039-020-00521-9.

[125] F. Viklund and Y. Wang, The Loewner-Kufarev energy and foliations by Weil-Petersson quasicircles, *Proc. Lond. Math. Soc.* (3) **128** no. 2 (2024), e12582, 62 pp. MR 4704164. Zbl 1542.30007. https://doi.org/10.1112/plms.12582.

[126] Y. Wang, The energy of a deterministic Loewner chain: reversibility and interpretation via SLE_{0+}, *J. Eur. Math. Soc.* (*JEMS*) **21** no. 7 (2019), 1915–1941. MR 3959854. Zbl 1422.30031. https://doi.org/10.4171/JEMS/876.

[127] Y. Wang, Equivalent descriptions of the Loewner energy, *Invent. Math.* **218** no. 2 (2019), 573–621. MR 4011706. Zbl 1435.30074. https://doi.org/10.1007/s00222-019-00887-0.

[128] Y. Wang, A note on Loewner energy, conformal restriction and Werner's measure on self-avoiding loops, *Ann. Inst. Fourier* (*Grenoble*) **71** no. 4 (2021), 1791–1805. MR 4398248. Zbl 1486.30063. https://doi.org/10.5802/aif.3427.

[129] Y. Wang, From the random geometry of conformally invariant systems to the Kähler geometry of universal Teichmüller space, *Notices Amer. Math. Soc.* **71** no. 6 (2024), 732–739. MR 4767165. Zbl 1546.60033.

[130] H. Wei and K. Matsuzaki, Parametrization of the p-Weil-Petersson curves: holomorphic dependence, *J. Geom. Anal.* **33** no. 9 (2023), Paper No. 292, 32. MR 4610351. Zbl 1522.32034. https://doi.org/10.1007/s12220-023-01338-2.

[131] B. White, On the compactness theorem for embedded minimal surfaces in 3-manifolds with locally bounded area and genus, *Comm. Anal. Geom.* **26** no. 3 (2018), 659–678. MR 3844118. Zbl 1394.53065. https://doi.org/10.4310/CAG.2018.v26.n3.a7.

(Received: December 22, 2020)
(Revised: September 17, 2024)

Christopher J. Bishop
Stony Brook University, Stony Brook, NY
E-mail: bishop@math.stonybrook.edu
https://orcid.org/0000-0002-8459-5448