TREE-LIKE DECOMPOSITIONS OF SIMPLY CONNECTED DOMAINS

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Abstract. We show that any simply connected rectifiable domain Ω can be decomposed into Lipschitz crescents using only crosscuts of the domain and using total length bounded by a multiple of the length of $\partial\Omega$. In particular, this gives a new proof of a theorem of Peter Jones that such a domain can be decomposed into Lipschitz disks.

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1. Introduction

Can every domain be efficiently decomposed into nice pieces? Of course, this depends on what the words "efficiently" and "nice" mean, but one possible answer was given by Peter Jones who proved in [8] that every simply connected plane domain Ω has a decomposition into Lipschitz disks $\{\Omega_k\}$, such that $\sum_k \ell(\partial\Omega_k) = O(\ell(\partial\Omega))$. However, Jones' proof is based on the conformal mapping from the disk onto Ω, so that the construction of these pieces might not be very efficient from a computational point of view. In this note we will give a simpler proof of a stronger result, replacing conformal maps by an object from computational geometry: the medial axis.

Theorem 1.1. *There is an $M < \infty$ so that every simply connected plane domain Ω has a collection of disjoint circular arc crosscuts $\Gamma = \cup\gamma_k$ with $\sum_k \ell(\gamma_k) \leq M\ell(\partial\Omega)$ and so that each connected component of $\Omega \setminus \Gamma$ is an M-Lipschitz crescent.*

A crosscut is a Jordan arc in Ω with distinct endpoints on $\partial\Omega$. Since Γ consists of crosscuts, the components of $\Omega \setminus \Gamma$ form the vertices of a tree under the obvious adjacency relation. This is analogous to the idea of taking a triangulation of a polygon using only existing vertices of the polygon, as opposed to allowing new vertices (called Steiner points) in the interior of the polygon. The pieces in our decomposition are not quite Lipschitz disks, but they satisfy a slightly weaker condition we call being a Lipschitz crescent (see Section 2). A Lipschtiz crescent is easily decomposed into Lipschitz disks with the correct length bounds, so Jones' theorem follows trivially from our result. Moreover, each crosscut γ in our construction is a circular arc which lies in a disk D contained inside the domain, and γ lies within an M-neighborhood of the hyperbolic geodesic (for either D or Ω) with the same endpoints. Distinct crosscuts are uniformly separated in the hyperbolic metric of Ω. Finally, the construction is invariant under Möbius transformations of Ω.

The medial axis of a domain is the subset of points that are equidistant from two or more boundary points. For a simply connected plane domain, it is a real tree and for polygons it is a finite tree. Briefly, our construction works by taking a medial axis disk and moving it along arms of the medial axis until a certain angle between the moving disk and the starting disk becomes too large. Then we insert a crosscut and start the process again. This is similar in spirit to the construction in [8] which uses a stopping time based on the growth of the derivative of the conformal map $f : \mathbb{D} \to \Omega$. Computation of the medial axis is easier in many cases (e.g., linear time for n-gons) and Stephen Vavasis has suggested using tree-like decompositions in the numerical computation of conformal maps, so using conformal maps to construct the decompositions would be circular. His idea is explored further in [3]. Our result is also an illustration of the idea that results that are proven using conformal mapping can sometimes be obtained from constructions using the medial axis. This may be of interest since the medial axis makes sense for any domain, even in higher dimensions.

This paper is one of three related papers that were prompted by questions of Stephen Vavasis. He asked whether a tree-like decomposition into "nice" pieces always exists, and he conjectured that such a decomposition could be used to construct an approximately conformal map to the disk. He also suggested that tree-like decompositions could be used to give bounds for the L^2 norm of harmonic conjugation $\partial\Omega$. In this note, we answer his first question afirmatively. In [3] we answer the second question by showing that a tree-like decomposition into uniformly chord-arc subdomains can be used to define a simple map $\partial\Omega \to \partial\mathbb{D}$ which has a uniformly quasiconformal extension to the interiors. In [1] we answer the third question by bounding the norm of harmonic conjugation using tree-like decompositions.

In Section 2 we recall some definitions related to rectifiable domains and in Section 3 we discuss the medial axis. In Section 4 we define a distance on the medial axis using angles between medial axis disks and use it to partition the medial axis into subtrees. In Section 5 we partition Ω into pieces corresponding to these subtrees and show they are Lipschitz crescents. In Section 6 we prove a technical lemma and in Section 7 we complete the proof of Theorem 1.1 by showing the lengths of our crosscuts have the correct length sum.

2. Background

Given a set E in the plane we define its 1-dimensional measure as

$$\ell(E) = \lim_{\delta \to 0} \inf\{\sum 2r_j : E \subset \cup B(x_j, r_j), r_j \leq \delta\}$$

where the infimum is over all covers of E by open balls. We denote it by $\ell(E)$, since if E is a Jordan curve, this agrees with the usual notion of length. We say that a simply connected domain Ω has a rectifiable boundary if $\ell(\partial\Omega) < \infty$. In this case

$\partial\Omega$ is locally connected and may be parameterized by a Lipschitz map from the unit circle which is at most 2-to-1 almost everywhere on the circle. It is also convenient to define

$$\tilde{\ell}(\partial\Omega) = \int_{\mathbb{T}} |f'(z)||dz|,$$

where f is a conformal map from the disk onto a simply connected domain Ω. For Jordan domains this equals $\ell(\partial\Omega)$ and in general $\tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega)$. This measures the length of the boundary "with multiplicity". For example, if $\Omega = \mathbb{D} \setminus [0, 1)$, then $\ell(\partial\Omega) = 2\pi + 1$ and $\tilde{\ell}(\partial\Omega) = 2\pi + 2$.

A set is called regular (or sometimes Ahlfors-regular or Ahlfors-David regular) if there is a constant $M < \infty$ so that

$$\ell(E \cap B(x, r)) \leq Mr,$$

for every disk in the plane. If $E = \Gamma$ is a Jordan curve, we say it is chord-arc (or Lavrentiev) if the length of any subarc is bounded in terms of the corresponding chord, i.e. if there is a constant $C < \infty$ so that

$$\ell(\Gamma_{x,y}) \leq C|x - y|,$$

where Γ_{xy} is the arc between x and y (or shortest arc in the case that Γ is a closed Jordan curve).

A real valued function is called M-Lipschitz if

$$|f(x) - f(y)| \leq M|x - y|$$

for all x, y in its domain. A curve Γ in the plane is called a M-Lipschitz graph if it is an isometric image (e.g., rotation and translation) of a set of the form

$$\{(x, f(y)) : a \leq x \leq b\}$$

where f is a M-Lipschitz function.

A bounded domain Ω in the plane is called a Lipschitz disk if it is isometric to a domain whose boundary is a Lipschitz function in polar coordinates, i.e.,

$$\Omega = \{z = re^{i\theta} : r < g(\theta)\},$$

where g is a 2π-periodic M-Lipschitz function on $\mathbb{R}$ so that $\frac{1}{M+1} \leq g \leq 1$ (this is the definition given in Chapter X.3 of [7]). We will say that Ω is a Lipschitz crescent (or more precisely an M-Lipschitz crescent) if it has the form

$$\Omega = \{re^{i\theta} : 0 < r < \infty, 0 < \theta < f(r)\},$$

where f is an M-Lipschitz function such that $1/M \leq f(r) \leq \pi/2$. We will also call any bounded Möbius image of such a domain a Lipschitz crescent. The following simple results are left to the reader.

Lemma 2.1. *Any Lipschitz crescent is chord-arc.*

Lemma 2.2. *Any Lipschitz crescent Ω can be partitioned into Lipschitz disks $\{\Omega_k\}$ so that $\sum_k \ell(\partial\Omega_k) \leq C\ell(\partial\Omega)$.*

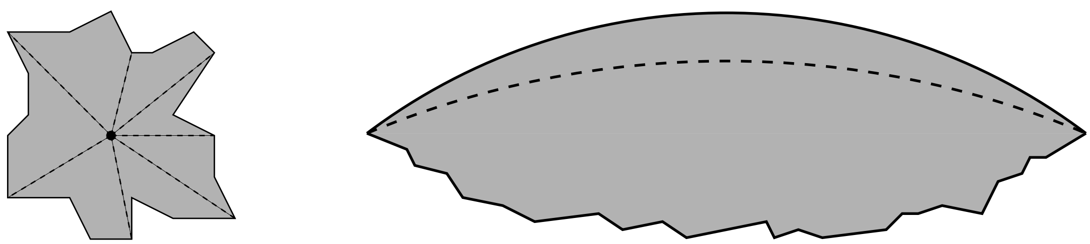

FIGURE 1. A Lipschitz disk and a Lipschitz crescent.

3. THE MEDIAL AXIS

Suppose Ω is a simply connected planar domain. A medial axis disk is an open disk D in Ω so that $\partial D \cap \partial\Omega$ contains at least two points. The medial axis of Ω is the set of all centers of such disks. A point of the medial axis is called a vertex if the boundary of the corresponding disk hits $\partial\Omega$ in three or more points. A point which is not a vertex is called an interior edge point (and the corresponding disk hits ∂D in exactly two points). The medial axis is always a union of countably many rectifiable arcs (see [6]; also [4], [5]). In particular, Fremlin shows in [6] that there is a $\lambda < \infty$, so that for each point z in the medial axis there is a ball $B(z, r)$ so that any point $w \in B(z, r)$ can be connected to z by a path of length $\leq \lambda\delta$.

If Ω is a polygon with n sides or if Ω is a union of n disks, then the medial axis is a finite tree with at most $O(n)$ vertices and whose edges are either straight lines or parabolic arcs (these only occur for polygons). It is easy to see by a limiting argument that if we prove Theorem 1.1 for one of these special classes (with a uniform bound) then it follows for general simply connected domains with the same bound. Therefore, in what follows, the reader may assume Ω has one of these forms (or may add any details needed to apply the argument to the general case).

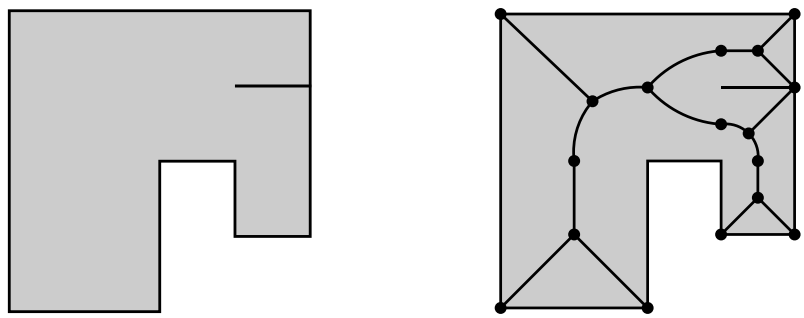

FIGURE 2. A polygon and its medial axis.

If T is a connected subset of the medial axis, let Ω_T be the union of medial axis disks with center in τ. This is a simply connected subregion of Ω and (see [2])

$$\tilde{\ell}(\partial\Omega_T) \leq \tilde{\ell}(\partial\Omega). \tag{3.1}$$

We call this the length decreasing property of the medial axis.

Given a medial axis disk D of Ω let $C = C_D$ be the hyperbolic convex hull (in D) of $E = \partial D \cap \partial\Omega$. This is simply the region bounded by replacing each arc in $\partial D \setminus E$ by a circular arc in D with the same endpoints and perpendicular to ∂D. See Figure 3. One can easily check that these sets are disjoint for distinct medial axis disks. Thus there can be at most countably many with interior, i.e., there are at most a countably many vertices of the medial axis.

Given a point p of the medial axis we also define a "thickened" version of the convex hull of $\partial D_p \cap \partial\Omega$ adding a crescent of angle θ along each face (if the convex hull is an arc, we add a crescent along both sides) . See Figure 4. Such a piece will be used as the root of our decomposition.

For an interior edge point of the medial axis, the set E has exactly two points and C is the circular arc in D with endpoints E and perpendicular to ∂D. Given an angle $0 < \theta < \pi/2$ we will let γ_θ to be the circular arc with the same endpoints but making angle θ with ∂D. Thus $\gamma_{\pi/2} = C$. If $\theta \neq \pi/2$ then there is some ambiguity about which of two possible arcs we mean. However, if we choose a fixed basepoint p_0 on the medial axis, then for any distinct medial axis non-vertex point, the corresponding

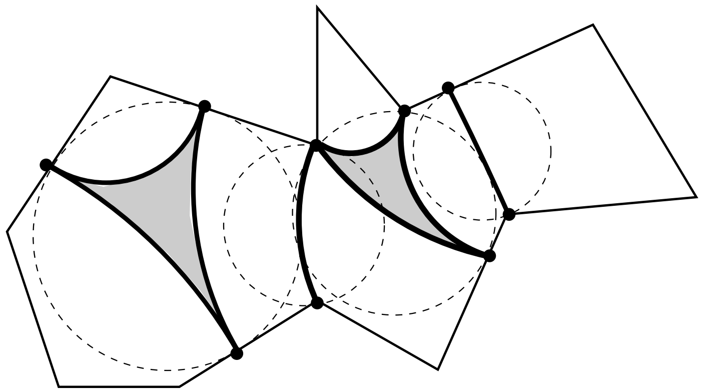

FIGURE 3. The convex hulls of $\partial D \cap \partial\Omega$ can either be arcs or hyperbolic polygons. Disjoint points of the medial axis give disjoint convex hulls (disjoint in Ω) they may have common points on the boundary).

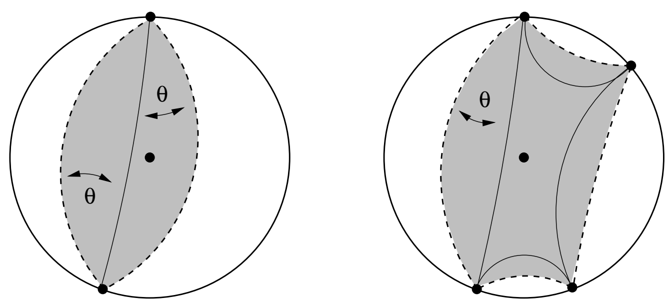

FIGURE 4. These show thickenings of the convex hull by adding crescents of angle θ along each boundary geodesic of the convex hull. If the root is not a vertex, then its medial axis disk meets $\partial\Omega$ in two points and the decomposition piece is a crescent with internal angle 2θ. If the root is a vertex, then the decomposition piece has several sides.

geodesic γ_θ divides Ω into two components only one of which hits p_0. We choose γ_θ to be in the other component, i.e., γ_θ "bends away" from p_0. See Figure 5.

If p is a vertex point in the medial axis, then the set $E = \partial D \cap \partial\Omega$ is a closed set with at least three points. If p is not the root, then $\partial D \setminus E = \cup_j I_j$ is a union of at least three (and possibly countably many) open intervals, and exactly one of these has the property that it is closest to the root p_0 in the sense that there is a crosscut with the same endpoints separates the other intervals from the root in Ω. Let this special interval be denoted I_0. For each of the remaining intervals, I_j, define a circular crosscut on D with the same endpoints as I_j and making angle θ with I_j

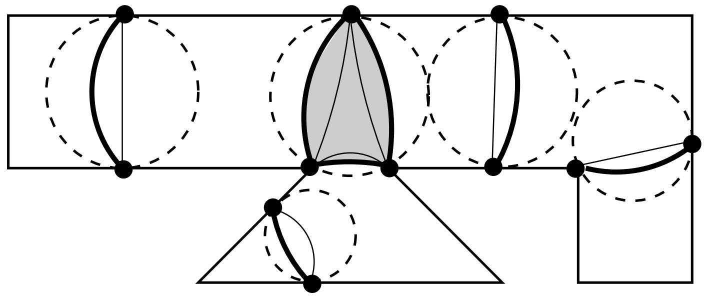

FIGURE 5. We cut the domains by circular crosscuts that have their endpoints in the set $\partial D \cap \partial\Omega$, where D is the medial axis disk corresponding to a chosen division point. The crosscut is not a geodesic in D but it makes a fixed angle with the geodesic and "bends away" from the root of the medial axis. In the picture the thin crosscuts are the hyperbolic geodesics and the thicker ones are the "bent" crosscuts.

and bending away from the root as above. The union of these circular crosscuts will be the γ_θ corresponding to the vertex point p.

The general idea is as follows. Fix some angle $\theta \in [\frac{1}{4}\pi, \frac{3}{8}\pi]$, $\epsilon > 0$ and a root p of the medial axis. The root of our decomposition is the thickened convex hull corresponding to p. We will define a distance function on the medial axis and use it to partition the medial axis into subtrees of diameter $\simeq \epsilon$. For each division point between adjacent subtrees we insert the crosscut γ_θ described above, giving the other pieces of the decomposition. In the remaining sections we explain:

- How to parition the medial axis.
- Why each decomposition piece is a Lipschitz crescent.
- Why the total length of the crosscuts is $O(\ell(\partial\Omega))$.

4. How to choose the subtrees

The most natural distance on the medial axis might be the hyperbolic metric of Ω, but it turns out that using this to construct our decompositions pieces leads to "bad shapes". In order to get "nice shapes", we introduce a more complicated distance defined in terms of the angles. The first step is to define an angle between two circular arcs (which don't necessarily intersect each other).

Suppose I and J are circular arcs (or line segments). Apply a Möbius transformation σ to both arcs, chosen so that I is mapped to the real segment $[-1, 1]$. If I is already a line segment, we take σ linear. If I is an arc of circle C, then we take σ so that it maps the point q of C that is opposite the center of I to ∞. Then we define the angle between I and J at $w \in J$ to be the angle $\sigma(J)$ makes with the horizontal at $\sigma(w)$. The angle between I and J is the maximum angle $\sigma(J)$ makes with the horizontal. See Figure 6.

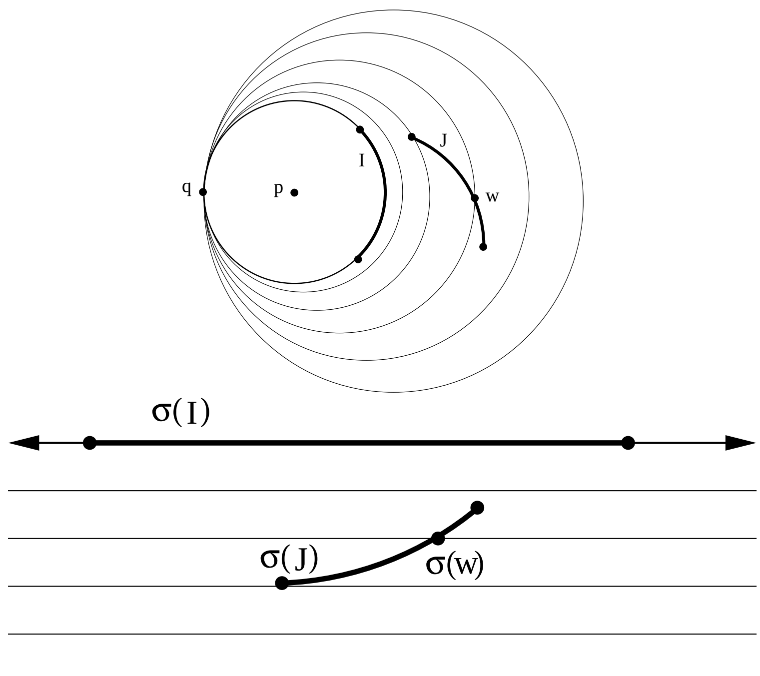

FIGURE 6. We define angles with respect to a family of circles tangent to D at a point opposite from I. If we map the disk to a half-plane this just becomes the angle between J and the horizontal.

Next we use this notion of angles to define a function on pairs of points z, w in the medial axis. Let D_z, D_w be the corresponding medial axis disks. Note that $\partial D_z \cap \Omega$ has at least two components, exactly one of which hits D_w or separates D_w from z. This arc is called the near arc of z with respect to w and its complement in ∂D_z is called the far arc of z with respect to w. Let I be the near arc of w with respect to z and let J be the far arc of z with respect to w (see Figure 7). Let $d(w, z)$ be the angle between I and J, as defined above and let $D(w, z) = \sup d(w, x)$ where the supremum is over all points x on the unique path in the medial axis between x and z. This is the function we will use to decompose the medial axis.

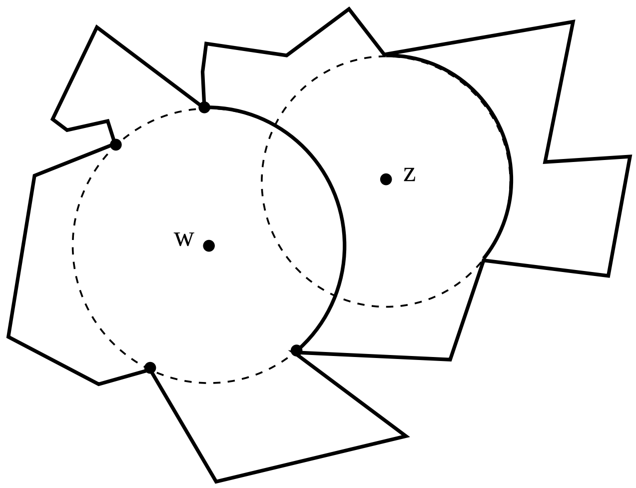

FIGURE 7. If w, z are in the medial axis $d(w, z)$ is the angle of the far arc of ∂D_z with respect to w with respect to the near arc of ∂D_w with respect to z.

Choose a root p of the medial axis. This point our first subtree. Removing it breaks the medial axis into connected components. Suppose $\epsilon > 0$ is fixed. For each connected component, we take all the points z so that $D(p, z) \leq \epsilon$. By the definition of D, this is a connected set. At each step, we choose connected components of points whose D-distance from what we have already chosen is $\leq \epsilon$. This breaks the medial axis into countably many subtrees. Each subtree has leaves that are either (1) leaves of the medial axis or (2) interior points of the medial axis. In former case we do nothing and in the latter case we divide Ω using the corresponding γ_θ (which is a single circular arc unless z is a vertex of the medial axis, when it is a collection of such arcs). The idea is illustrated in Figure 8. Figure 9 shows why we take $\theta < \pi/2$; if $\theta = \pi/2$ then cusps may form in certain situations.

Given the domain Ω and its medial axis, we want to subdivide the medial axis into subtrees with disjoint interiors and then cut Ω with circular crosscuts, one for each point were two subtrees meet. The idea is illustrated in Figure 8.

It is useful to note that $D(w, z)$ is a Lipschitz function with respect to the hyperbolic metric. This is because in the normalized situation, moving z by a small amount δ in the hyperbolic metic can change the angle of the bottom edge by only $O(\delta)$. This means that at each stage of our construction our subtrees cover a fixed

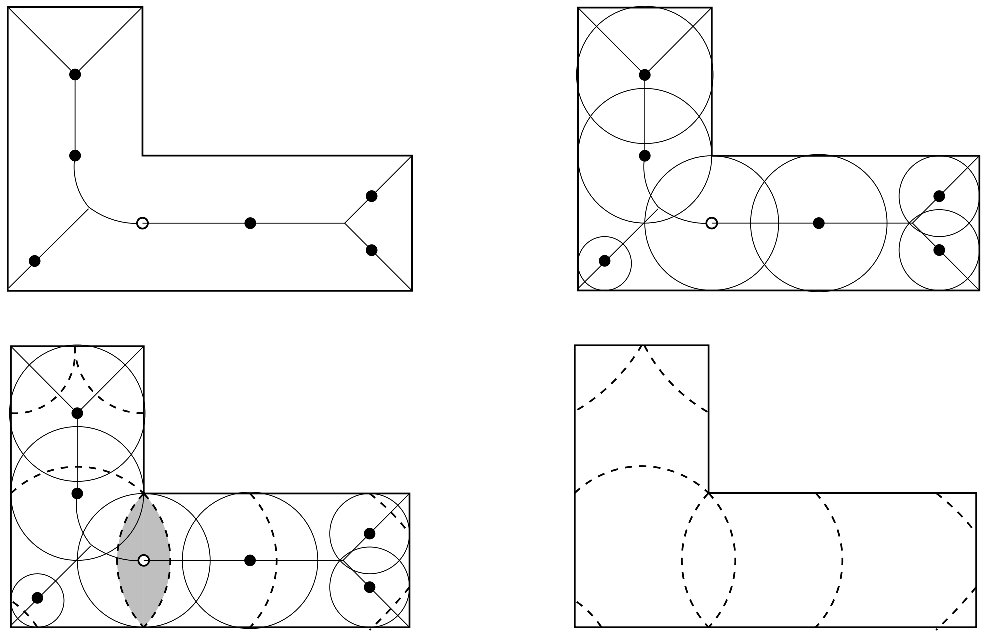

FIGURE 8. The top left shows a polygon, its medial axis and a collection of points that divide the medial axis into subtrees. The one chosen as root has a white interior. On the top right we show the medial axis disks corresponding to these points. On the lower left are the "bent" geodesics we use for crosscuts. Note the shaded region which is a thickened convex hull and is the decomposition piece corresponding to the root. On the lower right we erase the medial axis and disks to show just the resulting decompositions.

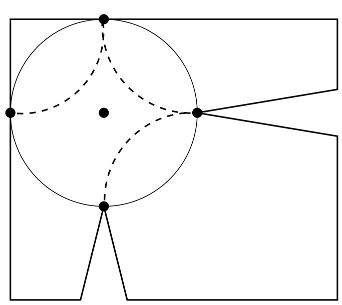

FIGURE 9. Here is a situation when we want to avoid using crosscuts that are perpendicular to the corresponding medial axis circles. If a leaf of the subtree is a vertex of the medial axis, then we may add crosscuts to two bottom arcs that are adjacent and then the resulting piece will have a cusp and the crosscuts will not be uniformly separated in the hyperbolic metric.

hyperbolic neighborhood of the previous step, which imples that eventually we cover the whole medial axis (the medial axis is path connected).

5. Decomposition pieces are Lipschitz crescents

In this section we will show that the decomposition pieces $\{W_n\}$ we have constructed are Lipschitz crescents. This is easy to see for the root piece, so we will only deal with non-root pieces. Each such piece corresponds to a subtree of the medial axis which is rooted at the point p_n closest to the root of the medial axis. Lipschitz crescents are invariant under Möbius transformations, so it suffices to show W_n is such a crescent after we normalize by mapping the medial axis disk D_n centered at p_n to the upper half-plane. After this normalization, W_n looks like the region illustrated in Figure 10.

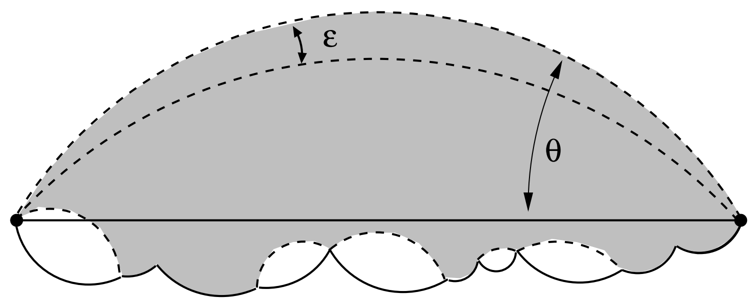

Figure 10. Here is a typical decomposition piece where we have normalized the medial axis disk of the root of the subtree by sending it to the upper half-plane. The large dashed arc at the top is the crosscut corresponding to the the root of the subtree. Along the bottom are arcs corresponding to the other leaves of the subtree (dashed if the leaf is an interior point of the medial axis, solid if it is also a leaf of the medial axis). The bottom arc makes angle with the horizontal which is bounded between $-\epsilon$ and $\theta - \epsilon$. The top and bottom arcs of W_n are separated by a crescent of fixed angle ϵ.

When we normalize, we see that the each piece has a top edge which is the circular crosscut corresponding to the root of the subtree. We claim that the lower edge of W_n is a Lipschitz graph. This is certainly true on the circular arc crosscuts corresponding to stopping points (because we stopped the first time the angle $= \epsilon$ anywhere on the arc). On the other hand, at any other points of the lower edge correspond to paths on the medial axis where we never stopped, and such a point is the tip of a cone with sides of angle ϵ (with the horizontal). See Figure 11. When we add in the crosscuts γ_θ to get the decomposition piece W, the crosscuts form an angle between $-\epsilon$ and $\theta - \epsilon$ with the horizintal.

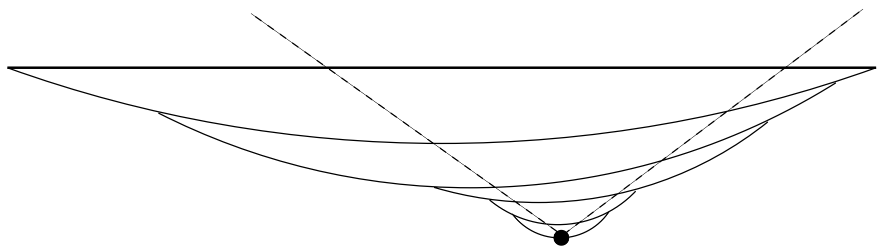

FIGURE 11. At a boundary point of the normalized piece were we never stop, consider the union of medial axis disks corresponding to the path leading to this point. Each bottom edge of each corresponding disk makes angle $\leq \epsilon$ with the horizontal and hence the point is vertex of a cone in the piece with sides of angle ϵ. Thus the bottom edge of W is a Lipschitz graph.

Finally, we only have to check that the top and bottom edges are separated by a crescent of angle ϵ. See Figure 10. Let C be the crescent of angle ϵ with endpoints ± 1 and top edge equal to the top edge of W. We claim that $C \subset W$, i.e., the bottom edge of W does not hit this crescent. By taking a finite approximation of the medial axis we can assume the lower edge of W is a finite union of circular arcs and each is either an arc of $\partial\Omega$, or a circular crosscut that was added at a leaf of the subtree. If the arc is a boundary arc of Ω then it lies in the lower half-plane and so does not hit C (assuming $\epsilon < \theta$). If the arc corresponds to an interior point of the medial axis, then there is a point in this arc where it makes angle ϵ with the horizontal. This point must be an endpoint of the arc, and then the corresponding arc γ_θ makes angle $\theta - \epsilon$ with the horizontal at this point. If this arc crosses into the upper half-plane, then at the points where it crosses, it must make angle $\leq \theta - \epsilon$ with the horizontal. Thus it lies beneath the circular arc from -1 to 1 which makes angle $\theta - \epsilon$ with the real axis, as claimed. From this is is clear that the normalized domain W is a Lipschitz crescent, as desired. By definition, any bounded Möbius image is also a Lipschitz crescent, so the unnormalized piece is as well.

6. A length estimate

Suppose we have a decomposition of the medial axis into subtrees $\{T_n\}$ where T_0 is the "root" and consists of a single point. Let $\Omega \setminus \Gamma$ denote the corresponding decomposition of Ω using the crosscuts γ_θ described earlier.

For any n, we let Ω_n be the union of all medial axis disks centered in T_n. For each n we let W_n be the component of $\Omega \setminus \Gamma$ which is inside Ω_n. These are the pieces of our decomposition. We let γ_n denote the crosscut which separates W_n from its parent. If $n \neq 0$, let $U_n = \Omega_n \setminus \Omega_{n^*}$ where n^* denotes the index of the parent domain of Ω_n (i.e., Ω_{n^*} separates Ω_n from Ω_0). ∂U_n has one circular arc edge in common with its parent U_{n^*}. We will call this the top edge of U_n and is denoted τ_n. Note that τ_n and γ_n are both circular arcs with the same endpoints, but that they bound a crescent with interior angle θ. Because the angles used to define the crosscuts in Γ have angles bounded between $\frac{1}{4}\pi$ and $\frac{3}{8}\pi$, then length of the crosscut γ_n dividing W_n from its parent is at most a uniform multiple of the distance between its endpoints. This distance, in turn, is at most the length of τ_n (possibly much shorter in some cases), i.e.,

$$\ell(\gamma_n) = O(\ell(\tau_n)). \tag{6.1}$$

Note that $\tau_0 = \emptyset$ since the root piece has no top edge. See Figure 12. The domain U_n is introduced because it will be easier to estimate its boundary length, and then use this to control the boundary lengths of our decomposition pieces, W_n.

The rest of ∂U_n is called the "bottom" edge and will be denoted β_n. This is a Jordan arc. By the length decreasing property of the medial axis we have

$$\ell(\tau_n) \leq \ell(\beta_n). \tag{6.2}$$

Fix some $\delta > 0$. We say that U_n is *boundary-like* if

$$\ell(\beta_n \cap \partial\Omega) \geq \delta \ell(\beta_n), \tag{6.3}$$

and is *interior-like* otherwise. Since $\ell(\tau_n) \leq \ell(\beta_n))$, in boundary-like pieces we also have

$$\ell(\beta_n \cap \partial\Omega) \geq \delta \ell(\tau_n). \tag{6.4}$$

We will bound the length of Γ using:

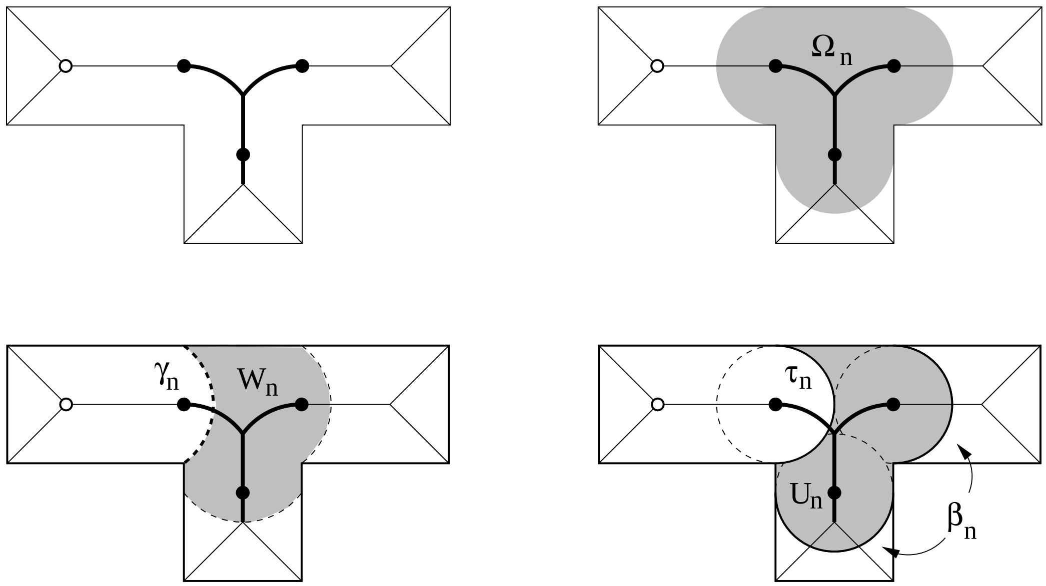

FIGURE 12. The upper left shows a domain and its medial axis. The selected root is shown as a white dot. A subtree T_n has been indicated by the darker edges. The top right shows the union of medial axis disks for this subtree; this is Ω_n. The lower left shows the corresponding decomposition piece W_n. It is bounded by arcs that deviate from the hyperbolic geodesic with the same endpoints by a fixed angle. Note that they bend away from the root. The lower right shows U_n.

Lemma 6.1. *Suppose that every non-root, interior-like subdomain U_n satisfies*

$$\ell(\beta_n) \geq (1+\delta)\ell(\tau_n). \tag{6.5}$$

Then $\sum \ell(\gamma_n) = \ell(\Gamma) \leq \frac{C}{\delta}\tilde{\ell}(\partial\Omega)$ *for some absolute* $C < \infty$.

Proof. Assume that the subdomains $\{W_n\}$ are indexed so that $n^* < n$ for all n (i.e., every subdomain comes somewhere after its parent in the list). Let $V_n = \cup_{k \leq n}\Omega_n$. Then $V_0 \subset V_1 \subset \cdots \cup_n V_n = \Omega$ and $\tilde{\ell}(\partial V_0) \leq \tilde{\ell}(\partial V_1) \leq \cdots \leq \tilde{\ell}(\partial\Omega)$ by (3.1). By (6.1) it is enough to show that

$$\sum_k \ell(\tau_k) \leq \frac{C}{\delta}\ell(\partial\Omega).$$

We will prove this by induction. Our hypothesis will be

$$\sum_{k\leq n} \ell(\tau_k) \leq \frac{1}{\delta}[\tilde{\ell}(\partial V_n) + \sum_{k \leq n} \ell(\beta_k \cap \partial\Omega)]. \tag{6.6}$$

This will suffice since $\tilde{\ell}(\partial V_n) \leq \tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega)$ by (3.1), and

$$\sum_{k\leq n} \ell(\beta_k \cap \partial\Omega) \leq \tilde{\ell}(\partial\Omega) \leq 2\ell(\partial\Omega).$$

The induction hypothesis (6.6) is trivial for $n = 0$ since the left hand side is zero (recall that $\tau_0 = \emptyset$). Assume the hypothesis for n and consider $n+1$. If U_{n+1} is boundary-like, then by the induction hypothesis, (6.4) and (3.1) we have,

$$\begin{aligned}\sum_{k\leq n+1} \ell(\tau_k) &\leq \ell(\tau_{n+1}) + \sum_{k\leq n} \ell(\tau_k) \\ &\leq \frac{1}{\delta}\ell(\beta_{n+1} \cap \partial\Omega) + \frac{1}{\delta}[\tilde{\ell}(\partial V_n) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)] \\ &\leq \frac{1}{\delta}[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n+1} \ell(\beta_k \cap \partial\Omega)],\end{aligned}$$

as desired. If U_{n+1} is interior-like, then by (6.5),

$$\ell(\tau_{n+1}) \leq \frac{1}{1+\delta}\ell(\beta_{n+1}) \quad = \quad \frac{1}{1+\delta}(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n) + \ell(\tau_n)),$$

so

$$(1 - \frac{1}{1+\delta})\ell(\tau_{n+1}) \leq \frac{1}{1+\delta}(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n)),$$

which gives

$$\ell(\tau_{n+1}) \leq \frac{1}{\delta}(\tilde{\ell}(\partial V_{n+1}) - \tilde{\ell}(\partial V_n)).$$

Hence

$$\begin{aligned}\sum_{k\leq n+1} \ell(\tau_k) &\leq \ell(\tau_{n+1}) + \sum_{k\leq n} \ell(\tau_k) \\ &\leq \frac{1}{\delta}(\tilde{\ell}(\partial V_{n+1}) - \tilde{e}ll(\partial V_n)) + \frac{1}{\delta}[\tilde{\ell}(\partial V_n) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)] \\ &\leq \frac{1}{\delta}[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n} \ell(\beta_k \cap \partial\Omega)] \\ &\leq \frac{1}{\delta}[\tilde{\ell}(\partial V_{n+1}) + \sum_{k\leq n+1} \ell(\beta_k \cap \partial\Omega)].\end{aligned}$$

Thus in either case, the induction hypothesis is verified and the lemma is proven. □

7. Decomposition pieces have the correct length bounds

To finish the proof of Theorem 1.1, it is now enough to check that (6.5) is satisfied for the non-root, interior-like pieces. We first need some simple geometric facts.

Lemma 7.1. *Suppose γ is a circular arc contained in the upper half-plane and contains at least one point where the tangent makes angle $\geq \epsilon > 0$ with the horizontal. Then it makes an angle $\geq \epsilon/2$ with the horizontal on at least $1/3$ of its length.*

Proof. If γ is a line segment, there is nothing to do since the slope is constant. Otherwise γ is an arc of a proper circle and subtends some angle θ with respect to the center of this circle. There are two arcs of this circle where the tangent makes angle $\leq \epsilon/2$ with the horizontal and each subtends angle ϵ with respect to the center of the circle. Each is separated by arcs of angle measure $\epsilon/2$ from the arcs of the circle where the angle to the horizontal is $\geq \epsilon$. Thus γ either has angle $\geq \epsilon/2$ on its whole length or it contains a subarc of angle measure $\geq \epsilon/2$ on which it makes angle $\geq \epsilon/2$ with the horizontal. This arc must account for at least 1/3 of its length, so we are done. □

Lemma 7.2. *Suppose γ is a circular arc contained in the upper half-plane and contains at least one point where the tangent makes angle $\geq \epsilon > 0$ with the horizontal. Then $\ell(\gamma) \geq (1 + c\epsilon^2)\ell(I)$ where I is the vertical projection of γ onto the real line and $c > 0$ is a fixed constant.*

Proof. Let $\gamma' \subset \gamma$ be the subarc where γ makes an angle of at least $\epsilon/2$ with the horizontal and let I' be its projection. Then $\ell(I') \geq \ell(I)/3$, so

$$\ell(\gamma) = \ell(\gamma \setminus \gamma') + \ell(\gamma') \geq \ell(I \setminus I') + \ell(I')/\cos(\epsilon^2) \geq \ell(I)(1 + c\epsilon^2).$$

□

Lemma 7.3. *Suppose $0 < \epsilon \leq 1/4$ and let $R = [0, 1] \times [0, \epsilon]$ and suppose $\epsilon \leq \frac{1}{2}$ and suppose γ is a circular arc in the upper half-plane, which is part of a circle centered in the lower half-plane and which connects the two short sides of R within R. Then γ makes an angle of at most $O(\epsilon)$ with any horizontal line.*

Proof. This is left to the reader. □

Lemma 7.4. *Suppose σ is a Möbius transformation that fixes both -1 and 1 and maps 0 to $-iR$ (i.e., σ is an elliptic rotation around ± 1 with angle $\leq \pi/2$). Let γ be a circular arc in the rectangle $[-1/R, 1/R] \times [-\eta/R, 0]$ which makes an angle $\geq \epsilon$ with the horizontal at some point and whose σ-image is in the lower half-plane. Let I be the vertical projection of γ onto the real line. Then there is a $c > 0$, so that $\ell(\sigma(\gamma)) \geq (1 + c\epsilon^2)\ell(\sigma(I))$.*

Proof. The Möbius transformation σ is of the form

$$\sigma(z) = \frac{z + \mu}{\mu z + 1},$$

where $\mu = -iR$. The derivative is $\tau'(z) = \frac{1-\mu^2}{(\mu z+1)^2}$, so

$$|\tau'(z)| = |\frac{1 + R^2}{R^2}| \frac{1}{|z - (-i/R)|}.$$

Thus $|\tau'(x - iy)|$ is an increasing function of $y \in [0, 1/R)$ for any $x \in [-1, 1]$. In particular, if γ is as in the lemma, $z = x - iy \in \gamma$, then $|\tau'(z)| > |\tau'(x)|$. Since γ makes an angle $\geq \epsilon/2$ with the horizontal along at least a third of its length, we get

$$\ell(\tau(\gamma)) \geq (1 + c\epsilon^2)\ell(\tau(I)).$$

□

So now we have to show (6.5) is satisfied for the non-root, interior-like pieces U. First we do this assuming that the corresponding decomposition piece W has been normalized as in Figure 10 (the medial axis disk of its root is the upper half-plane) and later we will verify the estimate for any Möbius image of W. Let β be the bottom edge of U and let τ be its top edge (which is a segment on the real line). By assumption, a fixed fraction δ of the length of β consists of interior arcs of Ω, and each of these arcs has a point where the angle with the horizontal is $\geq \epsilon$. By Lemma 7.2 this implies the length of the arc is strictly longer than its vertical projection onto τ by a factor depeding only on ϵ, δ. Thus a normalized U satisfies (6.5).

A general U' is simply a Möbius image of a normalized U. Since we may ignore Euclidean similarities, we can assume this transformation σ is of the from in Lemma 7.4 and the top edge of U is $\tau = [-1, 1]$. Then $\Gamma = \sigma(\tau)$ is a circular arc in the lower half-plane with endpoints ± 1. The image of β also has endpoints ± 1 and lies in the lower half-plane outside Γ. See Figure 13.

Let c be a small positive constant and define three adjacent rectangles

$$R_0 = [-c/R, c/R] \times [-c\eta/R, 0],$$

$$R_1 = [-3c/R, -c/R] \times [-c\eta/R, 0],$$

$$R_2 = [c/R, 3c/R] \times [-c\eta/R, 0].$$

The σ images of these are circular arc quadrilaterals of diameter $\simeq R$ that lie between Γ and Γ_η. See Figure 13.

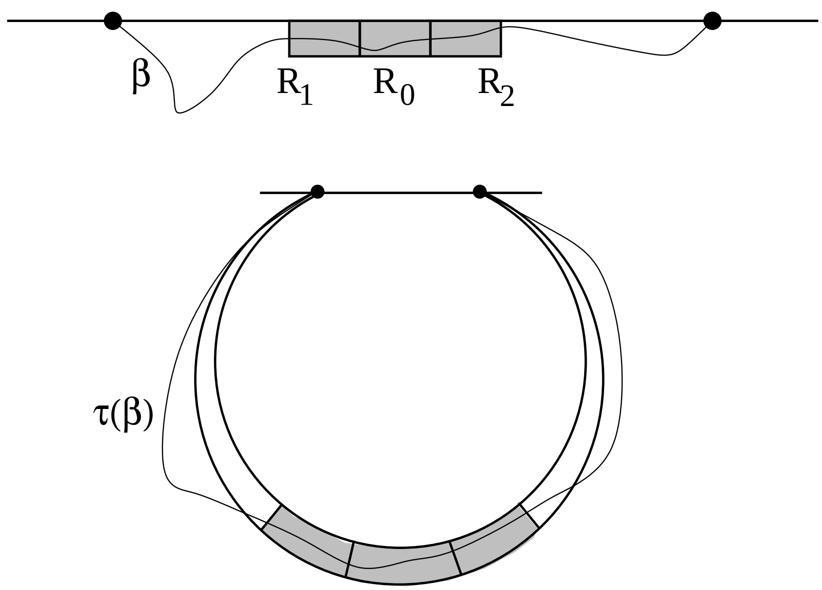

FIGURE 13. Three small rectangles map to three quadrilaterals with diameter comparable to R.

As the curve β goes from -1 to $+1$, it crosses from the vertical line $x = -2c/r$ to the line $x = 2c/R$. Either it stays entirely inside $R_0 \cup R_1 \cup R_2$ or it does not. If it does not, then β contains a point w between these lines but below the rectangles. Thus $\sigma(\beta)$ contains a point w which is at least distance $C\delta R$ from Γ. Thus the length of $\sigma(\beta)$ is at least the length of the shortest path connecting ± 1 in the lower half-plane, outside Γ and containing w. This at least $(1 + \mu)\ell(\Gamma)$, for some fixed $\mu > 0$ (see Figure 14).

Otherwise β lies entirely inside the union of the three rectangles. We may also assume β consists of a finite number of circular arcs. Suppose one of these arcs crosses R_1. Then by Lemma 7.3 it must make angle $\leq 2\pi\eta$ with the horizontal along

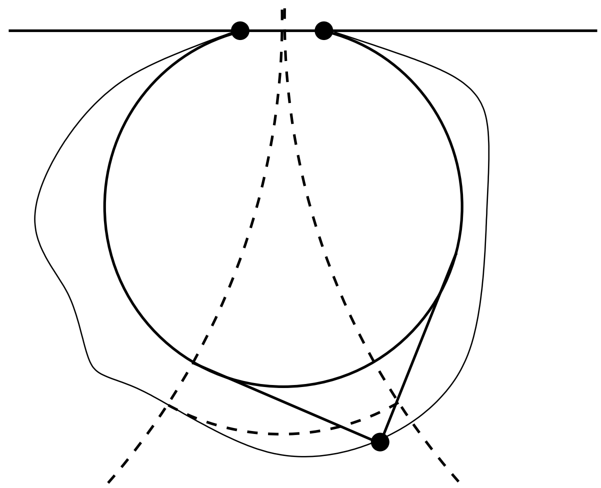

FIGURE 14. If β contains a point below $R_0 \cup R_1 \cup R_2$, then its image contians a point at least distance ηR from Γ. Then the lengh of $\tau(\beta)$ is at least the minimum length of a path in the lower halfplane connecting the endpoints of Γ and containing the point. The minimum length path looks follows Γ until it sees the point and then follows a straight line. Its easy to check this path has length $\geq (1 + c\eta^2)\ell(\Gamma)$.

its whole length. If $2\pi\eta < \epsilon$ then such an arc must correspond to a leaf of the medial axis since it does not satisfy the stopping rule we used to partition the medial axis. But the image of this arc under τ has length $\simeq R \simeq \ell(\Gamma) \geq \delta\ell(\Gamma)$ if δ is small enough, which contradicts the assumption that the piece U is interior-like. Thus no arc of β crosses R_1. Similarly, no arc crosses R_2. Thus any arc in β which hits R_0 cannot leave $R_0 \cup R_1 \cup R_2$.

Consider the union of the stopped arcs in β which hit R_0. First suppose that at most half the length of β in R_0 consists of these stopped arcs. Then just as in the previous paragraph the σ image of the complement of these arcs has length $\simeq R$ and we deduce that the piece in boundary-like, not interior-like. Thus at least half the length of β in R_0 must be from stopped arcs, so by Lemma 7.4, we are done. This proves the desired estimate and completes the proof of Theorem 1.1.

8. Further remarks

The Lipschitz crescents described in the previous proof are built using two boundary arcs which are Lipschitz graphs, so why aren't the domains themselves always Lipschitz disks? There are two things that can go wrong.

The first problem is illustrated in Figure 15. Suppose that the point labeled D is the right endpoint of $I_0 = [-1, 1]$ and the upper edge of W makes angle θ with the real axis at D. Suppose that near D, the boundary of Ω_0 looks like the dashed curves in Figure 15, each of these is a crescent of angle ϵ. We get ∂W_0 by taking circular arcs which have the same endpoints and which make angle θ with these arcs. Thus the solid arc from points A to B is an arc in W_0 and it makes angle $\theta - \epsilon$ with the horizontal at A and angle θ at B. Thus there cab be a sequence of points converging to D where the upper and lower boundaries of the crescent have the same slope. Thus there is no neighborhood of D in which ∂W_0 is a Lipschitz graph. Moreover, the domain can't be star-shaped; given any base point in the interior, there will always be points near D which can't be seen.

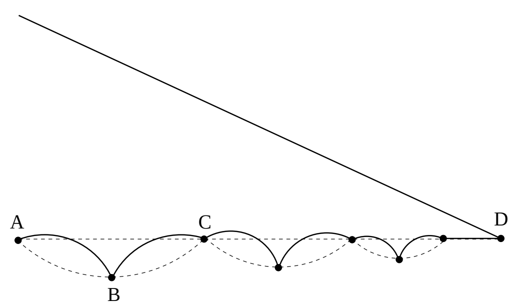

Figure 15. This shows that that the Lipschitz crescent we construct need not be a Lipschitz domain near a vertex.

The second problem is that even if W is Lipschitz with a uniform constant, a Möbius image of it need not be. Consider the case when W is a crescent with interior angle $< \pi/2$ and bottom edge equal to $[-1, 1]$. By applying an elliptic transformation fixing ± 1 and rotating by an angle slightly less than π we can map the bottom edge to a circular arc of very large diameter (as large as we want) while the top edge limits on a circular arc of fixed size. See Figure 16.

Clearly, the large crescent is not star-shaped since there is no interior point which sees both vertices. While it is true that every boundary point of the big crescent has

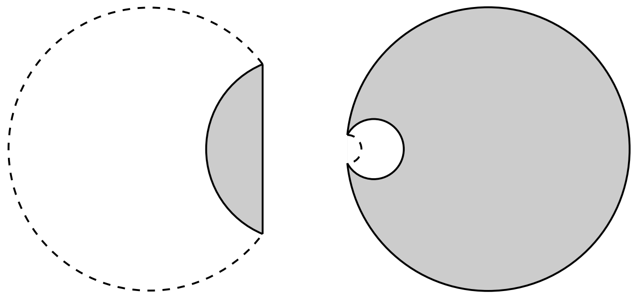

FIGURE 16. On the left is a Lipschitz crescent which is also a Lipschitz disk and on the right is a Möbius image which is not a Lipschtiz disk (but it is a Lipschitz crescent by definition).

some neighborhood in which the boundary is a Lipschitz graph, these neighborhoods can't have diameter comparable to the diameter of the whole domain (otherwise a neighborhood of one vertex would contain the other vertex). Thus we can't expect to get Lipschitz disks unless we give up the Möbius invariance of the construction.

Is it always possible to find a tree-like decomposition of a simply connected domain into Lipschitz disks so that $\ell(\Gamma) = O(\partial\Omega)$? If so, is it possible to do this using only circular arc crosscuts? Straight line crosscuts?

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