A RANDOM WALK IN ANALYSIS

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Abstract. I discuss the impact various papers have had on my own work.

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Introduction

Which papers have had a big impact on my own work? When Antonio Córdoba and José Luis Fernández asked me to write about this, I started by making a list of some topics I've worked on and drew arrows to indicate when one idea led to another. The final version of my diagram is in Figure 11 and it includes three papers besides my own: Nick Makarov's paper on the dimension of harmonic measure, Peter Jones' traveling salesman paper and Dennis Sullivan's paper on hyperbolic convex hulls. Below I'll try to explain why each of these caught my attention and how it pushed my work in new directions.

Harmonic Measure

I'm a Chicago Ph.D., but spent two years at Yale when my advisor, Peter Jones, moved there and I briefly shared an office with Stephen Semmes and Tim Steger who were Gibbs instructors. Stephen told me about his construction of a non-rectifiable closed curve such that the harmonic measures ω_1, ω_2 for opposite sides had a bounded ratio (i.e., $\log d\omega_1/d\omega_2$ is bounded. If you don't know what ω is, just think of a random path running until it hits the curve and ω is the probability distribution of that first hitting point.) His paper [58] was hard for me to follow, but while trying to sort through it, I built a curve with dimension > 1 and the same bounded ratio property (giving me the first part of a thesis). This is a useful technique: fail to understand what some smart person has done and prove a different result with a simpler technique instead. (Applying this method to Tim Steger's description of his work resulted in our joint paper [10] about Fuchsian groups, representations and rigidity.)

I told Peter about the curve when he returned from a visit to UCLA and it prompted him to share his conversations with Lennart Carleson and John Garnett about a related problem: harmonic measures ω_1, ω_2 corresponding to opposite sides of closed curve are mutually absolutely continuous on the tangent points, but what happens on the set of non-tangent points? Must the measures be singular there? Luckily, Nick Makarov had already invented the right tool to solve this problem.

The idea of Makarov's paper [51] is that harmonic measure on the boundary of a simply connected domain acts like a random walk. More precisely, if we consider a disk $D(x, r)$ with $x \in \partial\Omega$ and $r \searrow 0$, then $\log \omega(D)/r$ behaves like a random walk on $\mathbb{R}$ whose step size is related to the "flatness" of the boundary near x at scale t. At a.e. tangent point the boundary is very flat and this quantity approaches a finite limit because the steps become small. At non-tangent points we expect $\log \omega/r$ to oscillate between $+\infty$ and $-\infty$. Christian Pommerenke [55] proved $\limsup = +\infty$ soon after Makarov's paper, although $\liminf = -\infty$ took another twenty years (see the beautiful paper of Sunhi Choi [38]). The Ahlfors distortion theorem implies $\omega_1(D)\omega_2(D) = O(r^2)$, so for a disk where $\omega_1 \gg r$, we must have $\omega_2 \ll r$. Thus by Pommerenke's $\limsup = \infty$ result, ω_1 and ω_2 must be singular (written $\omega_1 \perp \omega_2$). on the non-tangent points. This gave me the second part of my thesis and a joint paper with Jones, Garnett and Carleson [9] (I still consider this paper a highlight of my career).

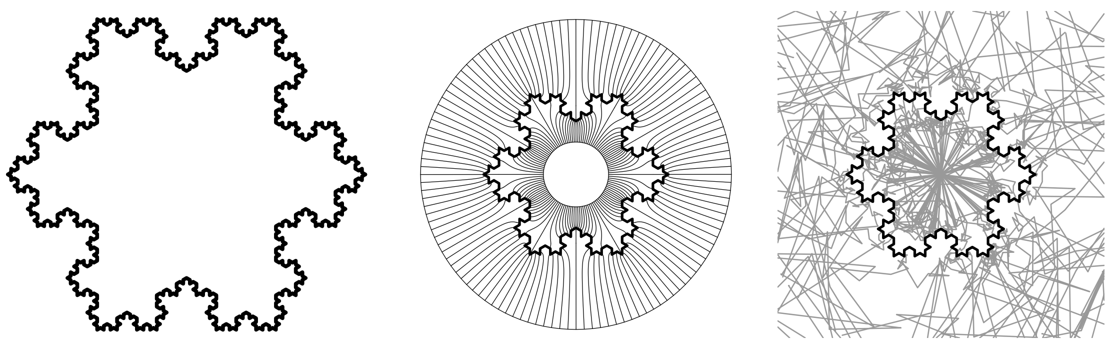

FIGURE 1. The von Koch snowflake has singular harmonic measures which we visualize in two ways. In the center we plot the images of 120 radials lines under conformal maps to the inside and outside of a 196-sided approximation of the snowflake. On the right we simulate 100 Brownian paths per side by a discrete random walk that steps the distance to the boundary. Using 10,000 such paths gives two 196-vectors whose normalized dot product is .0213 (so the vectors form an angle of 88.67°; almost perpendicular).

The final part of my thesis was an application of singular harmonic measures. If $f : \mathbb{C} \to \mathbb{D}$ is continuous and is holomorphic off a smooth curve γ, then it must be entire and hence constant (i.e., smooth curves are removable). However, using an indirect duality argument, John Wermer and Andrew Browder [33], [34], had proven that if $\omega_1 \perp \omega_2$, then there are many such non-constant functions. Moreover, every non-trivial example is "space-filling", i.e., it maps the curve to set that is the closure of its interior. Curious about what these functions looked like, I gave a new proof of the Browder-Wermer theorem using explicit constructions [12]; these methods later led to new results about function algebras [11], [19], [18], conformal welding [17], [24], [22] and Martin boundaries [14], [15].

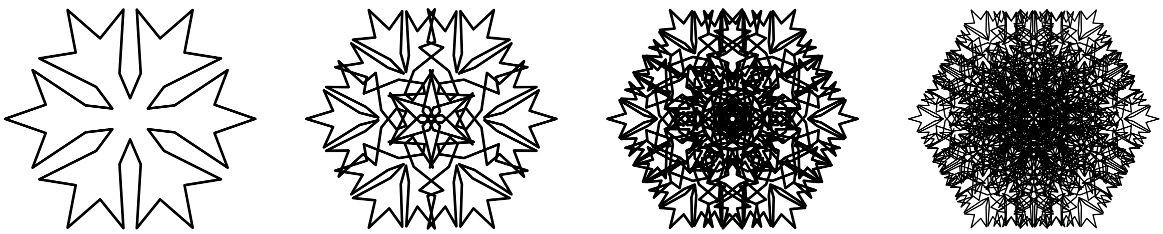

FIGURE 2. Polygons whose vertices are $v_k = 4^{-n} \sum_{j \neq k} (z_j - z_k)^{-1}$ where $\{z_k\}$ are the vertices of the nth generation of von Koch snowflake (this approximates the convolution of $1/z$ with Hausdorff measure on the snowflake). These converge to a space-filling image of the snowflake, an example of the functions given by the Browder-Wermer theorem.

Next I looked for other existence proofs that lacked an explicit construction. Don Sarason [45] had indirectly proven there are infinite Blaschke products in the little Bloch space (i.e., $|f'(z)| = o(1/1 - |z|)$ for $|z| < 1$) and had asked for an explicit example. I was able to build one [13], [16] using another idea from Makarov's paper: the radial behavior of harmonic functions on a disk is tied to the pointwise convergence of dyadic martingales on the boundary. To solve Sarason's problem, I constructed a martingale with certain smoothness properties on the unit circle and placed the zeros of the Blaschke product in Carleson boxes that corresponded to dyadic intervals where the martingale was zero.

As mentioned above, Makarov and Pommerenke proved that harmonic measure on the non-tangent points gives full mass to a set of zero length. What can we say about this set? Makarov proved it can't be too small (i.e., dimension < 1 is impossible) and Brent Øksendal conjectured that it must be big in the sense that it cannot be contained in any finite length curve. As a postdoc at MSRI and UCLA I thought a lot about this problem, but could only prove it in special cases (it's easy if Ω is a quasidisk). The difficulty is that most nice properties of a rectifiable curve γ only hold a.e.; how does a zero length subset of γ differ from a general zero length set? Fortunately, the answer became available right on schedule.

Traveling salesman and rectifiable sets

One summer I visited Peter Jones at Yale and he described his new "traveling salesman theorem" (TST) that estimates the length of the shortest path γ containing a given set E [47], [48]. For a disk $D = D(x,t)$, define

$$\beta_E(x,t) = \inf_L \sup_{z\in E\cap D} \operatorname{dist}(z,L),$$

where the infimum is over all lines L hitting D. Peter proved that

$$\ell(\gamma) \simeq \operatorname{diam}(E) + \iint \beta(x,t)^2 \frac{dxdt}{t}.$$

His proof was simplified by Kate Okikiolu [53] who extended the result to $\mathbb{R}^d$ and was extended to Hilbert space by Raanan Schul [56], [57].

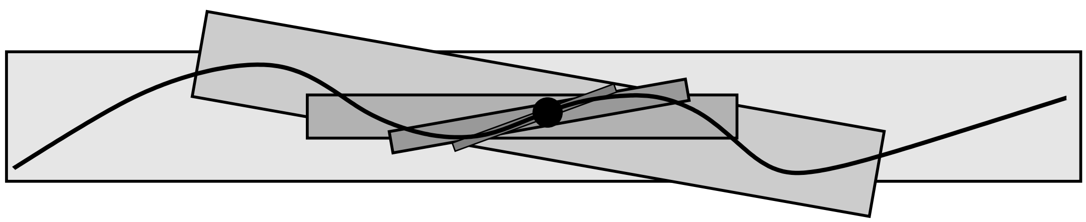

Figure 3. $\beta(x,t)$ measures the eccentricity of the narrowest rectangle covering $E\cap D(x,t)$. A curve is wiggly if $\beta > 0$ uniformly. x is a tangent point almost surely iff $\sum \beta(x,2^{-n})^2 < \infty$.

If a set E lies on a rectifiable curve, Jones' TST gives concrete bounds for how "flat" E must be and we turned these into bounds on the Green's function for the complement of E, and eventually into a proof of Øksendal's conjecture [28] and a generalization of the Hayman-Wu theorem. We wrote a sequel [29] that simplified the proof, extended work of Astala and Zinsmeister [6], [7] on BMO domains and gave an a.e. characterization of tangent points of a curve in terms the β's. See the excellent discussion in [44].

The TST allowed us to use Littlewood-Paley type estimates, but in place of the usual second derivatives of a function, our estimates involved the β-numbers and Schwarzian derivatives (the usual second derivative measures deviation from a linear function, the β's measure deviation from a line and Schwarzians measure deviation from a linear fractional transformation). The basic idea in Jones' TST is that sets can be analyzed by quadratic sums just as functions can be, and this fact distinguishes

Euclidean space in a way that I don't fully understand, but can illustrate with an example. At the 2005 Ahlfors-Bers colloquium in Ann Arbor, Juha Heinonen reminded me of the question of whether every A_1 weight on the plane is comparable to the Jacobian of some planar quasiconformal (QC) map. This problem is in his "33 Yes/No problems" paper [46] with Stephen Semmes, so I had seen it before, but I hadn't thought it was "up my alley". However, Juha's comments made me realize a counterexample would follow from a zero area set E with the property that every small-constant QC image of E contains a rectifiable curve. I constructed a Sierpinski carpet E where the holes are large enough to give zero area, but small enough (even after a QC mapping) so that we can construct rectifiable curves that avoid the holes [23] (the length is estimated using Jones' TST and the distribution of hole sizes). From this construction we can also obtain a quasisymmetric image of $\mathbb{R}^2$ in $\mathbb{R}^3$ that is not a biLipschitz image of $\mathbb{R}^2$. Hence characterizing Euclidean space up to biLipschitz equivalence is tied to understanding rectifiability and Jones' TST better.

Peter Jones and I also used his TST to prove "wiggly sets" have dimension > 1 [30] (a set is wiggly if it is connected and has β's uniformly bounded away from zero). This seems like an obvious result, but I still know no simpler proof than using the TST. Moreover, this basic result led to more subtle variations. A Brownian motion run for unit time defines a compact set in the plane and the complementary components are simply connected open sets, so their boundaries, called Brownian frontiers, are connected sets that look quite wiggly. Motivated by physical arguments, Benoit Mandelbrot had conjectured Brownian frontiers have dimension $4/3$ and this was later proven using SLE type techniques by Lawler, Schramm and Werner [49], [50]. At the time, only infinite length was known, but Peter Jones, Robin Pemantle, Yuval Peres and I were able to prove Brownian frontiers have dimension > 1. The β's are not bounded away form zero, but they do have positive probability of being non-zero with enough independence between different locations and scales to prove the result (but not without a few tricks, e.g., we used a fractal partition of the plane instead of the usual dyadic grid).

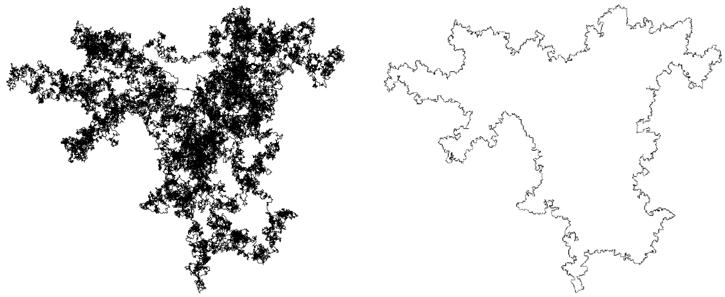

Figure 4. A Brownian path and its outer frontier. Jones' TST implied it has dimension > 1 and more recent work shows the exact dimension is $4/3$, verifying a conjecture of Benoit Mandelbrot.

Kleinian groups and the convex hull theorem

By this time I had moved to Stony Brook and started to learn about Kleinian groups. Ed Taylor, a student of Bernie Maskit, asked me if the limit set of a finite generated, geometrically infinite Kleinian group has dimension > 1, a problem that seemed related to wiggly sets. Briefly, a Kleinian group G is a discrete group of Möbius transformations acting as isometries on hyperbolic 3-space $\mathbb{H}$ (identified with the upper half-space $\mathbb{R}^3_+$). The limit set $\Lambda \subset \partial\mathbb{H} = \mathbb{R}^2 \cup \{\infty\}$ is the accumulation set of any orbit and usually has a fractal structure. $\Omega = \partial\mathbb{H} \setminus \Lambda$ is open and we define the "dome" S_Ω of Ω as the upper envelope in $\mathbb{H}$ of all hemispheres with base disk in Ω. The region above the dome is the hyperbolic convex hull of Λ, denoted $C(\Lambda)$. If G is finitely generated then $S_\Omega = \partial C(\Lambda)$ has finite hyperbolic area mod G, but $C(\Lambda)/G$ itself may have either finite or infinite hyperbolic volume. These cases are called geometrically finite and infinite respectively. Ed Taylor's question was a weaker version of a well known conjecture that limit sets of geometrically infinite groups must have dimension 2. Like Brownian frontiers, Kleinian limit sets need not be uniformly wiggly, but in the finitely generated case there are only countably many points at which β tends to zero, so it was possible prove $\dim(\Lambda) > 1$ using TST [30].

Eventually, I was able to prove $\dim(\Lambda) = 2$ as well. The critical exponent δ of a Kleinian group measures the exponential rate growth of the G-orbits (there are at most $O(e^{\delta n})$ orbits points in any hyperbolic ball of radius n). Peter Jones and I showed that $\delta \leq \dim(\Lambda)$ for any Kleinian group, so the conjecture reduces to the case when $\delta < 2$. Dennis Sullivan [60] had related δ to the base eigenvalues for the

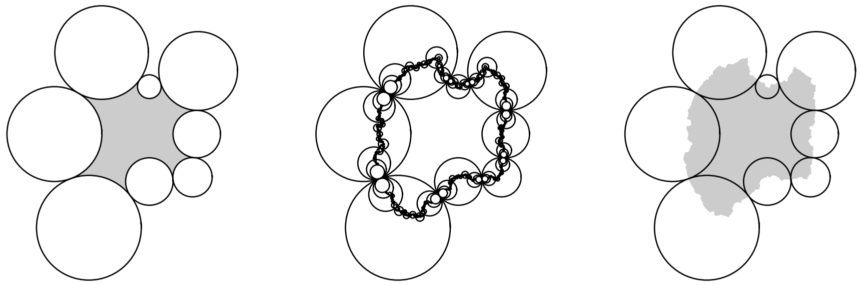

FIGURE 5. Here is a Kleinian limit set generated by circle reflections. The group is finitely generated and the limit set is not a circle, so must have dimension > 1. However, the β's are zero where generating circles touch and along the orbits of such points.

Laplacian on a hyperbolic manifold $M = \mathbb{R}^3_+/G$, and using this I showed that if $\delta < 2$ and G is geometrically infinite, then a Brownian motion started inside $C(\Lambda)$ has a positive probability of never crossing $\partial C(\Lambda)$. This implies Λ has positive area, hence $\dim(\Lambda) = 2$. (In fact, $\text{area}(\Lambda) > 0$ is impossible for finitely generated groups by later work of Danny Calegari, David Gabai [35] and Ian Agol [1] proving the Ahlfors measure conjecture.)

This Brownian motion argument uses the fact mentioned earlier that S_Ω/G has finite hyperbolic area because we estimate the probability a Brownian motion leaves $C(\Lambda)$ by integrating heat kernel bounds over $S_\Omega = \partial C(\Lambda)$. This fact is a consequence of the Ahlfors finiteness theorem (hyperbolic $\text{area}(\Omega/G) < \infty$) and Dennis Sullivan's convex hull theorem (CHT) [59]: Ω is biLipschitz equivalent to S_Ω with a universal bound K. The CHT holds for any simply connected domain, as established by David Epstein and Al Marden [40], [41]. Peter Jones and I avoided quoting the CHT by using an alternate argument in the $\dim(\Lambda) = 2$ paper, but it was not long before I needed to understand the CHT much better.

A Fuchsian group G is a Kleinian group that preserves the unit disk, $\mathbb{D}$. A deformation of G is a conformal map $f : \Omega \to \mathbb{D}$ that conjugates G to a Kleinian group $G' = f^{-1} \circ G \circ f$ acting on Ω. If the group is cocompact (i.e., $R = \mathbb{D}/G$ is compact) then Rufus Bowen [32] proved $\partial\Omega$ is either a circle or has dimension > 1. This is "Bowen's dichotomy". Dennis Sullivan extended it to cofinite groups (R has finite area), and Kari Astala and Michel Zinsmeister [3], [2], [5] showed it fails whenever G

is convergence type (R is a surface with a Green's function). This left open the case when R has infinite area but no Green's function (divergence type groups).

Thurston had observed that the hyperbolic path metric on the dome S_Ω is isometric to the hyperbolic unit disk (geometrically, the dome is just a hyperbolic disk that has been folded along certain geodesics). Composing Sullivan's map $\sigma : \Omega \to S_\Omega$ with this isometry gives a hyperbolically biLipschitz (hence QC) map from Ω to $\mathbb{D}$ with uniform constants. We call this the iota map. I observed (perhaps others had as well) that iota is locally Lipschitz $\Omega \to \mathbb{D}$ and deduced a factorization theorem: any conformal map $f : \Omega :\to \mathbb{D}$ is the composition of a locally Euclidean Lipschitz QC map $\varphi : \Omega \to \mathbb{D}$ and a hyperbolically biLipschitz map $\psi : \mathbb{D} \to \mathbb{D}$, both with uniform constants (assuming Ω has inradius ≥ 1).

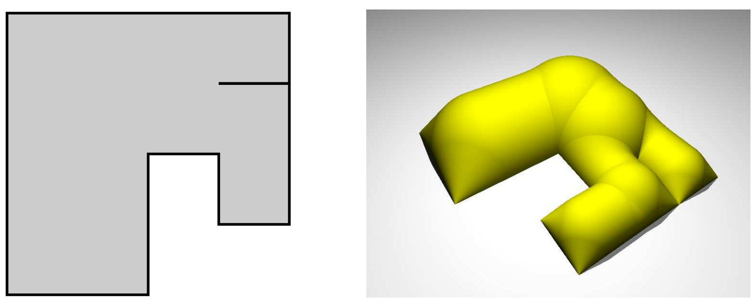

FIGURE 6. A polygon and its dome. The dome is the upper envelope of all hemispheres with base disk inside the polygon. The centers of hemispheres touching the dome form the medial axis of the polygon, a well studied object in computational geometry.

Why does this help with Bowen's dichotomy? Suppose we have a non-circular deformation of a divergence type group G. We can show the β's for $\partial\Omega$ are large a.e. with respect to harmonic measure, but we need them large on positive length to get dim > 1. Since Makarov showed harmonic measure can be concentrated on a zero length set, the strategy seems to fail, but Sullivan's CHT saves the day. The factorization theorem implies that a conformal deformation of G via f is also a QC deformation of the divergence type group $G' = \psi \circ G \circ \psi^{-1}$ via the map φ (divergence type is a QC invariant by Pfluger, [54]). Moreover, $\psi^{-1} : \mathbb{D} \to \Omega$ is locally expanding; this implies the β's are large on positive length, as desired [20].

The factorization theorem implies that if $f : \Omega \to \mathbb{D}$ is conformal, then $|f'| = |\varphi'| \cdot |\psi'|$ where $\varphi' \in L^\infty$ and ψ' is in weak-L^p for $p = 2K/(K-1)$ by a celebrated result

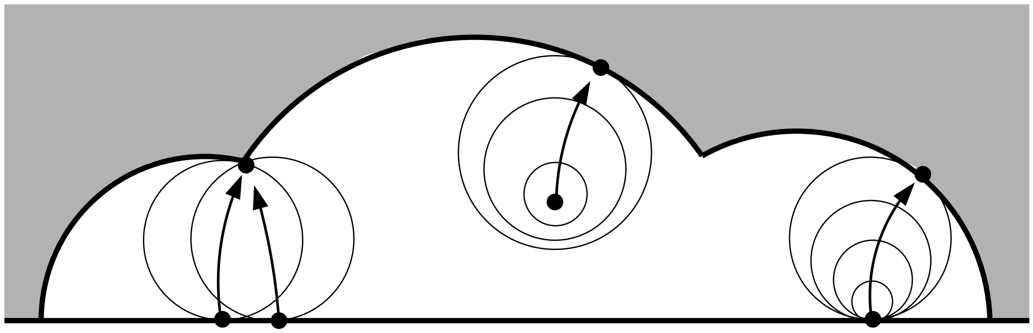

FIGURE 7. Here is a "side view" of a hyperbolic dome. The nearest point retraction is defined by expanding a hyperbolic ball until it hits the dome. For points in the base domain we expand a horoball instead. This defines a quasi-isometry, giving Sullivan's theorem. The map is not necessarily a homeomorphism since two distinct points can map to the same point.

of Kari Astala [4]. This is reminiscent of Brennan's conjecture that $f' \in L^{4-\epsilon}(dxdy, \Omega)$ for any $\epsilon > 0$. In fact, if Sullivan's theorem held with constant $K = 2$, Brennan's conjecture would follow. This motivated me to try to give the best explicit constant I could. In [21] I proved $K < 7.82$ by carefully examining the Epstein-Marden proof of CHT in [40]. Unfortunately, Epstein and Markovic found a logarithmic spiral domain for which $K > 2.1$, [42]. It is still possible that every simply connected domain has a 2-QC, locally Lipschitz map to the disk (this would imply Brennan's conjecture), but iota itself doesn't always work.

Computational conformal geometry

For polygons, the Riemann map is given by the Schwarz-Christoffel formula, but this involves unknown parameters, namely, the points on the circle that get mapped to the polygon's vertices. Solving for these can be quite difficult. On the other hand, the iota map can be applied to every vertex of an n-gon in time $O(n)$. This depends on the close relation between the dome of a domain and its medial axis. The medial axis is a term from computer science [31] that refers to the the centers of subdisks of Ω whose boundaries hit $\partial\Omega$ in ≥ 2 points (Erdös [43] called the same set M_2 twenty years earlier). The medial axis of an n-gon can be computed in time $O(n)$ by a result of Chin, Snoeyink and Wang [36], [37] and iota can be computed in linear time from the medial axis. Thus iota gives a "fast" map to the disk that is uniformly close to conformal by the CHT. Dennis Sullivan told me he originally thought of the CHT as a "constructive version of the Riemann mapping theorem".

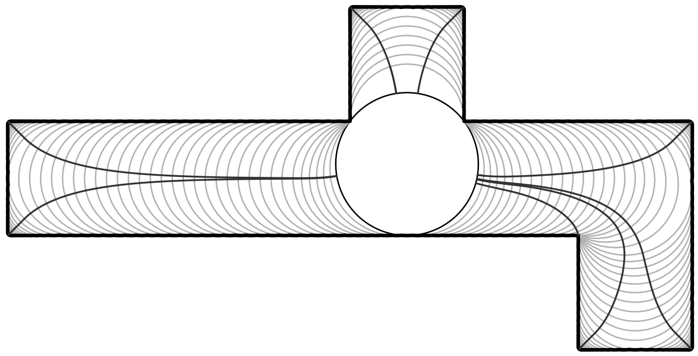

FIGURE 8. A polygon is foliated by arcs of medial axis disks; the orthogonal flow gives the iota map from the polygon to a circle (it can also be computed algebraically).

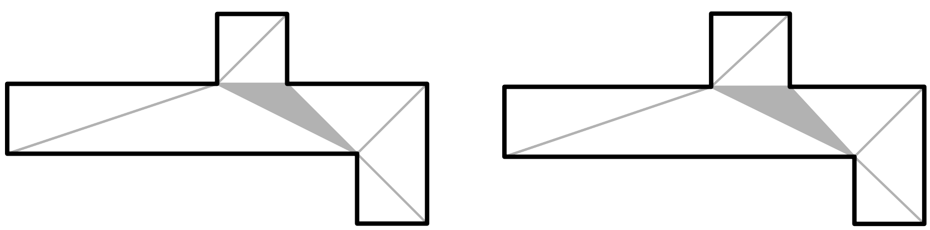

FIGURE 9. A polygon and the Schwarz-Christoffel image using the correct angles but pre-vertices guessed using iota. By the CHT there is a K-QC, vertex preserving map between the them. We can get an upper bound for K by triangulating both polygons and computing the maximum dilatation of the corresponding piecewise linear map; $|\mu| \leq .108$ is this case. The most distorted triangle is shaded.

In fact, the iota map was also discovered in numerical analysis, but under a different name. While trying to numerically compute the best K in Sullivan's theorem, I came across the paper [39] by Toby Driscoll and Steve Vavasis. It describes their CRDT algorithm, a numerical conformal mapping method that uses a map from simple n-gons to n-tuples on the circle defined in terms of cross ratios and the Delaunay triangulation of the polygon (hence the name) and while reading this paper (for the fourth or fifth time), I realized the CRDT map was a version of iota and I was able to prove the same uniform QC bounds as for the "real" iota [25]. (Actually, Vavasis had sent me a preprint of the CRDT paper a few years earlier, but I hadn't appreciated it without knowing about CHT and iota, and had forgotten about it. My discussions with Driscoll and Vavasis after I "rediscovered" their paper led to a workshop, a joint grant with Vavasis and several results about domain decomposition and conformal maps.)

CRDT uses the Schwarz-Christoffel formula, so each iteration gives a conformal map onto an approximate domain. The algorithm tries to improve this domain at each step, but the dependence on the parameters is so subtle that no proof of convergence is known (at least to me). Failing to prove CRDT converges, I tried a different approach: consider QC maps $\mathbb{D} \to \Omega$ and iterate by approximately solving a Beltrami equation that lowers the QC constant at each step. This method can compute a $(1+\epsilon)$-QC map from the disk onto any n-gon in time $O(n \log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon})$ [26]. The maps are held in memory using $O(n)$ series, each of length $p = \log \frac{1}{\epsilon}$. The iteration has quadratic convergence, so using iota as a starting point, Sullivan's CHT implies only $O(\log\log \frac{1}{\epsilon})$ iterations are needed to reach accuracy ϵ, independent of the domain.

The CHT is used in other parts of this algorithm as well. A key ingredient is the idea of a thick/thin decomposition of a polygon analogous to the thick/thin decomposition of a Riemann surface. Thin parts of a polygon are certain generalized quadrilaterals with a pair of sides whose extremal distance inside the polygon is less than ϵ. Decomposing a polygon into its thick and thin parts makes various mapping and meshing problems easier to understand. The iota map and Sullivan's CHT allow us to compute extremal distances (up to a bounded factor) in linear time and this leads to a linear time algorithm to find all the thin parts.

Marshall Bern and David Eppstein (two "p"'s this time; not the same David Epstein mentioned before) had proven in [8] that any simply n-gon has a quadrilateral mesh with $O(n)$ elements and no angle bigger than $120°$ (and this is sharp). They asked if a lower angle bound was possible, and using thick/thin decompositions and the mapping theorem above, I showed [27] we could also take all new angles $\geq 60°$ (small angles in the original polygon must remain).

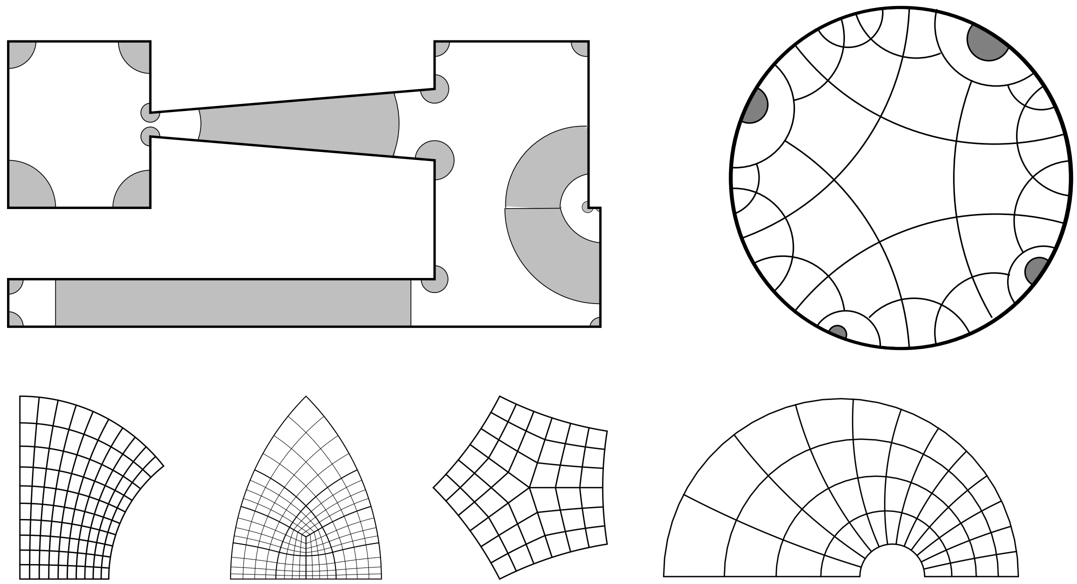

FIGURE 10. We decompose a polygon into thick and thin pieces (white and shaded respectively). Thin parts are meshed "by hand" and the Riemann map of the polygon sends each thick part to a region that can be subdivided into four types of hyperbolic polygons as shown. Each type has a mesh with angles in $[60^\circ, 120^\circ]$ that we transfer back to the thick part by the conformal map.

This quickly leads to a more general problem. A planar straight line graph (PSLG) is any finite, disjoint collection of line segments and points (polygons are a special case where the edges meet end-to-end). A mesh of a PSLG is a mesh of its convex hull whose vertices and edges covers all the vertices and edges of the PSLG. I was able to show that any PSLG has a quadrilateral mesh with $O(n^2)$ elements and the same angle bounds as above (and n^2 is sharp). By adding diagonals to the quadrilaterals, we get a $O(n^2)$ triangulation of any PSLG with all angles $\leq 120^\circ$, improving the bound 157.5° by Scott Mitchell [52] and 132° by Tiow-Seng Tan [61]. In fact, 120° can be replaced by any bound $> 90^\circ$ (but the constant in $O(n^2)$ grows) and there is even a polynomial algorithm for nonobtusely triangulating a PSLG (all angles $\leq 90^\circ$). The proof uses thick/thin decompositions and a foliation of the thin parts similar to those used by Epstein and Marden in their proof of CHT. In each thin part, the leaves are just circular arcs, but when joined together the leaves can become quite complicated. If every path hits at most $O(n)$ thin parts, we get an $O(n^2)$ nonobtuse triangulation. In general, I show that by adding $O(n^{1.5})$ extra paths and bending the original paths slightly we can cause collisions which terminate every path after crossing at most $O(n)$ thin parts; this gives an $O(n^{2.5})$ triangulation. The best lower bound is $O(n^2)$, so a gap remains open (I either need to understand CHT a bit better or it is time for another serendipitous result to appear; Dennis Sullivan suggested looking at closing lemmas in dynamics).

That's the story so far: Nick Makarov's paper helped me write my thesis and led to various problems including Øksendal's conjecture; Peter Jones' traveling salesman theorem was the key to solving that conjecture and involved me with Brownian motion, geometric measure theory and Kleinian groups; the $\dim(\Lambda) = 2$ problem for limit sets led me to Dennis Sullivan's convex hull theorem, which then solved Bowen's dichotomy and pushed me towards new results in numerical conformal mappings and computational geometry. Each paper was first useful because it contained a fact I needed, but their real value lay in the new problems they inspired.

Postscript

This essay is an edited version of an even more rambling previous attempt. Trying to compress it further, I projected into a lower dimension rhyming space:

Under logs are measures walking
along paths with betas stalking
over domes of bounded bending
questions answered and unending

Projecting into the even lower dimensional haiku space gives

flatness abandoned
deep origamic thunder
echos off my pen

This may be useful if NSF proposal limits ever drop from 15 pages to 17 syllables.

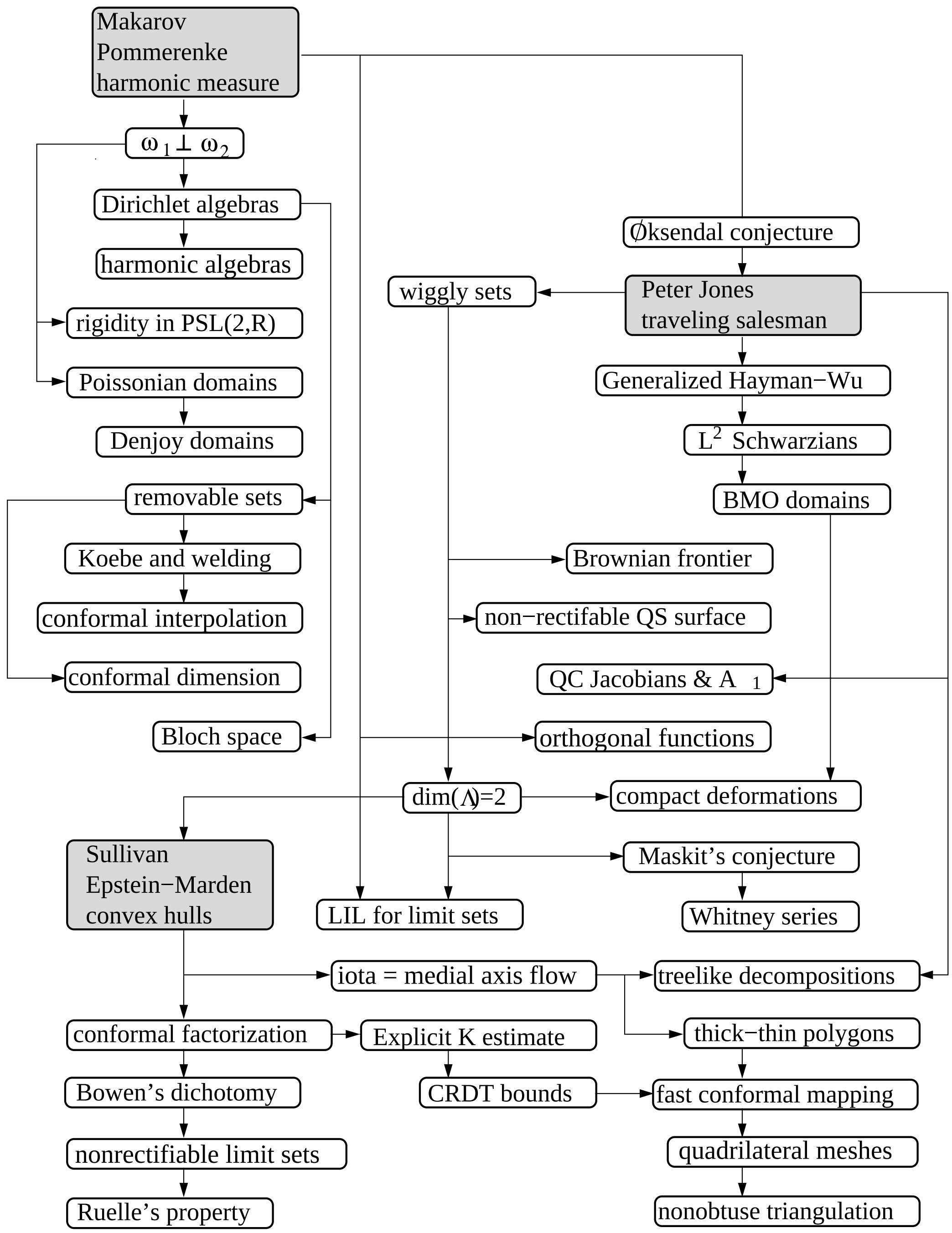

FIGURE 11. Some of my work as a directed graph

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