BOUNDARY INTERPOLATION SETS FOR CONFORMAL MAPS

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ABSTRACT. We show that a compact subset of the unit circle is an interpolation set for conformal mappings iff it has logarithmic capacity zero.

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1. Introduction

Suppose $E \subset \mathbb{T}$ is a compact subset of the unit circle and $f : \mathbb{D} \to \Omega \subset \mathbb{R}^2$ is a conformal mapping which extends continuously to E. Which mappings of E can occur in this way? Since f is a homeomorphism of $\mathbb{D}$, there is an obvious topological restriction: level sets $f^{-1}(z)$ and $f^{-1}(w)$ must either coincide or be contained in disjoint arcs of $\mathbb{T}$. The purpose of this note is to show that this is the only restriction iff E has logarithmic capacity zero.

Theorem 1. *Suppose $E \subset \mathbb{T}$ is compact. Then the following are equivalent.*

1. *E has logarithmic capacity zero.*
2. *Given any homeomorphism $g : \mathbb{D} \to \Omega \subset \mathbb{R}^2$ which extends continuously to $\mathbb{T}$, there is a conformal map $f : \mathbb{D} \to \Omega$ which extends continuously to $\mathbb{T}$ and $f|_E = g|_E$.*
3. *Given any continuous map $g : E \to \mathbb{R}^2$ such that $z \neq w$ implies $g^{-1}(z)$ and $g^{-1}(w)$ lie in disjoint arcs of $\mathbb{T}$, there is a conformal map $f : \mathbb{D}$ into Ω which extends continuously to $\mathbb{T}$ and $f|_E = g|_E$.*

We will also prove a fourth equivalent condition that requires a few more definitions to state. A decomposition of compact set K is a collection $\mathcal{C}$ of pairwise disjoint closed sets whose union is all of K. A decomposition $\mathcal{C}$ of K is called upper semi-continuous if a collection of elements which converges in the Hausdorff metric must converge to a subset of another element. A decomposition of the unit circle is called separated if any two distinct sets are contained in disjoint intervals. We say the decomposition is realized by a function $f : K \to S^2$ if it consists of the level sets $\{f^{-1}(z) : z \in S^2\}$ of f. If $K \subset \mathbb{T}$ and f is the boundary values of a conformal map on $\mathbb{D}$, we will call the decomposition conformal. It would be very interesting (and probably very difficult) to characterize which decompositions of subsets of the circle are conformal; as a special case this contains the difficult problem of determining which circle homeomorphisms are conformal weldings (see e.g., [6], [1]). However, we can do this if the subset is small enough.

Theorem 2. *A compact set $E \subset \mathbb{T}$ has zero logarithmic capacity iff*

4. *A decomposition of E is conformal iff it is upper semi-continuous and separated.*

That any of (2), (3) or (4) implies (1) follows from a well known result of Beurling: a univalent map cannot be constant on a set $E \subset \mathbb{T}$ of positive logarithmic capacity. Also (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) follows from topological results of R.L. Moore (we will discuss these more precisely in Section 3). Thus the main point for us will be to show (1) $\Rightarrow$ (2). This will follow from the following special case:

Theorem 3. *Suppose $E \subset \mathbb{T}$ is a closed set of zero logarithmic capacity and $h : \mathbb{T} \to \mathbb{T}$ is an orientation preserving homeomorphism. Then there is a conformal map $f : \mathbb{D} \to \Omega \subset \mathbb{D}$ onto a Jordan domain Ω so that $f|_E = h|_E$.*

The proof of this follows from Evan's theorem from potential theory and an explicit geometric construction. Indeed, if we only want a quasiconformal mapping (with a uniform constant) which does the interpolation, then we can essentially draw a picture of it. To obtain a conformal map we combine this picture with an iterative construction which solves a Beltrami equation at each step to keep the map conformal. This proof will be given in Section 2; Theorem 1 will be proven in Section 3.

Given a closed Jordan curve Γ, let Ω and Ω^* denote the bounded and unbounded complementary components and let $\mathbb{D} = \{|z| < 1\}$ and $\mathbb{D}^* = \{|z| > 1\}$. A closed set will be called a conformal welding set if for every orientation preserving homeomorphsim $h : \mathbb{T} \to \mathbb{T}$ there is a closed Jordan curve Γ and conformal maps of $f : \mathbb{D} \to \Omega$ and $g : \mathbb{D}^* \to \Omega^*$ such that $f = g \circ h$ for all $x \in E$. It is known that the unit circle is not such a set (e.g., see Remark 1 of [1]), but it is an immediate consequence of Theorem 3 and Theorem 8 of [1] that

Corollary 4. *Every compact set $E \subset \mathbb{T}$ of zero logarithmic capacity is a conformal welding set.*

Are there conformal welding sets of positive logarithmic capacity? Results in [1] imply that every orientation preserving homeomorphism of the circle satisfies $f = g \circ h$ for maps corresponding to some Γ on some set E of positive logarithmic capacity, so the question is whether this set can be chosen independent of h.

Finally, we mention that the interpolation results in Theorem 1 imply various similar results. For example, the separation condition in (3) is trivially satisfied by any homeomorphism g of E into the plane, so there is a conformal map $f : \mathbb{D} \to \Omega$ onto a Jordan domain so that $f|_E = g$. Moreover, given any Cantor set F in the plane, we can always write it as the homeomorphic image $F = g(E)$ of some zero capacity Cantor set E on the circle. Given any homeomorphism h of F we can also apply the theorem to $h \circ g : E \to h(F)$ to deduce

Corollary 5. *Suppose $F \subset \mathbb{R}^2$ is any compact, totally disconnected set and $h : F \to F' \subset \mathbb{R}^2$ is any homeomorphism. Then there is a Jordan domain Ω with $F \subset \partial\Omega$ and a conformal map $f : \Omega \to \Omega'$ to another Jordan domain such that $f|_F = h$.*

Part of this paper was written during a visit to the Mittag-Leffler institute and I thank the institute for its hospitality and the use of its facilities. The content of this paper originally appeared in the preprint [1] on conformal welding; Theorem 3 was used in the proof of the main result there, but was later made unnecessary by an alternate approach. I thank Bob Edwards and Mladen Bestvina for pointing out Moore's theorem on quotients of the plane to me. I also thank David Hamilton for reading the first draft of [1] and generously providing numerous helpful comments: historical, mathematical and stylistic. I also appreciate the encouragement and comments I received at various stages from Kari Astala, John Garnett, Juha Heinonen, Nick Makarov, Vlad Markovic, Bruce Palka, Stefan Rohde and Michel Zinsmeister.

2. Interpolation of circle homeomorphisms: proof of Theorem 3

In this section we prove Theorem 3. The proof is quite geometric; we can easily describe in pictures a quasiconformal map which takes the required boundary values. However, if we then solve the Beltrami equation to make the map conformal, we may alter the boundary values. Thus our proof is an inductive construction in which each step involves building an explicit quasiconformal map which "almost" interpolates, followed by a quasiconformal correction to make it conformal.

It is convenient to move to the upper half plane. The following result does not require Ω to be Jordan, but we will fix this in Corollary 8.

Theorem 6. *Suppose $E \subset \mathbb{R}$ is a compact set of zero logarithmic capacity and $h : E \to \mathbb{R}$ is a non-decreasing map. Then there is a conformal map $f : \mathbb{H} \to \Omega \subset \mathbb{H}$ such that $f|_E = h|_E$.*

Proof. The starting point is Evans' theorem (e.g., Theorem E.2 of [5], Theorem 5.5.6, page 156 of [11], or Chapter 7 of [7]) which states that any compact set of zero logarithmic capacity supports a probability measure μ so that the potential

$$u(z) = \int_{\mathbb{R}} \log \frac{1}{|z-x|} d\mu(x),$$

tends to $+\infty$ at every point of E.

Let $v(z)$ be the harmonic conjugate of u in $\mathbb{H}$ such that $\lim_{x \to -\infty} v(x) = 0$ and $\lim_{x\to\infty} v(x) = \pi$. By symmetry u has normal derivative zero on $\mathbb{R} \setminus E$ and hence v is constant on each component of $\mathbb{R} \setminus E$ (more explicitly, v is given on $\mathbb{R} \setminus E$ by $v(x) = \pi\mu(-\infty, x)$. We will call a domain a slit strip domain if it is of the form $S \setminus \cup_n L_n$, where $S = \{x+iy : -\infty < x < \infty, 0 < y < 1\}$ is the strip and $L_n = \{x + iy_n : x_n \leq x < \infty\}$ are the horizontal slits, for some sequence of points $\{(x_n, y_n)\} \subset S$ such that all the y_n's are distinct and the $x_n \to \infty$. See Figure 1. We will also assume $x_n \geq 1$ for all n. Then $f_0 = \frac{1}{\pi}(u+iv)$ is a conformal map of $\mathbb{H}$ onto a slit strip, with $u(z) \to +\infty$ as $z \to E$. Let $W \subset S$ denote this slit strip.

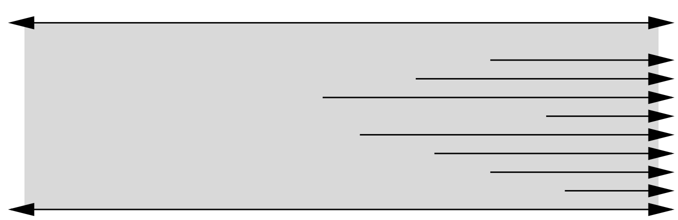

FIGURE 1. A slit strip domain

In order to prove the theorem, it clearly suffices to assume $E \subset [0,1]$ and $h([0,1]) \subset [0,1]$. Given any increasing map $h : [0,1] \to [0,1]$ are going to build a conformal map $f : W \to \mathbb{H}$ so that $\lim_{x\to\infty} f(x+iy) = h(y)$ as long as $y \notin \{y_n\}$. For future convenience we will think of h as being defined on all of W by $h(x+iy) = h(y)$.

Given a real number s let $H(s)$ denote the horizontal line $\{x+iy : y = s\}$ and let $V(s)$ denote the vertical line $\{x+iy : x = s\}$. If $I \subset \mathbb{R}$ then $H(I,s) = \{x+iy : x \in I, y = s\} \subset H(s)$. Similarly for $V(I,s) \subset V(s)$. Given $s < t$ let $W(s,t) = W \cap \{x+iy : s \leq x \leq t\}$. Given an interval $I \subset [0,1]$ let $W(I,s,t) = W(s,t) \cap \{x+iy : y \in I\}$.

Given $s \in \mathbb{R}$ let $\{I_j^s\}$ be the (finitely many) components of $W \cap V(s)$. Let $I_j^s(t) \subset V(t)$ be the orthogonal projection of I_j^s onto $V(t)$. Let J_j^s be the smallest closed interval (possibly a point) containing $h(I_j^s)$ (which might not be connected because h need not be continuous).

Given a sequence of real numbers $\epsilon_n \searrow 0$ we will construct sequences $\delta_n \searrow 0$ and $s_n \nearrow \infty$ and conformal maps f_n of $W_n = W(-\infty, s_n)$ into $\mathbb{H}$ such that if $I_j^n = I_j^{s_{n-1}}$, $J_j^n = J_j^{s_{n-1}}$ and a_j^n is the center of J_j^n, then

1. $\|f_n - f_{n-1}\|_\infty \leq \epsilon_n$ on W_{n-1}.
2. $f_n(I_j^n)$ is an interval of length $\delta_n|I_j^n|$ on the horizontal line $H(2^{-n-1})$ whose center projects vertically to a point less than ϵ_n away from a_j^n.
3. $f : I_j^n \to f(I_j^n)$ is quasisymmetric with bounds independent of n and j.

To start the induction we map $W(-\infty, 0)$ to the vertical strip $\Omega_0 = \{(x+iy) : 0 < x < 1, y > 1\}$ by a Euclidean similarity.

In general, suppose we have defined our sequences up to $n-1$. We will define a conformal map on W_n by first defining a quasiconformal map g_n on W_n and then applying the measurable Riemann mapping theorem to make it conformal. We define g_n in six steps.

Consider the intervals $\{I_j^n\} \subset W_{n-1}$. We are going to define the map g_n on each of the domains $W(I_j^n, s_{n-1}, s_n)$ (for some suitable large choice of s_n to be made later). For each j we will define a finite sequence of numbers

$$s_{n-1} = t_0 < t_1 < t_2 < t_3 < t_4$$

and define the map on $W(I_j^n, t_k, t_{k+1})$ for $k = 0, 1, 2, 3$. For convenience we will drop the n and j and write I_j^n simply as I.

Step 1: On W_{n-1} we let $g_n = f_{n-1}$.

Step 2: Choose $t_1 > t_0$ so that $W(I, t_0, t_1)$ can be quasiconformally mapped to the rectangle $\{x + iy : x \in f_{n-1}(I), \frac{7}{8}2^{-n} < y < 2^{-n}\}$ and so that g_n agrees with f_{n-1} along I and is affine on $I(t_1)$. This can be done by a map with uniformly bounded quasiconformal constant K_2 by hypothesis (3) above.

Step 3: Choose $t_2 > t_1$ large enough so that $W(I, t_1, t_2)$ can be affinely mapped to a trapezoid of height $\frac{1}{8}2^{-n}$, top edge of length $|f_{n-1}(I)|$ and bottom edge $\delta_n|I|$. See Figure 2. This can be done with quasiconformal constant K_3 which is independent of δ_n, although the choice of t_2 does depend on δ_n.

Step 4: Choose $t_3 > t_2$ so large that $W(I, t_2, t_3)$ is can be mapped to a "tube" of width $\delta_n|I|$ which connects the bottom edge of the trapezoid in the previous step to the interval of length $\delta_n|I|$ on $H(\frac{5}{8}2^{-n})$ whose center is with in $\epsilon_n/2$ of a_j^n. We can clearly choose δ_n so small that the tubes for different j's can be chosen to be disjoint

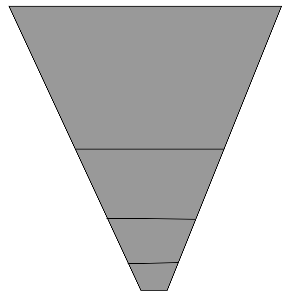

FIGURE 2. The third step

from one another. This is one condition that determines δ_n. Another will be given following Step 6. Also note that this construction can be done with quasiconformal constant K_4 which is independent of δ_n. See Figure 3.

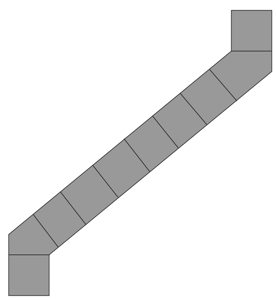

FIGURE 3. The fourth step

Step 5: Choose $t_4 > t_3$ so large that $W(I, t_3, t_4)$ can be mapped to a rectangle of height $\frac{1}{8}2^{-n}$ and width $\delta_n |I|$ and attach this to the bottom of the tube from the previous step (thus the bottom edge lies on the line $H(2^{-n-1})$). These five steps together are illustrated in Figure 4.

Step 6: Now let $s_n = \max_j t_4 = \max_j t_4(j)$. For the intervals I_j^n where this maximum is attained, do nothing. For intervals where $t_4(j) < s_n$, we replace the vertical rectangle of the fourth step above with a "constricted tube" as pictured in Figure 5, where the shape of the particular tube is chosen so that $W(I, t_4(j), t_n)$ can be mapped quasiconformally to the tube with uniformly bounded constant. We also assume the map is a similarity on the end squares. Again, the quasiconformal constant K_6 is independent of the other parameters.

This completes the definition of W_n and the construction of the quasiconformal map $g_n : W_n \to \mathbb{H}$ which extend f_{n-1}. To define f_n we take the Beltrami data associated

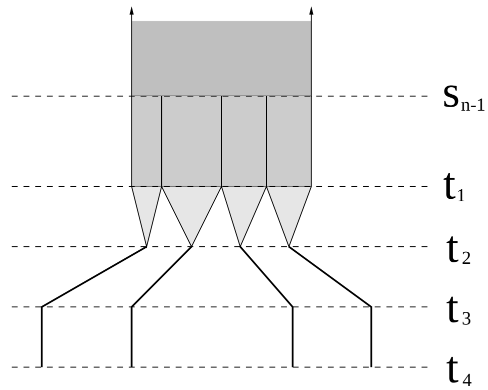

FIGURE 4. First five steps of constructing g_n

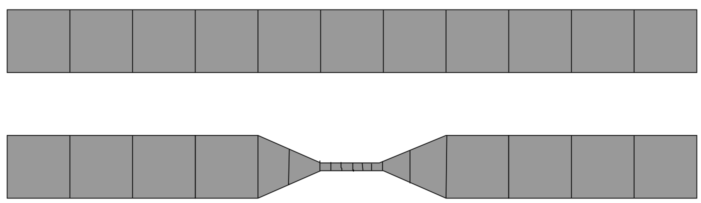

FIGURE 5. Replacing a tube by a conformally longer one

to g_n and reflect it across the horizontal line $H(2^{-n-1})$. Solve the Beltrami equation to obtain a quasiconformal map h_n which maps $H(2^{-n-1})$ to itself, and such that $f_n = h_n \circ g_n$ is conformal. Since the dilatation of g_n is uniformly bounded and is supported of a set of area $\leq 2^{-n}\delta_{n-1}$ (this is an upper bound for the area of the regions constructed in the four steps above), the map h_n is uniformly close to the identity on $\mathbb{H}$. In particular, given any ϵ_n we can choose δ_{n-1} so small that if μ is a complex dilatation of norm $\leq \lambda = (K-1)/(K+1)$, with $K = \max(K_2, K_3, K_4, K_6)$ and which is supported on a set of area $\leq 2^{-n}\delta_{n-1}$, then the corresponding solution h of the Beltrami equation satisfies $|h(z) - z| \leq \epsilon_n/2$ for all $z \in \mathbb{R}^2$. This is the final condition determining the choice of δ_n.

Conditions (1) and (2) of the induction are now clear. To see that (3) holds, note that each interval of the form $g_n(I_j^n)$ has length $\delta_n|I_j^n|$ and is at least distance $\delta_n|I_j^n|$ from the set where g_n has non-zero dilatation (by the explicit construction of the constricted tubes). This completes the induction. Passing to the limit we obtain a conformal map $f : W \to \Omega \subset \mathbb{H}$ with the desired properties. $\square$

Given a compact set $E \subset \mathbb{T}$ we will now define the associated "sawtooth" region W_E and a 2-quasiconformal map between W_E and $\mathbb{D}$ which keeps E fixed pointwise. Suppose $\{I_n\}$ are the connected components of $\mathbb{T} \setminus E$ and for each n let $\gamma_n(\theta)$ be the circular arc in $\mathbb{D}$ with the same endpoints as I_n and which makes angle θ with I_n (so $\gamma_n(0) = I_n$ and $\gamma_n(\pi/2)$ is the hyperbolic geodesic with the same endpoints as I_n). Let $C_n(\theta)$ be the region bounded by I_n and $\gamma_n(\theta)$, and let $W_E = \mathbb{D} \setminus \cup_n C_n(\pi/8)$. It is easy to check (see [1]) that we can msp $\mathbb{D}$ to W_E by a 2-quasiconformal map f which is the identity on E.

Lemma 7. *Suppose $E \subset \mathbb{T}$ is compact. Then there is a conformal map $g : \mathbb{D} \to W \subset \mathbb{D}$ onto a Jordan domain so that $g(E) = \partial W \cap \mathbb{T}$. Moreover, E has zero logarithmic capacity iff $g(E)$ does.*

Proof. Take $\varphi : \mathbb{D} \to W_E$ as above. Take the Beltrami data μ corresponding to φ and extend it across the unit circle by reflection to get a dilatation $\tilde{\mu}$. Solve the Beltrami equation $f_{\bar{z}} = \tilde{\mu} f_z$ to find a 2-quasiconformal selfmap h of the disk so that $g = h \circ \varphi$ is conformal. Since φ is bi-Lipschitz and h is bi-Hölder (since it is quasiconformal), we see that g is also bi-Hölder and hence $g(E)$ has zero logarithmic capacity. Moreover it is clear that $g(\mathbb{D})$ is a Jordan domain that only hits the circle at $g(E)$. □

Corollary 8. *If the map h in Theorem 6 is a homeomorphism then we can take Ω to be a Jordan domain such that $\partial\Omega \cap \partial\mathbb{H} = E$.*

Proof. Assume that $E, h(E) \subset [-1, 1]$ and let F be the conformal map of the vertical strip $\{x + iy : y > 0, -1 < x < 1\}$ to the hemisphere $\mathbb{D} \cap \mathbb{H}$. Let $G : \mathbb{H} \to W \subset \mathbb{H}$ be given by Lemma 7 (after conjugating from $\mathbb{D}$ to $\mathbb{H}$). Apply Theorem 6 to the map $F^{-1} \circ h \circ G^{-1}$ restricted to $G(E)$ to find a map f. Then $F \circ f \circ G$ is a conformal map of $\mathbb{H}$ to a Jordan domain in $\mathbb{H}$ which equals h on E. □

3. Boundary interpolation sets: proof of Theorem 1

In this section we will prove Theorem 1. The basic fact which we need is the following result of R.L. Moore.

Theorem 9 (Moore, [8])**.** *Suppose $\mathcal{C}$ is an upper semi-continuous collection of disjoint continua (compact, connected sets) in $\mathbb{R}^2$ each of which does not separate $\mathbb{R}^2$.*

Then the quotient space formed by identifying each set to a point is homeomorphic to $\mathbb{R}^2$.

Also see Daverman's book [2]. For an overview of Moore's life and work see [4] and [12] (reprinted in [3]). For another application of Moore's topological work (i.e. the Moore triod theorem) to conformal mappings, see Pommerenke's paper [10].

We will also need a few easier facts, starting with the following version of Lindelöf's theorem.

Lemma 10. *Suppose* $f : \mathbb{D} \to \Omega$ *is an orientation preserving homeomorphism which extends continuously to the boundary and which is non-constant on every boundary arc. If* $\gamma \subset \Omega$ *is an open arc with one endpoint on* $\partial\Omega$*, then* $\tilde{\gamma} = f^{-1}(\gamma)$ *has a well defined endpoint on* $\mathbb{T}$.

Proof. Since f is a homeomorphism, $\tilde{\gamma}$ leaves every compact subset of $\mathbb{D}$ and hence accumulates on $\mathbb{T}$. If the accumulation set is a point we are done. Otherwise it is an interval, and we deduce that γ accumulates on the image of this interval, which is not a single point by assumption. This is a contradiction and proves the lemma. □

Lemma 11. *Suppose* f, g *are both orientation preserving homeomorphisms of the disk onto a planar domain* Ω *which extend continuously to the boundary and which are non-constant on every boundary arc. Then* $h = f^{-1} \circ g$ *is a homeomorphism of the disk which extends continuously to the boundary and these boundary values are a homeomorphism of the circle.*

Proof. If h did not extend continuously, then there would be a curve γ which converges to a single boundary point, but such that $h(\gamma)$ does not. However, since g is continuous, $g(\gamma)$ does converge to a point of $\partial\Omega$ and by Lemma 10, then $h(\gamma)$ also terminates; thus h must be continuous

If h were not 1-to-1 on $\mathbb{T}$, then there would be distinct points $x, y \in \mathbb{T}$ such that $h(x) = h(y)$. If γ is a Jordan curve in $\mathbb{D}$ with endpoints at x and y (say the hyperbolic geodesic connecting them), then $h(\gamma)$ is a closed Jordan curve (both endpoints at $h(x) = h(y)$) and which encloses a region $W \subset \mathbb{D}$. Thus $f(h(\gamma)) = g(\gamma)$ is a Jordan curve in Ω with both endpoints equal on $\partial\Omega$ and whose bounded complementary component $f(W)$ is contained in Ω. But this implies the interval of $\mathbb{T}$ between x and

y is mapped to a single point by g, contrary to assumption. A continuous, 1-to-1 map of $\mathbb{T}$ to itself must be a homeomorphism, so we are done. □

The following two facts are easy and left to the reader.

Lemma 12. *Suppose $f : \mathbb{D} \to \Omega$ is a homeomorphism which extends continuously to the boundary. Then the corresponding decomposition of $\mathbb{T}$ is separated and upper semi-continuous.*

Lemma 13. *Suppose $\mathcal{C}$ is a decomposition of $\mathbb{T}$ which is separated and upper semi-continuous. Construct a decomposition $\mathcal{D}$ of the closed disk by adjoining to each (non-singleton) set $E \in \mathcal{C}$ its hyperbolic convex hull (i.e., the closure of $W_E(\pi/2)$). Then $\mathcal{D}$ is an upper semi-continuous decomposition of $\overline{\mathbb{D}}$.*

Proof of Theorem 1. We will prove (2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (2).

(3) $\Rightarrow$ (2): This is easy since if g satisfies the condition in (2) its restriction to E automatically satisfies the conditions in (3).

(2) $\Rightarrow$ (3): Suppose $g : E \to \mathbb{R}^2$ is a continuous function such that for any distinct points $z, w \in \mathbb{R}^2$, $g^{-1}(z)$ and $g^{-1}(w)$ lie is disjoint arcs of $\mathbb{T}$. Then the decomposition associated to g is separated and upper semi-continuous (by Lemma 12). Now use Lemma 13 to extend the decomposition of E to a upper semi-continuous decomposition of the plane by non-separating continua. Then by Moore's theorem, there is a homeomorphism h from the quotient space to the plane. Let $F = h(E)$. Then $H = g \circ h^{-1}$ is well defined, one-to-one and continuous on F. Hence H can be extended to a homeomorphism of the plane and so $H \circ h \circ 1/\bar{z}$ restricted to $\overline{\mathbb{D}}$ is continuous, a homeomorphism on the interior and equals g on E. By (2) there is a conformal map on $\mathbb{D}$ which equals g on E and we are done.

(2) $\Rightarrow$ (4): Suppose we have a decomposition of E which is separated and upper semi-continuous. Use Lemma 13 to extend this to upper semi-continuous decomposition of S^2 by continua which do not separate the plane. By Moore's theorem there is a homeomorphism from the quotient space to the sphere, so restricting this to $\mathbb{D}^*$ gives a homeomorphism of $\mathbb{D}^*$ which extends continuously to $\mathbb{T}$ and realizes the original decomposition. Since we are assuming (2) holds, this means there is a conformal map f equal to g on E.

(4) ⇒ (1): Take the trivial decomposition where of all of E is a single element. If there is a conformal map which realizes this, then E must have zero logarithmic capacity by a well known result of Beurling (e.g., Theorem 11.5, [9]).

(1) ⇒ (2): Suppose $E \subset \mathbb{T}$ is a compact set of zero logarithmic capacity and suppose g is a continuous map on $\overline{\mathbb{D}}$ which is a homeomorphism on $\mathbb{D}$. Let W_E be the sawtooth region associated to E (see Section 2) and let $F : \mathbb{D} \to W_E$ be the standard quasiconformal map which fixes E pointwise. By replacing g by $g \circ F$ if necessary, we may assume g is non-constant on every boundary arc (since E contains no intervals) and that $g(\Omega)$ omits some disk and hence is hyperbolic.

Let $\Omega = g(\mathbb{D})$ and let $f : \mathbb{D} \to \Omega$ be a conformal map. By Lemma 11, $f^{-1} \circ g$ is a homeomorphism of the circle and so by Theorem 3 there is a conformal map $\varphi : \mathbb{D} \to \Omega' \subset \mathbb{D}$ which equals $f^{-1} \circ g$ on E. Thus $f \circ \varphi$ is a conformal map which equals g on E, as desired. □

References

[1] Christopher J. Bishop. Conformal welding and Koebe's theorem. 2003. preprint.
[2] Robert J. Daverman. *Decompositions of manifolds*, volume 124 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986.
[3] Peter Duren, Harold M. Edwards, and Uta C. Merzbach, editors. *A century of mathematics in America. Part III*. American Mathematical Society, Providence, RI, 1989.
[4] Ben Fitzpatrick, Jr. Some aspects of the work and influence of R. L. Moore. In *Handbook of the history of general topology, Vol. 1*, pages 41–61. Kluwer Acad. Publ., Dordrecht, 1997.
[5] J.B. Garnett and D. Marshall. *Harmonic measure*. to appear.
[6] D.H. Hamilton. Conformal welding. In *The handbook of geometric function theory*. North Holland, 2002.
[7] L. L. Helms. *Introduction to potential theory*. Wiley-Interscience A Division of John Wiley & Sons, New York-London-Sydney, 1969.
[8] R. L. Moore. Concerning upper semi-continuous collections of continua. *Trans. Amer. Math. Soc.*, 27(4):416–428, 1925.
[9] Ch. Pommerenke. *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen, 1975. With a chapter on quadratic differentials by Gerd Jensen, Studia Mathematica/Mathematische Lehrbücher, Band XXV.
[10] Ch. Pommerenke. On the boundary continuity of conformal maps. *Pacific J. Math.*, 120(2):423–430, 1985.
[11] Thomas Ransford. *Potential theory in the complex plane*. Cambridge University Press, Cambridge, 1995.
[12] R. L. Wilder. The mathematical work of R. L. Moore: its background, nature and influence. *Arch. Hist. Exact Sci.*, 26(1):73–97, 1982.

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