ANOTHER BESICOVITCH-KAKEYA SET

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Theorem 1 (Besicovitch [2], [3])**.** *There is a compact set $K \subset \mathbb{R}^2$ that has zero area and contains a unit line segment in every direction.*

Proof. Let $\{a_k\}_0^\infty$ be dense in $[0,1]$ with $|a_{k+1} - a_k| \le \epsilon(k) \searrow 0$, let $g(t) = t - \lfloor t \rfloor$, set

$$f_k(t) = \sum_{m=1}^{k} \frac{a_{m-1} - a_m}{2^m} g(2^m t), \qquad f(t) = \lim_{k\to\infty} f_k(t),$$

and define $K = \{(a, f(t) + at) : a, t \in [0,1]\}$. Fixing t and varying a shows K contains unit segments of slope t for all $t \in [0,1]$. By telescoping series, $f_k(t)$ has slope $-a_k$ on every component I of $U = [0,1] \setminus 2^{-k}\mathbb{Z}$. Fix $a \in [0,1]$ and choose k so that $|a - a_k| < \epsilon(k)$. We have $f(t) - at = [f(t) - f_k(t)] + [f_k(t) - a_k t] + (a_k - a)t$, and

$$|f(t) - f_k(t)| \le \sum_{m=k+1}^{\infty} \frac{|a_{m-1} - a_m|}{2^m} g(2^m t) \le \epsilon(k) \sum_{m=k+1}^{\infty} 2^{-m} = \epsilon(k) 2^{-k},$$

hence each of the 2^k intervals I is mapped by $f(t) + at$ to a set of diameter $\le \epsilon(k)2^{-k} + 0 + \epsilon(k)|I| \le \epsilon(k)2^{-k+1}$. Thus the slice $\{(x,y) \in \overline{K} : x = a\}$ has length zero. Hence $\overline{K}$ has zero area and a union of four rotations of $\overline{K}$ proves the theorem. $\square$

This example was inspired by the random examples in [1] based on the Cauchy process. If we take $\epsilon(k) = 2^{-n}$ for $k \in [2^n, 2^{n+1})$ (as in Figures 1 and 2) then the intersection of K with the vertical strip $S_k = \{(x,y) : |x - a_k| \le \epsilon(k)\}$ can be covered by 2^{2k} squares of side length $\epsilon(k)2^{-k}$, giving total area $\epsilon^2(k)$. Thus all of K can be covered by squares of side length $\ge \delta = 2^{-n-2^{n+1}}$ with total area $\le 2^n \epsilon^2(2^n) = 2^{-n} = O(1/\log(1/\delta))$. This implies area$(\{z : \mathrm{dist}(z, K) < \delta\}) = O(1/\log(1/\delta))$, which is sharp by a result of Cordoba; see [4] or Proposition 1.5 of [1].

References

[1] Y. Babichenko, Y. Peres, R. Peretz, P. Sousi, and P. Winkler. Hunter, Cauchy rabbit, and optimal Kakeya sets. 2011. preprint available at http://arxiv.org/abs/1207.6389.

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[3] A. S. Besicovitch. On Kakeya's problem and a similar one. *Math. Z.*, 27(1):312–320, 1928.

[4] Antonio Córdoba. The fat needle problem. *Bull. London Math. Soc.*, 25(1):81–82, 1993.

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FIGURE 1. The graph of f.

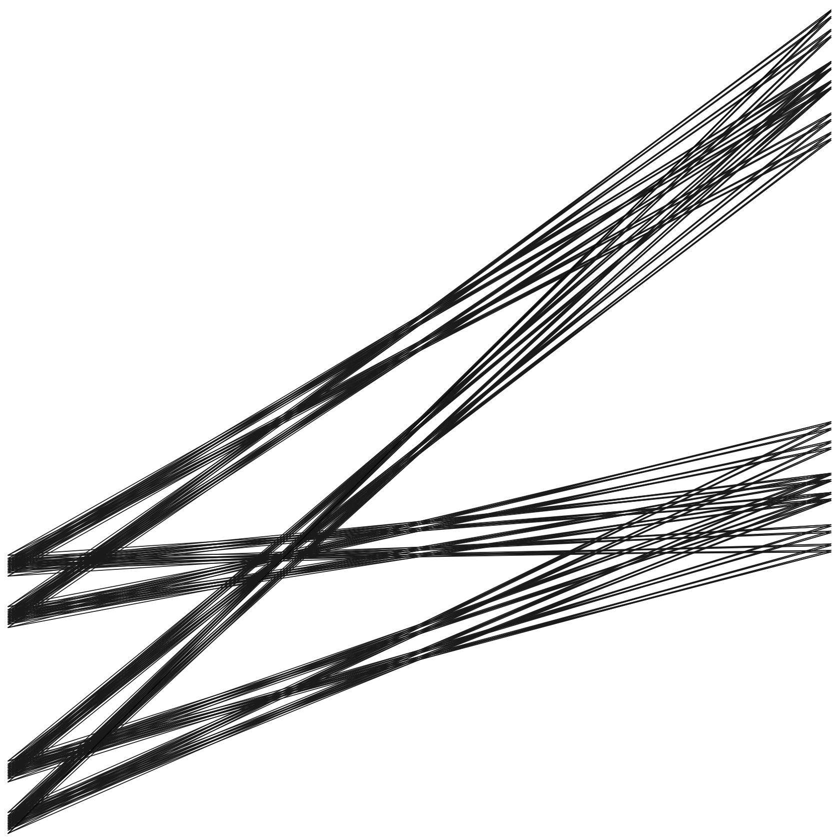

FIGURE 2. The corresponding set K.