A CENTRAL SET OF DIMENSION 2

CHRISTOPHER J. BISHOP AND HRANT HAKOBYAN

ABSTRACT. The central set of a domain D is the set of centers of maximal discs in D. Fremlin showed in [3] that the central set of a planar domain has zero area and asked whether it can have Hausdorff dimension strictly larger than 1. We construct a planar domain with central set of Hausdorff dimension 2.

1. INTRODUCTION

Let D be a domain in $\mathbb{R}^2$. A subdisc of D is *maximal* if it is not strictly contained in any other subdisc of D. The *central set* of D consists of the centers of maximal discs, i.e.,

$$C(D) = \{\, x \in D \,:\, D(x, d(x, \partial D)) \text{ is maximal in } D\},$$

where $D(x,r)$ denotes a disc of radius r centered at x and $d(A,B)$ denotes the Euclidean distance between subsets $A, B \subset \mathbb{R}^2$. The *skeleton* or *medial axis* of D is

$$M(D) = \{x \in D : \exists \text{ distinct } y, y' \in \partial D \text{ s.t. } d(x,y) = d(x,y') = d(x,\partial D)\}$$

It is easy to check that $M(D) \subset C(D)$, with equality for some domains (such as polygons), but not in general. For example, a non-circular ellipse contains two maximal discs which are each tangent to the boundary at only one point. Nevertheless, some sources in the literature mistakenly identify these sets and one purpose of this note is to emphasize how different they can be, even for quite reasonable domains.

In [4], Erdös proved that $M(D)$ has Hausdorff dimension 1 for planar domains. In [3] Fremlin gives many interesting further results, including the fact that any central set of a planar domain has zero area. He also gives an example of a domain so that the closure of

1991 *Mathematics Subject Classification.* Primary: 28A78 , Secondary: 28A75 .

Key words and phrases. Hausdorff dimension, central set, medial axis, skeleton, Lipschitz domain, maximal discs, disc trees.

The first author is partially supported by NSF Grant DMS 04-05578.

its medial axis covers a disc. Given the previous fact, we see that the central set does not always contain the closure of the medial axis. Fremlin asks whether the central set can have Hausdorff dimension strictly bigger than 1. How big can the gap between the dimensions of the medial axis and central set be? We answer this by proving

Theorem 1.1. *There is a domain $D \subset \mathbb{R}^2$ with $\dim_H(C(D)) = 2$.*

Moreover, our domain is close to the unit disc in the following sense. For any $\epsilon > 0$ we can take $D(0,1) \subset D \subset D(0, 1+\epsilon)$, and we can take ∂D to be an ϵ-Lipschitz graph, i.e.,

$$D = \{re^{i\theta} : 0 \le r < f(\theta)\},$$

where $f : [0, 2\pi] \to [1, 1+\epsilon]$ satisfies $|f(s) - f(t)| \le \epsilon|s - t|$. The construction also gives something better than just dimension 2. We will show that we can take $H_\varphi(C(D)) > 0$ for measure functions φ so that $\varphi(t)/t^2 \nearrow \infty$ as slowly as we wish, as $t \to 0$.

Recall the definitions of Hausdorff measures and Hausdorff dimension. Given a subset X of the plane and a continuous increasing function $\varphi : [0,\infty) \to [0,\infty)$, one defines Hausdorff φ-measure of X as

$$H_\varphi(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \varphi(r_i) : X \subset \bigcup_{i=1}^{\infty} D(x_i, r_i),\ r_i < \varepsilon \right\}.$$

When $\varphi(t) = t^s$, for some $s > 0$, this is called s-dimensional measure and is denoted by H_s. The Hausdorff dimension of X is

$$\dim_H(X) = \inf\{s : H_s(X) = 0\} = \sup\{s : H_s(X) = \infty\}.$$

The standard way to prove a lower bound on dimension is to use:

The mass distribution principle. If $X \subset \mathbb{R}^2$ supports a positive measure μ such that

$$\mu(D(x,r)) \le C\varphi(r),$$

for a fixed constant $C > 0$ and for all $x \in \mathbb{R}^2$ and $r > 0$ then $H_\varphi(X) \ge 0$ (see [7]).

Central sets and the medial axis arise naturally in various parts of analysis and computer science, e.g., [2], [6], [8] (see [1] for a connection to conformal mapping and its references for further applications of the medial axis).

We thank the referee for their careful reading of the original manuscript and for suggesting several changes which corrected mistakes and improved the exposition.

The rest of the paper is organized as follows. In Section 2 we define a class of domains called "disc trees" for which the medial axes are trees and impose some conditions which imply the closure of the medial axis is contained in the central set. In Section 3 we construct domains of this type so that $\overline{M(D)}$ has dimension 2.

2. Disc Trees

Our domains are unions of discs arranged with the structure of an infinite rooted tree. We will construct them inductively as an increasing union $D_0 \subset D_1 \subset D_2 \dots$ whose union is the desired domain D. We start with D_0 being the unit disc. In this case the skeleton M_0 of D_0 is just one point, the origin.

Let $\mathcal{G}_1 = \{D_{1,i}\}_{i=1}^{n_1}$ be the "children" of D_0. This is a collection of finitely many discs with centers in D_0 such that the corresponding crescents $C_{1,i} = D_{1,i} \setminus \overline{D_0}$ are mutually disjoint. Let $D_1 = D_0 \cup \bigcup_{i=1}^{n_1} D_{1,i}$. The skeleton M_1 of D_1 is obtained from M_0 by adding (radial) segments connecting M_0 to the centers of the discs $D_{1,i}$. The corresponding *bending points* are defined as $\{b_i^+, b_i^-\} = \partial D_0 \cap \partial D_{1,i}$. There is a 1-parameter family of subdiscs of D_1 which hit ∂D_1 exactly at these two points and whose centers sweep out the interval between the centers of D_0 and $D_{1,i}$.

Since the crescents $C_{1,i}$ are mutually disjoint the maximal discs of $D_0 \cup D_{1,i}$ are still maximal in D_1. Hence M_1 is a tree with one vertex of degree n_1 and n_1 vertices of degree one. Let

$$B_1 = \{x \in D_1 | \operatorname{dist}(x, \partial D_1) = \operatorname{dist}(x, \partial D_0 \cap \partial D_1)\}.$$

Then $D_1 \setminus B_1$ can be written as a disjoint union of circular sectors $S_{1,i}$ of $D_{1,i}$ corresponding to the crescents $C_{1,i}$ (see white regions in Figure 1). $B_1 \setminus M_1$ consists of two type of points: those for which the closest points on ∂D_1 are bending points (the dark grey triangular regions in Figure 1) and the rest (light grey circular sectors, which could possibly degenerate to a line segment if two successive sectors have a common bending point).

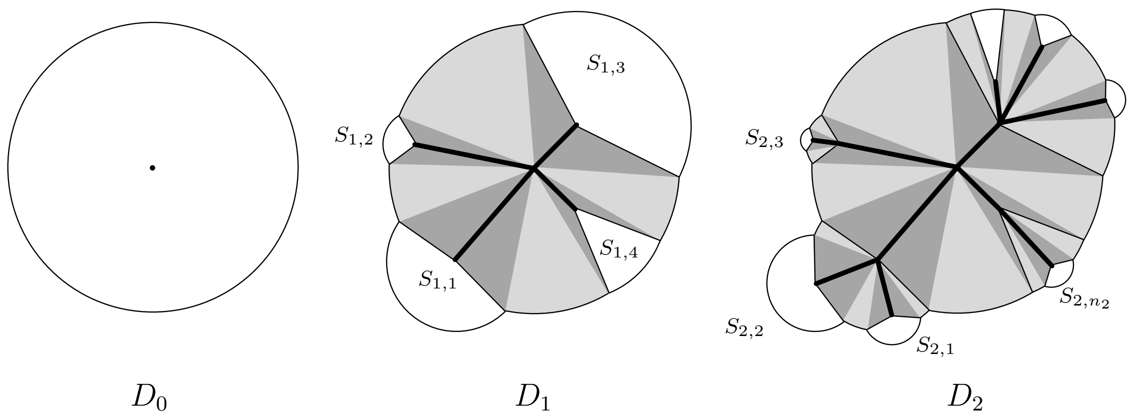

FIGURE 1. Here are D_0, D_1 and D_2. Black represents the medial axis, the union of the grey regions is B, the dark grey triangles are regions closest to bending points, light gray point have a unique closest boundary point and the white regions are the sectors S.

In general, suppose D_k has been constructed from D_{k-1} by adding discs $D_{k,i}$, $i = 1, \dots, n_k$ and let $S_{k,i}$ denote the corresponding sectors. Let $\mathcal{G}_{k+1} = \{D_{k+1,i}\}_{i=1}^{n_{k+1}}$ be a collection of discs with centers in $\cup_{i=1}^{n_k} S_{k,i}$. Denote by $\tilde{D}_{k+1,j} \in \mathcal{G}_k$ the (k-th generation) disc which contains the center of $D_{k+1,j}$ (i.e., the "parent" of $D_{k+1,j}$). Assume that $k+1$ generation discs satisfy the following conditions:

(i) $C_{k+1,i} := D_{k+1,i} \setminus D_k = D_{k+1,i} \setminus \tilde{D}_{k+1,i}$;
(ii) $C_{k+1,i} \cap C_{k+1,j} = \emptyset$ whenever $i \neq j$.

Let $D_{k+1} = D_k \cup_{i=1}^{n_{k+1}} D_{k+1,i}$. The skeleton M_{k+1} of D_{k+1} is obtained from M_k by adding edges connecting the centers of discs of $\mathcal{G}_{k+1}$ to the corresponding degree one vertices of M_k. Let

$$B_{k+1} = \operatorname{int}\{x \in D_{k+1} |\, \operatorname{dist}(x, \partial D_{k+1}) = \operatorname{dist}(x, \partial D_k \cup \partial D_{k+1})\}.$$

Just as before $D_{k+1} \setminus B_{k+1} = \cup_{i=1}^{n_{k+1}} S_{k+1,i}$, is a union of disjoint sectors and $B_{k+1} \setminus M_{k+1}$ is a union of "bending" triangles and circular sectors. So for every edge e of M_k there are two triangles T^+ and T^- which have e as a common edge and the corresponding bending points b^+ and b^- as vertices, respectively.

In this way we obtain an increasing sequence of domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$. Let $D = \bigcup_{i=1}^{\infty} D_i$. We will call a domain a *disc tree* if it can be constructed as above. We will also impose three extra conditions for the remainder of this paper. First, we require that α_k, the maximum angle of a sector in the kth generation, tends to zero. Second, we require that the medial axis of D_k remains uniformly bounded away from ∂D_k with an estimate independent of k. Thirdly, we assume that each sector is contained in the cone defined by its parent sector (this is satisfied if the α_k tend to zero fast enough).

With these assumptions it is fairly easy to see that the closure of the medial axis is contained in the central set, but we will give the details for completeness.

Let D be a disc tree and let $B = \bigcup_{k=1}^{\infty} B_i$ and $L = D \setminus B$. Note that if $x \in B$ then $x \in B_k$ for some k and so, by construction, the segment connecting x to a nearest point of the boundary is also in B.

Given a sector $S_{k,i}$ from the construction, let $\tilde{S}_{k,i}$ be the "extended sector" $S_{k,i} \subset \widetilde{S}_{k,i} \subset D$ so that $\partial \widetilde{S}_{k,i} \cap D = \partial S_{k,i} \cap D$, i.e., $\widetilde{S}_{k,i}$ is $S_{k,i}$ plus the part of D separated from the origin by $S_{k,i}$. By construction, $\widetilde{S}_{k,i}$ is in the infinite cone obtained by extending the edges of $S_{k,i}$.

Note that L is an intersection of finite unions of extended sectors (closed in D), $L = \cap_k \cup_{i=1}^{n_k} \widetilde{S}_{k,i}$, and hence L is closed in D. Moreover, every point $x \in L$ is contained in an infinite, decreasing sequence of closed extended sectors, whose angles decrease to zero. Therefore each connected component of L is a line segment in D, touching ∂D at one end and $\overline{M(D)}$ at the other. Moreover, since $\overline{M(D)} \setminus M(D)$ is contained in the union of kth generation extended sectors for each k, it must be contained in L. Since distinct sectors of the kth generation may only touch on ∂D and since $M(D)$ is bounded away from ∂D, the only way for a sequence in $M(D)$ to approach a point of $\overline{M(D)} \setminus M(D)$ is through a sequence of nested extended sectors and the only possible limit point is an endpoint of a connected component of L. Thus $\overline{M(D)} \setminus M(D)$ contains exactly one point in each connected component of L, and this point must be an endpoint of that component (which is a segment).

Finally, we want to show every point of $\overline{M(D)} \setminus M(D)$ is in $C(D)$, the central set. Suppose not, i.e., suppose there is a point $x \in \overline{M(D)} \setminus (M(D) \cup C(D))$. Then $x \in L$, has a unique closest point y on ∂D but the disc $D(x, |x - y|)$ is not maximal in D. Therefore this disc is contained in a larger subdisc of D, which must be centered at a point x' which lies on the line through y and x. The point x' must be in L for otherwise it would be in B and hence so would x by our remark following the definition of L. This implies x in an interior point of a component of L and hence not in $\overline{M(D)}$, a contradiction.

3. Proof of theorem 1.1

We will now construct a particular disc tree and prove $\overline{M(D)} \setminus M(D)$ has dimension 2. We actually describe how to build an infinite rooted tree in the plane. It is easy to place discs at the vertices of this tree so that the tree becomes the medial axis of the union of discs (this will be explained below).

Consider an increasing function φ on $(0,1)$ so that $\lim_{t\to 0} \varphi(t)/t^2 = \infty$. Let $\phi(t) = \varphi(t)/t^2$. For example, if we show the central set has positive measure for the measure function $\varphi(t) = t^2\phi(t) = t^2 \log \frac{1}{t}$, then it certainly has Hausdorff dimension 2.

The construction is by induction. Suppose we have two strictly increasing integer sequences $\mathbf{p} = \{p_i\}$ and $\mathbf{n} = \{n_i\}$ such that p_i divides n_i. Let $q_i = p_i + 1$ and set

$$N_k = \prod_{i=1}^{k} n_i, \quad P_k = \prod_{i=1}^{k} p_i, \quad Q_k = \prod_{i=1}^{k} q_i.$$

Let $r_k = 1/Q_k N_k$ and assume these sequences satisfy

$$Q_{k+1} \geq Q_k N_k = r_k^{-1} \tag{3.1}$$

$$\phi(r_k) = \phi(Q_k^{-1} N_k^{-1}) \geq q_k^2 Q_k^2. \tag{3.2}$$

Note that since $\phi(t) \nearrow \infty$ as $t \to 0$ we can make the left hand side of (3.2) as large as we want by taking $N_k \to \infty$, while keeping q_k, p_k, Q_k fixed. Taken together, these conditions imply $Q_{k+1} \geq (\phi^{-1}(q_k^2 Q_k^2))^{-1}$, so that $\{Q_k\}$ grows very quickly if ϕ grows slowly. For example, if $\phi(t) = \log \frac{1}{t}$, then $Q_{k+1} \geq \exp(q_k^2 Q_k^2)$.

Initial step: Let 0 be the root of the tree. Divide the plane into n_1 disjoint sectors with vertex at 0 and all with angle $\alpha_1 := 2\pi/n_1$. On the bisector of the jth sector place segments of the form $[0, z]$ where $|z| = (1 + (j \mod p_1))/Q_1$ for $j = 0, \dots, n_1 - 1$. Thus these segments increase in length by $1/Q_1$ at each step until they reach length p_1/Q_1 and then start at $1/Q_1$

again, see Figure 2. We let V_1 denote the non-zero endpoints of these segments. Note that V_1 can be thought of as consisting of p_1 rows, or annular shells, in which the vertices which are equidistant from 0, and each row contains n_1/p_1 vertices.

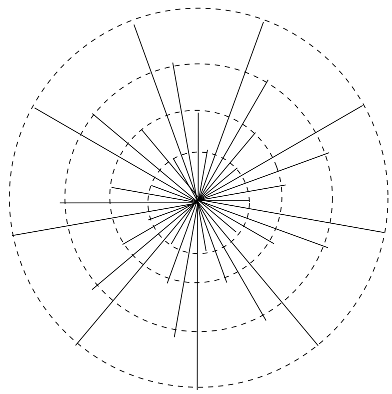

FIGURE 2. The first generation segments. Here $p_1 = 4$, $n_1 = 36$ and $\alpha_1 = 10°$.

General step: Suppose the kth generation edges and vertices V_k have been constructed. For each $v \in V_k$ let $r(v)$ be the ray starting at $\tilde{v}$, the parent of v, and passing through v. Also let $r^{\pm}(v)$ be the rays starting at v and making an angle $\pm\alpha_k/2$ with $r(v)$, where $\alpha_k = \alpha_{k-1}/n_k = 2\pi/N_k$. Let $\mathcal{C}(v)$ be the cone with sides $r^{\pm}(v)$ and angle α_k at v.

Divide $\mathcal{C}(v)$ into n_{k+1} congruent cones of opening $\alpha_{k+1} = 2\pi/N_{k+1}$. On the bisectors of these cones place the $(k+1)$st generation segments of lengths $(1 + (j \mod p_{k+1}))/Q_{k+1}$ for $j = 0, \ldots, n_{k+1} - 1$. The new endpoints can be divided into p_{k+1} rows with n_{k+1}/p_{k+1} vertices per row. The collection V_{k+1} is given by doing this construction for every vertex in V_k. Continue by induction. Denote the resulting tree by $\Gamma = \Gamma(\mathbf{p}, \mathbf{n})$.

To construct a domain for which Γ is the medial axis one needs to start with a disc D_0 of radius R strictly larger than 2. For each first generation vertex $v \in V_1$ consider the disc $D(v)$ centered at v such that $\partial D(v) \cap \partial D_0 = (r^+(v) \cup r^-(v)) \cap \partial D_0$. Define $D_1 = D_0 \bigcup_{v \in V_1} D(v)$. For a $v \in V_2$ denote by $D(v)$ the disc centered at v such that $\partial D(v) \cap \partial D_1 = (r^+(v) \cup r^-(v)) \cap \partial D_1$. Then $D_2 = D_1 \cup_{v \in V_2} D(v)$. Continuing by induction we get a sequence of domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$ and get $D = \bigcup_{k=0}^{\infty} D_k$. By construction, all sector angles tend to zero uniformly. According to the previous section, Γ is the medial axis of D and $C(D)$ contains $\overline{\Gamma}$. We shall see below that $\Gamma \subset D(0,2) \subset D_0$ which implies Γ is bounded away from ∂D, as desired. Note that if R is large, then the boundary of the domain lies in a thin annulus between radii R and $R+\epsilon$ and is Lipschitz with a small constant. Thus rescaling gives the claim following the statement of Theorem 1.1.

Let $\widetilde{\Gamma} = \overline{\Gamma} \setminus \Gamma$. To estimate the dimension of $\widetilde{\Gamma}$ we consider a special covering of it by circular sectors. To do that first note that if v is a vertex of generation k then all its descendants are contained in a subsector of the cone $\mathcal{C}(v)$. We are interested in the radius of the smallest such subsector. To find it, we note that the children of v are at most p_{k+1}/Q_{k+1} away from v, the grand children are at most $\frac{p_{k+1}}{Q_{k+1}} + \frac{p_{k+2}}{Q_{k+2}}$ away and so on. Hence, we see that any descendant of v is at most $l_k := \sum_{i=k+1}^{\infty} p_i/Q_i$ away. Let us denote by $C(v)$ the circular sector centered at v of angle α_k and radius $\sum_{i=k+1}^{\infty} p_i/Q_i$. Then

$$\widetilde{\Gamma} = \bigcap_{i=1}^{\infty} \bigcup_{v \in V_i} C(v).$$

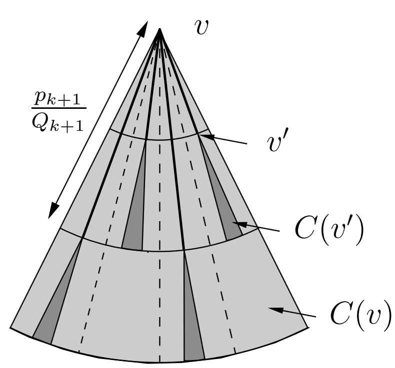

FIGURE 3. *The covering of $\widetilde{\Gamma}$ by circular sectors.*

Lemma 3.1. *With notation as above $l_k = \frac{1}{Q_k}$.*

Proof. First, note that

$$l_k = \lim_{n\to\infty} \sum_{i=k+1}^{n} \frac{p_i}{Q_i} = \frac{L_{k+1}}{Q_k}, \tag{3.3}$$

where $L_{k+1} := \lim_{n\to\infty}\left[\frac{p_{k+1}}{q_{k+1}} + \ldots + \frac{p_n}{q_{k+1}\cdots q_n}\right]$. We claim $L_k = 1$, for every k. Indeed, the general term of the sequence can be rewritten using the fact that $q_i = p_i + 1$ as follows

$$\begin{aligned}(3.4)\qquad &\frac{p_k}{q_k} + \frac{1}{q_k}\frac{p_{k+1}}{q_{k+1}} + \ldots + \left(\frac{1}{q_k}\frac{1}{q_{k+1}}\cdots\frac{1}{q_{n-1}}\right)\frac{p_n}{q_n}\\ &= \frac{p_k}{q_k} + \left(1-\frac{p_k}{q_k}\right)\frac{p_{k+1}}{q_{k+1}} + \ldots + \prod_{i=k}^{n-1}\left(1-\frac{p_i}{q_i}\right)\frac{p_n}{q_n}.\end{aligned}$$

Now, given a sequence of numbers $c_i < 1$, induction on n implies

$$(3.5)\qquad c_k + (1-c_k)c_{k+1} + \ldots + \left[\prod_{i=k}^{n-1}(1-c_i)\right]c_n = 1 - \prod_{i=k}^{n}(1-c_i).$$

Applying this in our case we get

$$(3.6)\qquad L_k = \lim_{n\to\infty}\left[1 - \frac{1}{q_k\ldots q_n}\right] = 1,$$

since $q_i = p_i + 1 > 2, \forall i$. $\square$

Now we are ready to calculate the Hausdorff dimension of $\widetilde{\Gamma}$. We will use the mass distribution principle. Define a probability measure μ on $\widetilde{\Gamma}$ by distributing it evenly among all the sectors of the same generation, i.e.,

$$\mu(C(v)) = \frac{1}{N_i}\,, \quad \forall v \in V_i.$$

For a ball $B \subset \mathbb{R}^2$ and $i \in \mathbb{N}$ let $\nu_i(B)$ be the number of i-th generation sectors which have positive μ-mass when intersected with B, or

$$(3.7)\qquad \nu_i(B) = \#\{C(v) : \mu(C(v)\cap B) > 0, v \in V_i\}.$$

Recall that $r_k = l_k/N_k = Q_k^{-1}N_k^{-1}$. This is approximately the length of the circular arc edge of $C(v)$ in the kth generation. The sectors of the next generation which are contained in $C(v)$ are actually contained in truncated sector obtained by removing all points within l_k/q_k of v. The remaining region has two long radial edges (with respect to v) and two circular arc edges, one of length about r_k (the one farther from v) and one of length about r_k/q_k. Thus, if B is a ball of diameter $\le r_k$, it can hit at most $O(q_k)$ kth generation sectors in positive measure. (With a little more work one can show only $O(1)$ sectors can hit B, but the weaker estimate is easier and sufficient for us.)

Clearly $r_k = l_k/N_k < l_k/q_k < l_k$ (since $N_k > n_k \geq q_k$). Also note that by (3.1), we have $l_{k+1} = Q_{k+1}^{-1} \leq Q_k^{-1} N_k^{-1} = r_k$ so

$$\cdots < r_{k+1} < l_{k+1} < r_k < l_k/q_k < l_k < \ldots .$$

Fix a ball B, let $|B|$ denote its diameter and choose an index k so that $r_{k+1} < |B| \leq r_k$. As noted above, B hits at most $O(q_k)$ sectors in the kth generation and it is enough to estimate the mass coming from one of them. Fix such a sector, $C(v)$, hitting B and consider the $(k+1)$st generation subsectors of $C(v)$. They are arranged into p_k levels, according to their distance from the point v (which are multiples of l_k/q_k). Since $|B| < l_k/q_k$, B can hit at most two of these levels. Inside each level, there are n_{k+1}/p_{k+1} $(k+1)$st generation sectors equidistributed in a row which is l_k/q_k "high" and w "wide" where w is at most r_k (for the row farthest from v) and at least r_k/q_k (for the row closest to v). Thus the number of $(k+1)$st generation sectors that hit B is approximately n_{k+1}/p_{k+1} times $|B|/w$, and so is at most $|B| q_k n_{k+1}/r_k p_{k+1}$.

Next, each row of $(k+1)$st generation sectors is divided into q_{k+2} bands of $(k+2)$nd generation sectors. (We use the term "bands" instead of "rows" since the situation is slightly different than before; the $(k+2)$nd generation subsectors of fixed $(k+1)$st generation sector do lie in rows equidistant from the vertex of the sector, exactly as before, but the union of the subsectors over different $(k+1)$st sectors are not all equidistant from a single point. However, they are arranged in obvious bands which are close to being equidistant from the vertex of the kth generation sector containing them.)

The height of each of these bands is approximately $u = l_k/(q_k q_{k+1})$ and so at most $1 + O(|B|/u)$ rows can hit B (and this is less than $O(|B|/u) = O(|B| q_k q_{k+1} Q_k))$. Each row contains n_{k+2}/p_{k+2} sectors. Thus the total number of $(k+2)$nd generation sectors that hit B is less than a bounded multiple of

$$|B|^2 q_k^2 q_{k+1} Q_k^2 N_k n_{k+2} n_{k+1} p_{k+2}^{-1} p_{k+1}^{-1} \leq |B|^2 q_k Q_k^2 N_{k+2}.$$

(Recall that $p_{k+1} \geq p_k + 1 = q_k$ since the sequence is strictly increasing.)

Every $(k+2)$nd generation sector has mass N_{k+2}^{-1}, and B hits at most $O(q_k)$ kth generation sectors in positive mass, so the total mass of the $(k+2)$nd generation sectors hitting B is bounded by a constant times

$$\frac{q_k}{N_{k+2}} \cdot (|B|^2 q_k Q_k^2 N_{k+2}) \leq |B|^2 \phi(|B|) \frac{q_k^2 Q_k^2}{\phi(|B|)} \leq |B|^2 \phi(|B|) = \varphi(|B|),$$

by (3.2). Thus $H_\varphi(\widetilde{\Gamma}) > 0$ by the mass distribution principle, which proves Theorem 1.1.

References

1. C. J. Bishop, *Conformal mapping in linear time*, preprint 2006.
2. H.I. Choi, S.W. Choi and H.P. Moon. *Mathematical theory of medial axis transform*, Pacific J. Math. (1) **181** (1997) 57–88.
3. D.H. Fremlin, *Skeletons and central sets*, Proc. London Math. Soc. (3) **74** (1997), no. 3, 701–720.
4. P. Erdös, *Some remarks on measurability of certain sets*, Bull. Amer. Math. Soc. **51** (1945), 728-731.
5. W.D. Evans, D.J. Harris, *Sobolev Embeddings for Generalized Ridged Domains*, Proc. London Math. Soc. (3) **54** (1987), 141–175.
6. L.A. Caffarelli, A. Friedman, *The Free Boundary for Elastic-Plastic Torsion Problems*, Trans. Amer. Math. Soc. **252** (1979), 65–97.
7. P. Mattila, *Geometry of sets and measures in the Euclidean spaces: fractals and rectifiability,* Cambridge studies in advanced mathematics, vol. 44, Cambridge University Press, Cambridge, 1995.
8. T.W. Ting, *The ridge of a Jordan domain and completely plastic torsion*, J. Math. Mech. **15** (1966), 15–47.