FALCONER'S (K, d) DISTANCE SET CONJECTURE CAN FAIL FOR STRICTLY CONVEX SETS K IN $\mathbb{R}^d$

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ABSTRACT. For any norm on $\mathbb{R}^d$ with countably many extreme points, we prove there is a set $E \subset \mathbb{R}^d$ of Hausdorff dimension d whose distance set with respect to this norm has zero linear measure. This was previously known only for norms associated to certain finite polygons in $\mathbb{R}^2$. Similar examples exist for norms that are very well approximated by polyhedral norms, including some examples where the unit ball is strictly convex and has C^1 boundary.

Date: September 3, 2019.

2010 *Mathematics Subject Classification.* Primary: 28A78, Secondary: 52A20.

Key words and phrases. Falconer's conjecture, Hausdorff dimension, convex sets, distance sets, polyhedral norms.

The first author is supported in part by NSF Grant DMS-1906259.

1. Introduction

We will call $K \subset \mathbb{R}^d$ a "norm ball" if it is compact, convex, symmetric, and contains the origin in its interior. Under these conditions there is an associated norm $\|\cdot\|_K$ defined on $\mathbb{R}^d$ by

$$\|x\|_K = \inf\{\lambda > 0 : x/\lambda \in K\}.$$

If $E \subset \mathbb{R}^d$ then the K-distance set of E is

$$\Delta_K(E) = \{\|x - y\|_K : x, y \in E\} \subset [0, \infty).$$

Motivated by [12] and [15], we say Falconer's (K, α)-conjecture holds if for any set $E \subset \mathbb{R}^d$ with $\dim(E) = \alpha$, the set $\Delta_K(E)$ has positive 1-dimensional Lebesgue measure, also referred to as length; here and below "dim" refers to Hausdorff dimension. In this note we give new examples where this fails for $\alpha = d$.

When K is the usual closed unit ball $\mathbb{B}$ in $\mathbb{R}^d$, $d \in \mathbb{N} = \{1, 2, \dots\}$, we shall denote $\Delta_{\mathbb{B}}(E)$ simply by $\Delta(E)$. Falconer's conjecture is a refinement of a well known result of Steinhaus that $\Delta(E)$ contains an interval whenever $E \subset \mathbb{R}$ has positive Lebesgue measure. Falconer [6] proved that the $(\mathbb{B}, \alpha)$-conjecture is true for all $\alpha > (d+1)/2$ and he asked if it holds for all $\alpha > d/2$. Falconer's result was subsequently improved by Bourgain [2], Wolff [19] and Erdogan [5] and very recently there has been much activity by various authors including Du, Guth, Iosevich, Ou, Wang, Wilson and Zhang [3], [4], [10]. See Iosevich's brief survey [13] for a summary of the history of this problem, the best currently known bounds, the ideas behind these results, and the close connection between Falconer's conjecture and the Erdős distance conjecture for finite sets (recently solved in the plane, [11]).

When K is not the round ball, much less is known. If ∂K is smooth and has non-vanishing curvature, then Iosevich and Łaba [12] proved the (K, α)-conjecture is true for $\alpha > (d+1)/2$, but Konyagin and Łaba [16] proved that the $(K, 2)$-conjecture is false for various finite polygons in $\mathbb{R}^2$, e.g., when the slopes of the sides are algebraic. In [15] they extend this to polygons where the slopes belong to a certain set of full measure, and prove that the (K, α) conjecture always fails if $\alpha > N/(N-1)$, where N is the number of sides of the polygon K. Corollary 4 of Falconer's paper [7] claims that the (K, d) conjecture fails for all finite polyhedral norm balls $K \subset \mathbb{R}^d$, but the proof contains a gap, explained in Section 2. We will fill this gap by proving a slightly stronger result:

Theorem 1.1. *If K is a norm ball with countably many extreme points, then the (K, d)-conjecture fails, i.e., there is a compact $E \subset \mathbb{R}^d$ of Hausdorff dimension d such that $\Delta_K(E)$ is a null set.*

A null set in $\mathbb{R}$ is a set of zero Lebesgue measure. Recall that $x \in \partial K$ is an extreme point of K if it does not lie on any open line segment between distinct points of K. For a finite polygon, these are exactly the vertices. We say that K is strictly convex if every point of ∂K is an extreme point, i.e., the boundary contains no line segments. Every point x on the boundary of a convex set $K \subset \mathbb{R}^d$ lies on a $(d-1)$-plane that misses the interior of K. The boundary of K is C^1 if and only if there is only one such plane at each $x \in \partial K$; see Lemma 4.3.

Theorem 1.2. *There is a strictly convex norm ball $K \subset \mathbb{R}^d$ with C^1 boundary such that the (K, d)-conjecture fails.*

So far as we know, Falconer's (K, α)-conjecture was not previously known to fail for any strictly convex set K and $\alpha > d/2$.

2. Dimensions of intersections

Theorem 1 of [7] (see also Theorem 8.3 of [8]; 8.2 in earlier editions), claims that if E and F are Borel subsets of $\mathbb{R}^d$, then the set of homotheties σ on $\mathbb{R}^d$ (compositions of dilations and translations) such that

$$\dim(E \cap \sigma(F)) \geq \dim(E) + \dim(F) - d, \tag{2.1}$$

has positive measure in the group of all homotheties. However this claim is false: we will show that there are compact sets E and F so that (2.1) does not hold for any Euclidean similarity (a composition of dilations, translations, rotations and reflections).

For simplicity, consider the case $\dim(E) = \dim(F) = d = 1$. Let $\{I_n\}$ be the collection of closed intervals $[2^{-2^n}, 2^{\frac{1}{4}-2^n}]$, $n \in \mathbb{N}$, and for each n choose a compact set $E_n \subset I_n$ with $\dim(E) = 1 - 1/n$. Then set $E = \{0\} \cup \bigcup_{n \in \mathbb{N}} E_n$. Similarly, let $J_n = [2^{-3^n}, 2^{\frac{1}{4}-3^n}]$, choose compact $F_n \subset J_n$ with dimensions $1 - 1/n$ and set $F = \{0\} \cup \bigcup_{n \in \mathbb{N}} F_n$.

We claim that $\dim(E\cap\sigma(F)) < 1$ for any similarity σ. First note that $\dim(E\setminus U) < 1$ and $\dim(F \setminus U) < 1$ for any neighborhood U of zero. Thus if $\dim(E \cap \sigma(F)) = 1$, we must have $\sigma(0) = 0$, for otherwise there are disjoint neighborhoods U, V of 0 and $\sigma(0)$ and hence

$$\dim(E \cap \sigma(F)) \leq \max(\dim(E \setminus U), \dim(\sigma(F) \setminus V)) < 1.$$

The restriction $\sigma(0) = 0$ already implies that the set of similarities (or homotheties) such that $\dim(E \cap \sigma(F)) = 1$ has measure zero.

To show the set satisfying (2.1) is empty, we may assume $\sigma(0) = 0$, but that $E\cap\sigma(F) \neq \{0\}$. Then some I_n must intersect some $\sigma(J_m)$ (otherwise the intersection is just the point $\{0\}$). Therefore σ must be a dilation of the form $\sigma(x) = 2^{\lambda+2^n-3^m}x$, for some $\lambda \in [-\frac{1}{4}, \frac{1}{4}]$. We claim that only finitely many other pairs of the form I_k, $\sigma(J_j)$ can intersect. Assume (j, k) is such a pair and $j > m, k > n$. If $\sigma(J_j)$ hits I_k we must have

$$\lambda + 2^n - 3^m - 2^k = \lambda' - 3^j$$

for some $\lambda' \in [0, \frac{1}{4}]$. Because the powers of 2 and 3 are integers, we must have

$$2^n - 3^m - 2^k = -3^j,$$

or, equivalently,

$$2^n(2^{k-n} - 1) = 2^k - 2^n = 3^j - 3^m = 3^m(3^{j-m} - 1).$$

By unique factorization of integers, this implies

$$2^n = 3^{j-m} - 1.$$

Since n, m are fixed there is at most one j that can satisfy this equation. Similarly for k. Since only finitely many pairs of intervals can overlap and the dimension of E, F inside each of these intervals is strictly less than 1, we see that $\dim(E \cap \sigma(F)) < 1$.

It is easy to see that the same idea can be applied to sets in $\mathbb{R}^d$: there exists $E, F \subset \mathbb{R}^d$ both of dimension d, so that that $\dim(E \cap \sigma(F)) < d$ for every similarity σ of $\mathbb{R}^d$. Indeed, only slightly more work shows this holds for every diffeomorphism of $\mathbb{R}^d$ into itself.

The proof of Theorem 1 in [7] uses an induction argument on the dimension d which breaks down at the first step $d = 1$; this case is quoted from [17], but the result is not found there, and our example shows it is incorrect. There are correct versions of (2.1) under stronger hypotheses, e.g., Theorem 13.14 of [17] or [14].

3. The basic construction

Before giving our main construction, we recall a simple criterion for a set $E \subset \mathbb{R}^d$ to have dimension d. Suppose $b \in \mathbb{N}$, $b \geq 2$. An nth generation b-adic cube $Q \subset \mathbb{R}^2$ is a product of intervals of the form $[jb^{-n}, (j+1)b^{-n}]$, $j \in \mathbb{Z}$. Fix a set $S \subset \mathbb{N}$. Suppose E is defined as an intersection of sets E_n, where each E_n is a union of nth generation cubes. We assume E_0 is a union of unit (i.e., 0th generation) b-adic cubes in $\mathbb{R}^d$. In general, suppose we obtain E_{n+1} from E_n by taking all b^d subcubes if $n \notin S$, and by taking at least one child cube if $n \in S$. For example, for the construction of the middle thirds Cantor set in the real line one would take $b = 3$, $S = \mathbb{N}$, and would choose the leftmost and rightmost children among the $b = 3$ children of an interval in each generation.

We say S has zero density if

$$\lim_{n\to\infty} \frac{\#(S \cap [1, n])}{n} = 0.$$

Lemma 3.1. *With notation as above, if S has zero density then* $\dim(E) = d$.

Proof. Consider the subset E' of E constructed by choosing exactly one child cube of a cube of E_n whenever $n \in S$. More precisely, E' is the intersection of sets E'_n, where E'_n is a union of cubes such that $E'_0 = E_0$ and E'_{n+1} is obtained from E'_n by taking all b^d subcubes if $n \notin S$, and by taking exactly one child cube if $n \in S$. We shall show that $\dim(E') = d$, and thus $\dim(E) = d$.

Let μ be the measure on E' that assigns mass 1 to each unit cube in the construction and divides the mass of each cube evenly between its children. If Q_n is a cube of nth generation contained in a cube Q_{n-1} of $(n-1)$th generation, then by construction

$$\frac{\mu(Q_n)}{\mu(Q_{n-1})} = \begin{cases} 1, & n \in S \\ b^d, & n \notin S. \end{cases}$$

We therefore have,

$$\mu(Q_n) = b^{d(n-\#(S\cap[1,n]))}.$$

By Billingsley's lemma (see, for example, Lemma 1.4.1 of [1]), it follows that

$$\dim(E') \geq \lim_{n\to\infty} \left|\frac{\log \mu(Q_n)}{n \log b}\right| = d\left(1 - \lim_{n\to\infty} \frac{\#(S \cap [1,n])}{n}\right) = d. \qquad \square$$

The following strengthens Corollary 3 of [7] from finite collections of vectors to countable collections.

Lemma 3.2. *Suppose $\{\theta_1, \theta_2, \dots\}$ is a countable collection of vectors in $\mathbb{R}^d$. There is a compact set $E \subset \mathbb{B}$ with $\dim(E) = d$ and so that for every $n \in \mathbb{N}$, $E_n = \Delta(\Pi_n(E))$ has zero length, where Π_n is the orthogonal projection onto the line in direction θ_n. Moreover, there are closed null sets $\{D_n\} \subset [0, \infty)$, independent of the choice of $\{\theta_n\}$, so that $E_n \subset [0, 2] \cap D_n$ for every $n \in \mathbb{N}$.*

Proof. We start with a standard construction of a set $Y \subset \mathbb{R}$ of Hausdorff dimension 1 whose distance set has length zero. Choose a strictly increasing sequence of positive integers $\{m_k\}$ and set $n_k = m_1 + \cdots + m_k$. Set $I = [0, 1]$ and let $X_k = 3^{-n_k}(I + 3\mathbb{Z})$; this is an infinite union of closed intervals of length 3^{-n_k} separated by open intervals of length $2 \cdot 3^{-n_k}$. For $n \in \mathbb{N}$, set

$$Y_n = \bigcap_{k=1}^{n} X_k \subset Y_{n-1}, \qquad Y = Y(\{n_k\}) = \bigcap_{n=1}^{\infty} Y_n.$$

The set Y can also be described by an inductive construction using 3-adic intervals as in the setting described before Lemma 3.1: one starts with all integer unit intervals and replaces an interval by its three children if $n \notin S \coloneqq \{n_k\}$, and chooses only the leftmost interval if $n \in S$. If $m_k \to \infty$, then S has zero density and Lemma 3.1 shows that $\dim(Y) = d$.

The distance set $D_0 = \Delta(Y)$ of $Y \subset \mathbb{R}$ is the same as the projection of $Y \times Y \subset \mathbb{R}^2$ onto the real line via lines of slope 1. See Figure 1. By construction, the projection of $Y_{n+1} \times Y_{n+1}$ can be obtained from the projection of $Y_n \times Y_n$ by replacing each interval I in the latter set by a union of subintervals covering at most $2/3$ of the length of I. Thus, in the limit, the projection has zero length, and hence so does D_0.

For each $k = 1, 2, \dots$, define $Z_k = Y_k \times \mathbb{R}^{d-1}$ and let $Z = \cap_k Z_k = Y \times \mathbb{R}^{d-1}$. Then Z_k consists of infinitely many infinite, parallel "slabs" of thickness 3^{-n_k}. Each slab is a union of d-cubes of side length 3^{-n_k} and disjoint interiors. We call these the cubes associated to Z_k. The main observation we need is that we can choose a dilation

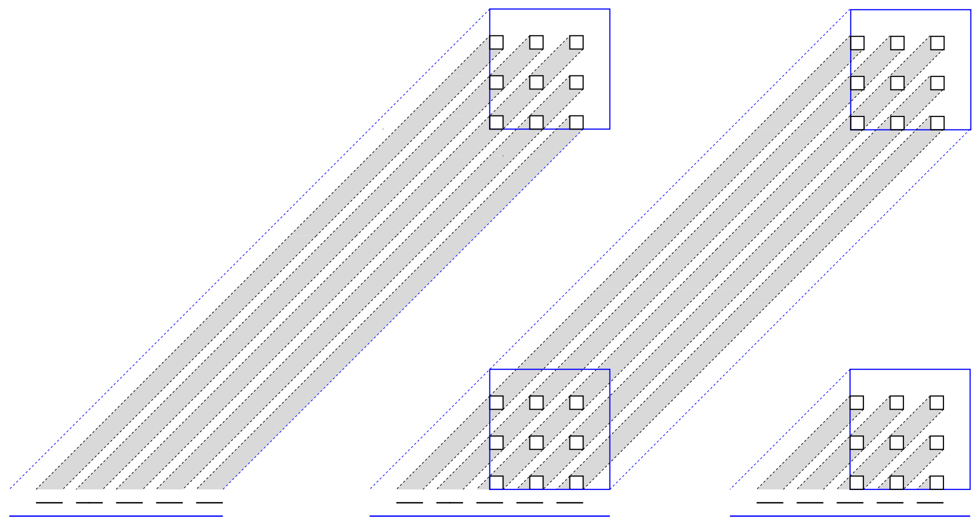

FIGURE 1. The reason why $D_0 = \Delta(Y)$ has zero length.

factor $0 < \lambda = 3^{-t} \leq 1/(2\sqrt{d+3})$, $t \in \mathbb{N}$, (see Figure 2) so that each cube associated to Z_n contains a cube associated to $\lambda\tau(Z_n)$, where τ is any rotation of $\mathbb{R}^d$ (this is also true for all rigid motions, but we don't need that much generality).

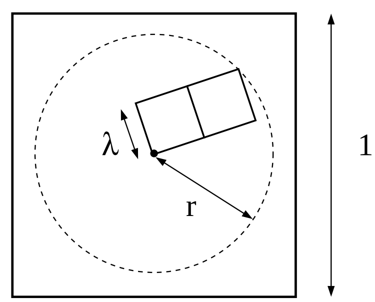

FIGURE 2. The center of a unit cube in Z_0 is contained in a λ-sized cube in $\lambda\tau(Z_0)$ and we either keep that cube or an adjacent one. In either case the kept cube is contained in disk of radius $r = \lambda\sqrt{d+3}$ around the center and hence it is inside the unit cube if $\lambda \leq 1/(2\sqrt{d+3})$. This holds even if the cubes come from grids that are rotated with respect to each other.

Set $F_n^k = \tau_k(\lambda^k Z_n)$, where τ_k is any rigid rotation of $\mathbb{R}^d$ that takes the first coordinate axis into the line L_k in direction θ_k. Set

$$F^k = \mathbb{B} \cap \bigcap_{n=1}^{\infty} F_n^k.$$

Note that the orthogonal projection of F^k into L_k is contained $\tau_k(\lambda^k Y)$ and hence its distance set is contained in $\lambda^k D_0$.

We now give the construction of E in the case of finitely many direction vectors, and then show how to adapt it to the countable case. If there are N direction vectors $\theta_1, \dots, \theta_N$ then $E = \bigcap_{k=1}^N F^k$ will work if the increasing sequence $\{m_k\}$ is chosen correctly. We require that $m_k \geq t(N+1)$; recall that $\lambda^{N+1} = 3^{-t(N+1)}$. Then each of the cubes Q_1 of side length 3^{-n_1} associated to F_1^1 contains *one cube* Q_2 with side length $\lambda 3^{-n_1}$ associated to F_1^2 (by the choice of λ), and so on until we reach *one cube* Q_N associated to F_1^N which has size $\lambda^N 3^{-n_1} = 3^{-n_1 - tN}$. We now consider a cube $\widetilde{Q}_1 \subset Q_N$ of side length $3^{-m_1 - t(N+1)} = \lambda 3^{-n_1 - tN}$ that is the product of intervals of the form $[j3^{-m_1-t(N+1)}, (j+1)3^{-m_1-t(N+1)}]$ and intersects the set F_2^1. Then we take *all the cubes* associated to F_2^1 that are contained in $\widetilde{Q}_1$; these have size $3^{-n_2} = 3^{-m_1-m_2} \leq 3^{-m_1-t(N+1)}$.

We repeat the construction above, taking *one cube* from F_2^k, $k = 1, \dots, N$, and then *all the cubes* associated to F_3^1. Continuing in this way defines nested collections of 3-adic cubes, whose intersection we call E. This procedure can also be described as an iterative construction on 3-adic cubes where we always choose all 3^d children, except for generations $n \in [n_k, n_k + t(N+1)]$. Since $m_k = n_{k+1} - n_k \to \infty$, the exceptional generations have zero density and so Lemma 3.1 proves that $\dim(E) = d$.

Now we modify the argument for countably many vectors. Each cube Q_1 of side length 3^{-n_1} associated to F_1^1 contains one cube Q_2 of side length $\lambda 3^{-n_1}$ associated to F_1^2. We now consider a cube $\widetilde{Q}_1 \subset Q_2$ of side length $\lambda^2 3^{-n_1} = 3^{-n_1-2t}$ that is the product of intervals of the form $[j\lambda^2 3^{-n_1}, (j+1)\lambda^2 3^{-n_1}]$ and intersects the set F_2^1. Then we take all the cubes associated to F_2^1 that are contained in $\widetilde{Q}_1$; these have size $3^{-n_2} = 3^{-n_1-m_2} \leq \lambda^2 3^{-n_1}$, provided that $3^{-m_2} \leq \lambda^2$. This completes the first step of the construction.

Let now R_1 be a cube of F_2^1 that has side length 3^{-n_2}. There exists a cube $R_2 \subset R_1$ of side length $\lambda 3^{-n_2}$ associated to F_2^2. *Now, we also choose a cube* $R_3 \subset R_2$ *of side* *length* $\lambda^2 3^{-n_2}$ *associated to* F_2^3. We end the second step by choosing a cube $\widetilde{R}_1$ of side length $\lambda^3 3^{-n_2} = 3^{-n_2-3t}$ that is the product of intervals of the form $[j\lambda^3 3^{-n_2}, (j+1)\lambda^3 3^{-n_2}]$ and intersects the set F_3^1. Then we take all the cubes associated to F_3^1 that are contained in $\widetilde{R}_1$; these have size $3^{-n_3} = 3^{-n_2-m_3} \le \lambda^3 3^{-n_2}$, provided that $3^{-m_3} \le \lambda^3$. This completes the second step of the construction.

We continue the construction in this way, choosing every time sufficiently large m_k, so that the construction can go through. This procedure can also be described as an iterative construction on 3-adic cubes where we always choose all 3^d children, except for generations $n \in [n_k, n_k + t(k+1)]$. We define $S = \cup_k [n_k, n_k + t(k+1)]$. If, in addition, m_k is chosen to be so large that $m_k/k \to 0$ as $k \to \infty$, then S has density zero, so $\dim(E) = d$.

The projection of E onto direction θ_n is contained in a copy of $\lambda^{a_n} Y(\{\widetilde{n}_k\})$ where a_n is the generation in which we first use a cube associated to F^n (for example, $a_3 = 2$ from the construction above) and $\{\widetilde{n}_k\}$ is the truncation of $\{n_k\}$ starting at index a_n. Thus the corresponding distance set, D_n, is a set of zero length that depends on our choice of $\{m_k\}$, but not on the $\{\theta_k\}$. □

Lemma 3.3. *Suppose K is a norm ball with countably many extreme points. Then there is a countable set of vectors $\{\theta_n\}$ such that for each $x \in \mathbb{R}^d$ there exists $n \in \mathbb{N}$ with*

$$\|x\|_K = |x \cdot \theta_n| = \max_k |x \cdot \theta_k|.$$

Moreover, if $B(0, R_1) \subset K \subset B(0, R_2)$ for some $0 < R_1 < R_2$, then $1/R_2 \le |\theta_n| \le 1/R_1$ for all $n \in \mathbb{N}$. If K is a polyhedron, then the vectors $\{\theta_n\}$ may be taken to be parallel to the normal vectors of the faces of K.

Proof. Every point x on the boundary of a convex set $K \subset \mathbb{R}^d$ lies on a $(d-1)$-plane P_x, called a *supporting hyperplane*, that misses the interior of K. In other words, there is a linear functional f_x so that $f_x(x) = 1$ and $f_x(y) \le 1$ for all $y \in K$, and $P_x = \{y : f_x(y) = 1\}$; see [18, Theorem 11.6].

Moreover, Carthéodory's theorem [18, Theorem 17.1] states that every non-extreme boundary point x is a convex combination $x = \sum_{j=1}^k p_j x_j$ of k extreme points with $0 < p_j < 1$ for $1 \le j \le k$, and $2 \le k \le d+1$. Let $E(x)$ denote the set of k extreme points associated to x. Then $f_x(y) = 1$ for every $y \in E(x)$ (otherwise $f_x(x) < 1$), and hence $f_x(y) = 1$ for every convex combination y of points in $E(x)$. Thus the plane P_x covers the convex hull of $E(x)$. Since there are only countably many k-tuples of a countable set, there are countably many $(d-1)$-planes that cover ∂K.

We take normal vectors ν_n, $n \in \mathbb{N}$, to these planes such that ν_n, as a point of $\mathbb{R}^d$, lies on the corresponding plane. If we define $\theta_n = \nu_n/|\nu_n|^2$, then these vectors have the desired properties. It suffices to check the claim whenever $\|x\|_K = 1$. Then $x \in P_x$ and suppose that ν_n is the normal to P_x. The vector $(x \cdot \nu_n/|\nu_n|)\nu_n/|\nu_n| = (x \cdot \nu_n/|\nu_n|^2)\nu_n$ is the projection of x to the direction ν_n, which is precisely the vector $1\nu_n$. Hence, $x \cdot \nu_n/|\nu_n|^2 = 1 = \|x\|_K$.

If P is some other supporting hyperplane not containing $\pm x$ with normal vector ν_m, then there exists a constant $\lambda \in (-1,1)$ such that $x \in P_\lambda := \nu_m^\perp + \lambda\nu_m$ and the hyperplane P_λ has normal $\lambda\nu_m \in P_\lambda$. Then $(x \cdot \nu_m/|\nu_m|)\nu_m/|\nu_m| = \lambda\nu_m$, so $|x \cdot \nu_m/|\nu_m|^2| = |\lambda| < 1 = \|x\|_K$.

For the last assertion note that $\nu_n \in \overline{B}(0,R_2) \setminus B(0,R_1)$ for all $n \in \mathbb{N}$. Hence $|\theta_n| = 1/|\nu_n| \in [1/R_2, 1/R_1]$. □

It will be crucial below that we have a maximum in the previous lemma and not just a supremum; the latter version is always true by taking a dense set of directions.

Proof of Theorem 1.1. This is the same as the proof of Corollary 4 in [7]. By Lemma 3.3, there are countably many vectors $\{\theta_n\} \subset \mathbb{R}^d$ so that for each $x, y \in \mathbb{R}^d$ there exists n such that

$$\|x-y\|_K = |(x-y)\cdot\theta_n|.$$

Let E and $\{D_n\}$ be the sets from Lemma 3.2. We have

$$\Delta(\Pi_n(E)) = \{|(x-y)\cdot\theta_n/|\theta_n|| : x, y \in E\} \subset D_n.$$

Then

$$\{\|x-y\|_K : x, y \in E\} \subset \bigcup_{n=1}^{\infty}\{|(x-y)\cdot\theta_n| : x, y \in E\} \subset \bigcup_{n=1}^{\infty}(|\theta_n| D_n),$$

which is a countable union of zero length sets. □

Given a Banach space X, a subset $B \subset X^*$ of its dual space is called a (James) boundary if for every $x \in X$, there is a $b \in B$ so that $\|x\|_X = b(x)$. For example, the unit sphere in X^* is such a boundary, as is the set of extreme points of the unit ball in X^*. Thus another way to state Lemma 3.3 is that if X is a finite dimensional

Banach space whose unit ball has countable number of extreme points, then X has a countable boundary. Certain infinite dimensional Banach spaces also have this property, e.g., c_0, the space of real valued sequences that tend to zero. Is there an interesting version of Theorem 1.1 for such spaces? What is the the correct notion of a "large" set whose distance set has zero length? Infinite Hausdorff dimension?

4. A strictly convex example

Given two subsets $E, F \subset \mathbb{R}^d$, recall that the Hausdorff distance between E and F is defined as

$$d_H(E,F) = \inf\{\varepsilon > 0 : E \subset N_\varepsilon(F) \text{ and } F \subset N_\varepsilon(E)\},$$

where $N_\varepsilon(E), N_\varepsilon(F)$ denote the open ε-neighborhoods of E and F respectively.

Lemma 4.1. *For each $0 < R_1 < R_2$ there is a function $\varphi : \mathbb{N} \to (0,1]$ so that the following holds. Suppose $K \subset \mathbb{R}^d$ is a norm ball and $\{K_n\} \subset \mathbb{R}^d$ is a sequence of finite polyhedral norm balls such that*

(a) *$B(0,R_1) \subset K, K_n \subset B(0,R_2)$ for all $n \in \mathbb{N}$, and*

(b) *K_n has s_n sides for each $n \in \mathbb{N}$, where s_n strictly increases to ∞ as $n \to \infty$.*

Moreover, consider the set E given by Lemma 3.2 and corresponding to the countably many normal directions of the sides of all polyhedrons K_n.

(i) *If $d_H(K,K_n) \le \varphi(s_n)$ for some $n \in \mathbb{N}$, then $\Delta_K(E)$ has length at most $1/s_n^2$.*

(ii) *If $d_H(K,K_n) \le \varphi(s_n)$ for all $n \in \mathbb{N}$, then the (K,d)-conjecture fails.*

In other words, the (K,d)-conjecture not only fails for finite polyhedrons, but also for any convex body that is "very well approximated" by finite polyhedrons.

Proof. **Claim 1:** Consider an infinite ray $\mathcal{R}$ emanating from 0 and hitting $\partial K_n, \partial K$ at points x, y, respectively. We first claim that there exists a constant $C_1 = C_1(R_1,R_2) > 0$ such that

$$|x-y| \le C_1 d_H(K,K_n).$$

Here, the roles of K and K_n are symmetric, so suppose that $y \in \partial K$ satisfies $|y| > |x|$. Let $z \in \partial K$ be the point closest to $x \in \partial K_n$, so $|x-z| \le d_H(K,K_n)$; see Figure 3.

If $|x-y| > C_1 d_H(K,K_n)$ for a constant $C_1 > 0$ then we would have $|x-z|/|x-y| < 1/C_1$. In the extreme case that $z = y$ we have $C_1 < 1$, so if we choose $C_1 \ge 1$, then

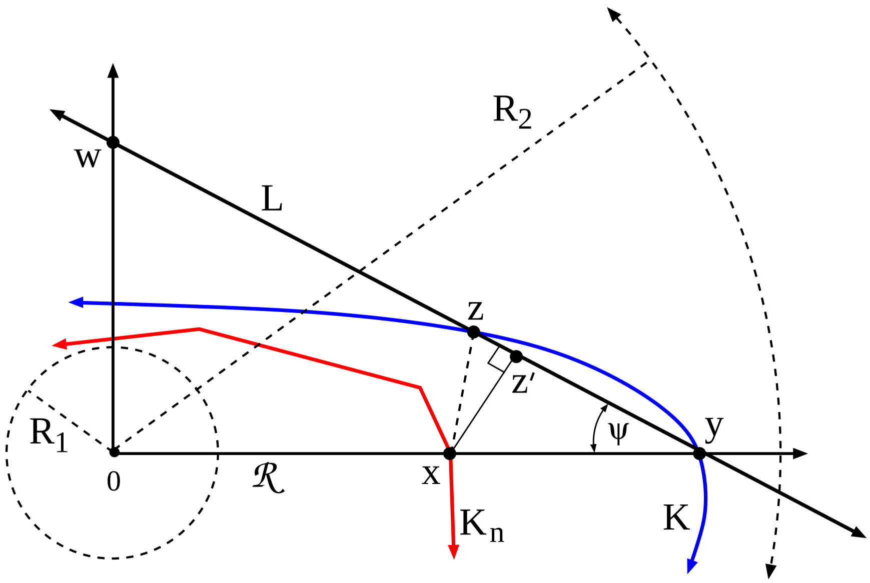

FIGURE 3. Illustration of the proof of Claim 1.

$z \neq y$. Consider the line L through y and z and let ψ be the angle between L and the ray $\mathcal{R}$. Then $\psi < \pi/2$, since $z \neq y$. Therefore, the line L hits a point w on the hyperplane that is perpendicular to $\mathcal{R}$ and passes through the origin, with $|w| = |y|\tan(\psi) \leq R_2 \tan(\psi)$. We claim that z is on the segment between w and y if C_1 is sufficiently large. Indeed, otherwise we have $|x-z| \geq R_1$ because $x \notin B(0, R_1)$, so $1/C_1 > R_1/|x-y| > R_1/(R_2-R_1)$. Therefore, if we choose $C_1 \geq (R_2-R_1)/R_1$, then our claim follows. Since z is on the segment between w and y, if the point w is in the interior of K, so is z by convexity, a contradiction. Hence, w is either on the boundary of K or is outside K. In either case $|w| \geq R_1$. We have $R_2\tan(\psi) \geq |w| \geq R_1$, which implies that $\tan(\psi) \geq R_1/R_2$.

Now, let z' be the point of the line L that is closest to x, so $|x - z'| \leq |x - z|$. It follows that $\sin(\psi) = |x - z'|/|x - y| \leq |x - z|/|x - y| < 1/C_1$. Hence, $C_1 < 1/\sin(\arctan(R_1/R_2))$, i.e., $C_1 < \sqrt{(R_2/R_1)^2 + 1}$. Therefore, if we choose $C_1 \geq \sqrt{(R_2/R_1)^2 + 1}$, then we have the desired conclusion.

Claim 2: Next, we claim that if $d_H(K, K_n) \leq \varphi$ for some number $\varphi > 0$, then for any Borel set $F \subset \mathbb{B} = \overline{B}(0,1)$ the distance set $\Delta_K(F)$ is contained in the $C_2\varphi$-neighborhood of $\Delta_{K_n}(F)$ for some constant $C_2 > 0$ depending only on R_1, R_2. Indeed, let $|x| \leq 2$ be arbitrary. Then there exist $\alpha, \alpha_n > 0$ such that $\alpha x \in \partial K$ and

$\alpha_n x \in \partial K_n$. In particular, $\|x\|_K = 1/\alpha$ and $\|x\|_{K_n} = 1/\alpha_n$. By Claim 1 we have $|\alpha x - \alpha_n x| \leq C_1 d_H(K, K_n) \leq C_1\varphi$. It follows that

$$|\|x\|_K - \|x\|_{K_n}| = \frac{|\alpha_n - \alpha|}{\alpha_n \alpha} \leq \frac{C_1\varphi}{\alpha_n\alpha|x|} \leq \frac{2C_1\varphi}{|\alpha_n x||\alpha x|} \leq \frac{2C_1}{R_1^2}\varphi = C_2\varphi.$$

Now, if $x, y \in F$, then $|x-y| \leq 2$, so $\Delta_K(F)$ lies in the $C_2\varphi$ neighborhood of $\Delta_{K_n}(F)$, proving Claim 2.

To continue the proof of Lemma 4.1, we need the following elementary lemma that is proved later. Here m_1 denotes the 1-dimensional Lebesgue measure.

Lemma 4.2. *Let $F \subset \mathbb{R}$ be a bounded set.*

(i) *For each $c, \delta > 0$ we have $m_1(N_{c\delta}(F)) \leq \max\{1, c\} \cdot m_1(N_\delta(F))$.*
(ii) *For each $r, \delta > 0$ we have $m_1(N_\delta(rF)) \leq \max\{1, r\} \cdot m_1(N_\delta(F))$.*

Let $\{D_n\}$ be as in Lemma 3.2 and set $W_k = [0,2] \cap D_k$, $k \in \mathbb{N}$. This is a compact null set for each $k \in \mathbb{N}$. Thus for each $n \in \mathbb{N}$ there is a $\delta_n > 0$ so that the δ_n-neighborhood of W_k has length less than $2^{-k}/(C_3 n)$ for $k = 1, \ldots, n$, where $C_3 = (1+C_2)(1+1/R_1)$ and our choice will be evident later. Here we have used the fact the length of the δ-neighborhood of a compact null set in $\mathbb{R}$ tends to zero with δ by the Lebesgue dominated convergence theorem. Let $\varphi(n) = \delta_{n^2}$; note that this definition depends only on R_1, R_2 and not on K, K_n.

Now suppose $d_H(K, K_n) \leq \varphi(s_n)$ as in the statement of Lemma 4.1(i). For $n \in \mathbb{N}$ let $t_n \coloneqq \sum_{k=1}^n s_k \leq n s_n \leq s_n^2$. Enumerate the normal vectors corresponding to the sides of $K_1, K_2, \ldots$ and let these be the θ_k in Lemma 3.3 and in Lemma 3.2; also let $E \subset \mathbb{B}$ be the d-dimensional set given by Lemma 3.2for this sequence. The enumeration of the θ_k is such that the normal vectors for the sides of K_n are accounted for among the first $t_n = s_n + t_{n-1}$ vectors in the list, and therefore, as in the proof of Theorem 1.1, we have

$$\Delta_{K_n}(E) \subset \bigcup_{k=1}^{t_n} |\theta_k| W_k \subset \bigcup_{k=1}^{s_n^2} |\theta_k| W_k.$$

Since K_n approximates K to within $\varphi(s_n)$, we deduce from Claim 2 that $\Delta_K(E)$ lies inside the $C_2\varphi(s_n)$-neighborhood of $\cup_{k=1}^{s_n^2} |\theta_k| W_k$. Hence, $\Delta_K(E)$ lies inside the union of the $C_2\varphi(s_n)$-neighborhoods of the sets $|\theta_k| W_k$, $k = 1, \ldots, s_n^2$. By Lemma 4.2, and using the fact that $|\theta_k| \leq 1/R_1$ by Lemma 3.3, we have

$$\begin{aligned} m_1(N_{C_2\varphi(s_n)}(|\theta_k|W_k)) &\leq \max\{1, C_2\} \cdot m_1(N_{\varphi(s_n)}(|\theta_k|W_k)) \\ &\leq \max\{1, C_2\} \cdot \max\{1, |\theta_k|\} \cdot m_1(N_{\varphi(s_n)}(W_k)) \\ &\leq (1 + C_2)(1 + 1/R_1) m_1(N_{\delta_{s_n^2}}(W_k)) \\ &\leq C_3 \frac{2^{-k}}{C_3 s_n^2} \end{aligned}$$

for $k = 1, \dots, s_n^2$. Since $C_3 > C_2$, it follows that

$$m_1(\Delta_K(E)) \leq \sum_{k=1}^{s_n^2} m_1(N_{C_2\varphi(s_n)}(|\theta_k|W_k)) \leq \frac{1}{s_n^2},$$

which completes the proof of part (i). Since $1/s_n^2 \to 0$, part (ii) also follows. $\square$

Proof of Lemma 4.2. If $c \leq 1$, then (i) is immediate since $N_{c\delta}(F) \subset N_\delta(F)$. Suppose that $c > 1$ and consider the δ-neighborhood of F, which is a bounded open set. Hence, it can be written as a finite union of bounded disjoint open intervals I_i, $i \in I$, each having length at least 2δ. We append to each I_i two closed intervals of length $(c-1)\delta$, disjoint from I_i, so that one is appended to the left endpoint and the other to the right endpoint of I_i. We let I_i', $i \in I$, be the resulting collection of intervals, which satisfy $m_1(I_i') = m_1(I_i) + 2(c-1)\delta$. Note that the union $\cup_{i \in I} I_i'$ covers $N_{c\delta}(F)$. It follows that

$$m_1(N_{c\delta}(F)) \leq \sum_{i \in I} (m_1(I_i) + 2(c-1)\delta) \leq \sum_{i \in I} c m_1(I_i) = c m_1(N_\delta(F)).$$

Part (ii) follows from (i) and the observation that $N_\delta(rF) = r N_{\delta/r}(F)$. $\square$

We will give a separate proof of Theorem 1.2 in the case $d = 2$, since it is quite visual and simple to state. However, the proof for general dimensions, given later, also applies to the planar case.

Proof of Theorem 1.2 for $d = 2$. We first build a strictly convex example and then explain how to make it C^1. We will construct K as as a limit of nested, increasing, finite convex polygons $\{K_n\}$. Let K_1 be a square centered at the origin. Suppose, in general, we are given a polygon K_n with 2^{n+1} sides. Let I_k be one face of ∂K_n, denote the midpoint of I_k by x_k, and define y_k as the outward normal vector to K_n at x_k. Let $z_k = x_k + \varepsilon_n y_k$, where $\varepsilon_n \leq \frac{1}{2}\varphi(2^{n+2})$; here φ is the function from Lemma 4.1, corresponding to a small fixed $R_1 > 0$ and a large fixed $R_2 > 0$, and the convex sets K_n, K are constructed so that $B(0, R_1) \subset K, K_n \subset B(0, R_2)$. Replace the edge I_k by an arc J_k consisting of two edges, connecting the endpoints of I_k to the point z_k. This gives the polygon K_{n+1}. See Figure 4. If ε_n is small enough (but positive) K_{n+1} will still be convex. Moreover, if $\varepsilon_{n+1} \leq \varepsilon_n/2$ is sufficiently small, then the limiting region K will not contain any line segments in its boundary so it will be a strictly convex norm ball which is approximated to that $2\varepsilon_n$ by K_n. Thus by Lemma 4.1, there is a compact set E of dimension 2 so that $\Delta_K(E)$ has length zero, as desired.

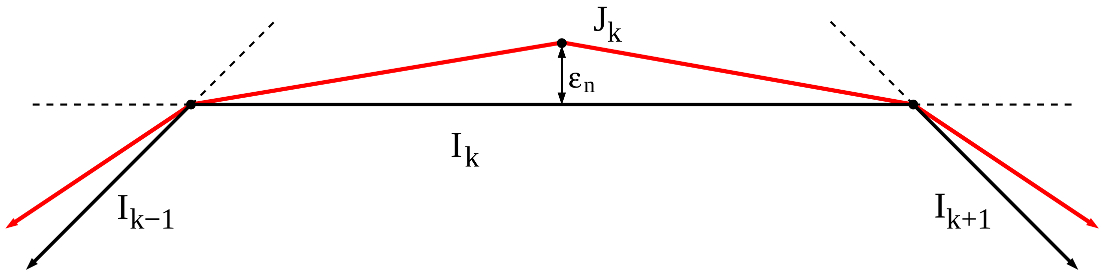

FIGURE 4. Constructing K_{n+1} from K_n to give a strictly convex example.

The example described above is not C^1 since there are countably many extreme points where the exterior angle is strictly larger than π. However we can eliminate these corners as follows. Instead of replacing each edge I_k by two edges as above, we replace it by a polygonal arc J_k with 4 edges, as illustrated in Figure 5.

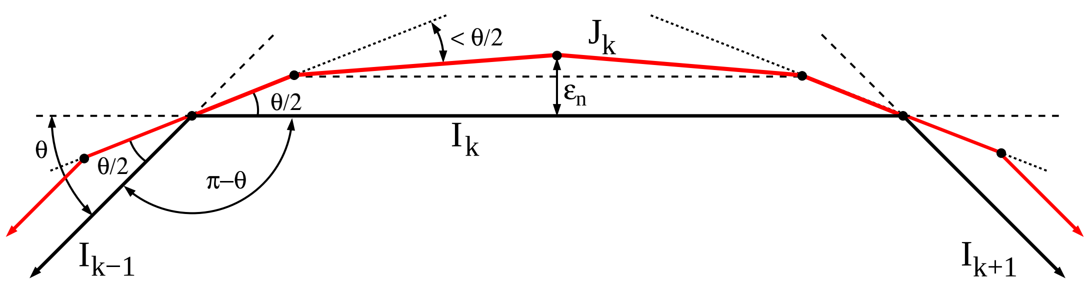

FIGURE 5. Constructing K_{n+1} from K_n to give a C^1 boundary in the limit.

If the two points near each endpoint of I_k are chosen correctly (see Figure 5), then the exterior angle at each of these points is approximately half the angle θ at the corresponding endpoint of I_k (and can certainly be chosen to be less than $2/3$ of that angle). The angle at the central vertex is as close to zero as we wish. Thus all the exterior angles for K_{n+1} are less than the maximum exterior angle for K_n by a fixed factor strictly less than 1. This implies ∂K is has a (unique) tangent at each point. By Lemma 4.3 below we conclude that ∂K is C^1. $\square$

Lemma 4.3. *Let $K \subset \mathbb{R}^d$ be a norm ball such that there exists a unique supporting hyperplane at each point of ∂K. Then ∂K is a C^1-smooth $(d-1)$-submanifold of $\mathbb{R}^d$.*

This lemma follows from Theorem 25.1 and Corollary 25.5.1 of [18], if one uses as local coordinates the projection from ∂K to a tangent hyperplane of ∂K. In fact, ∂K is homeomorphic to the $(d-1)$-sphere under the map $x \mapsto x/|x|$.

In our construction above, the boundary curvature of the limiting set K will be a measure μ supported ∂K that is singular to length measure. Can this measure have positive dimension? Is there some relation between whether the (K, α)-conjecture holds and the dimension of the measure μ, say $\alpha > 2 - \dim(\mu)$?

Proof of Theorem 1.2 for $d \geq 2$. We use Proposition 2.1 of [9] which states that any convex body $K \subset \mathbb{R}^d$ can be approximated as closely as we wish in the Hausdorff metric by a C^∞ strictly convex body that contains K. Although it is not explicitly stated there, the proof in [9] shows that if K is symmetric then the approximation will be too.

Start with a cube centered at the origin and approximate it by a smooth convex body S_1 to within $\varepsilon_1 > 0$, where ε_1 will be fixed below, subject to several additional conditions. Choose a finite, symmetric collection points that are sufficiently dense on ∂S_1 that the intersection of half-spaces containing S_1 and touching ∂S_1 at these points defines a polytope approximating S_1 within ε_1.

Since S_1 is strictly convex, its boundary contains no line segment and hence for any $\delta > 0$ there is a $\eta > 0$ with the following property: any segment of length δ that lies outside the interior of S_1 contains a point at least distance 2η from S_1. In particular, any convex body K_1 that contains S_1 and approximates it to within η cannot contain any δ-long segment in its boundary. Thus by taking ε_1 small enough we may assume that ∂K_1 contains no line segment of length $1/2$.

Using the smoothness of S_1, given any $\delta > 0$ we can also choose $\eta > 0$ so small that if a $(d-1)$-plane P that misses the interior of S_1 comes within η of a point $x \in \partial S_1$, then the normal to P is within angle δ of the normal to S_1 at x. This implies that any convex K that contains S_1 and approximates it to within η, has normals that approximate the normals to S_1. Again, by taking ε_1 small enough, we can assume that any ray from the origin intersects S_1 and K_1 at points where the normals agree to within angle $1/2$ (for K_1, there might be multiple choices of the normal direction at some points, but they all satisfy this estimate).

Using these arguments repeatedly, we obtain a sequence of smooth, strictly convex, symmetric bodies $\{S_n\}$, finite symmetric polytopes $\{K_n\}$, and positive numbers $\{\varepsilon_n\}$ so that

(1) K_n contains S_n and approximates S_n to within ε_n in the Hausdorff metric.
(2) S_{n+1} contains K_n and approximates K_n to within ε_{n+1} in the Hausdorff metric.
(3) $\varepsilon_{n+1} \leq \varphi(s_n)/2$ where s_n is the number of faces of K_n and φ is as in Lemma 4.1. Hence the distance set $\Delta_K(E)$ for any body K approximating K_n to within $2\varepsilon_{n+1}$ has length less than $1/s_n^2$.
(4) ε_n is so small that any $(d-1)$-plane that comes within $2\varepsilon_n$ of a point $x \in S_n$ without hitting S_n has normal direction that is within 2^{-n} of the normal to S_n at x.
(5) ε_n is small enough that any convex body that contains S_n and approximates it to within $2\varepsilon_n$ contains no segment of length 2^{-n} in its boundary.
(6) $\varepsilon_{n+1} \leq \varepsilon_n/4$.

Condition (6) implies the limiting body K approximates S_n to within $\varepsilon_n + \sum_{k \geq n+1} \varepsilon_k \leq 2\varepsilon_n$ and hence contains no segments at all by condition (5). Hence K is strictly convex. Conditions (6) and (3) imply the distance set $\Delta_K(E)$ has zero length. Conditions (6) and (4) imply that ∂K has a unique supporting hyperplane at each of its points. Indeed, if $x \in \partial K$ has two supporting hyperplanes with unit normals ν_1 and ν_2 respectively, then each of ν_1, ν_2 is within 2^{-n} of the normal of S_n at a point x_n. This implies that $\nu_1 = \nu_2$. By Lemma 4.3 we conclude that ∂K is a C^1, as desired. □

References

[1] Christopher J. Bishop and Yuval Peres. *Fractals in probability and analysis*, volume 162 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017.
[2] Jean Bourgain. Hausdorff dimension and distance sets. *Israel J. Math.*, 87(1-3):193–201, 1994.
[3] X. Du, L. Guth, H. Wang, B. Wilson, and R. Zhang. Weighted restriction estimates and application to Falconer distance set problem. 2018. preprint, arXiv: 1802.10186.
[4] X. Du and R. Zhang. Sharp L^2 estimate of Schrödinger maximal function in higher dimensions. 2018. preprint, arXiv:1805.02775.

[5] M. Burak Erdoğan. On Falconer's distance set conjecture. *Rev. Mat. Iberoam.*, 22(2):649–662, 2006.

[6] K. J. Falconer. On the Hausdorff dimensions of distance sets. *Mathematika*, 32(2):206–212, 1985.

[7] K. J. Falconer. Dimensions of intersections and distance sets for polyhedral norms. *Real Anal. Exchange*, 30(2):719–726, 2004/05.

[8] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.

[9] Mohammad Ghomi. Optimal smoothing for convex polytopes. *Bull. London Math. Soc.*, 36(4):483–492, 2004.

[10] L. Guth, A. Iosevich, Y. Ou, and H. Wang. On Falconer distance set problem in problem. 2018. preprint, arXiv: 1808.09346.

[11] Larry Guth and Nets Hawk Katz. On the Erdős distinct distances problem in the plane. *Ann. of Math. (2)*, 181(1):155–190, 2015.

[12] A. Iosevich and I. Łaba. K-distance sets, Falconer conjecture, and discrete analogs. *Integers*, 5(2):A8, 11, 2005.

[13] Alex Iosevich. What is Falconer's conjecture? *Notices Amer. Math. Soc.*, 66(4):552–555, 2019.

[14] Jean-Pierre Kahane. Sur la dimension des intersections. In *Aspects of mathematics and its applications*, volume 34 of *North-Holland Math. Library*, pages 419–430. North-Holland, Amsterdam, 1986.

[15] Sergei Konyagin and Izabella Łaba. Falconer's distance set conjecture for polygonal norms. In A. Miyachi and K. Tachizawa, editors, *Proceedings of Harmonic Analysis and its applications at Sapporo*, pages 43–55. Hokkaido University, Sapporo, 2005. arXiv math/0407503.

[16] Sergei Konyagin and Izabella Łaba. Distance sets of well-distributed planar sets for polygonal norms. *Israel J. Math.*, 152:157–179, 2006.

[17] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[18] R. Tyrrell Rockafellar. *Convex analysis*. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[19] Thomas Wolff. Decay of circular means of Fourier transforms of measures. *Internat. Math. Res. Notices*, (10):547–567, 1999.

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